Minimum stick number for knots and links

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Abstract

This paper looks at stick number of links and knots. We find an upper bound for n-integer Conway links, and classify all two component links with stick number of eight or less.

1 Introduction

The knot theory was born at the end of 19th century. It was believed that some of the atoms of a substance called ether could be represented by knots. However, this hypothes was incorrect and knot theory was left by chemists. This is when mathematicians got interested in knots. Knots were classified and put in the table according to specific notation which I will describe later. And then in 1980s, it was discovered that DNA molecules have a structures of knots. Finally knot theory could be applied to synthetic chemistry. Knot theory has also found some applications in theoretical physics.

2 Background

As the name knot theory suggests, this field of mathematics is talking about knots. A **knot** is a knotted loop of string, where the string has no thickness, and its cross-section is a single point. An example is a trefoil knot shown in Figure 1a below. A **link** is a set of knotted components all tangled up together, like the Hopf link shown in Figure 1b. Knots are the subset of links consisting of those with one component . A **crossing** of a link is a place where one of the strings goes over or under another depending on the projection. The **crossing** number is the minimum nubmer of crossings of a link L in any projection, denoted by $c[L]$. A **tangle** is just a region in the projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times.

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(a) (b)

Figure 1. (a) trefoil knot. (b) Hopf link.

A link can be represented using sticks. For example, an unknot can be represented as a triangle using three sticks. The **stick number**, denoted by $s[L]$, of a link L is the minimum number of sticks needed to form L.

Now that we have all these links a question that might arrise is how do we now which link is which, or is there a method of producing a link. The first time that the knots were tabulated was done by Peter Guthrie Tait, and he tabulated the knots with up to ten crossings. Right now knots of up to 16 crossings are tabulated, according to [5], and for example there are 9988 knots of 13 crossings. The number of knots with 17 or more crossings is not known. There are different notations that help us to make or identify a given link. The one that I will be dealing with in this paper is **Conway's notation**, which was invented by John H. Conway in 1969, see [3]. Before we get to Conway's notation consider Figure 2 below. We see two tangles, tangle (a) is 0-tangle and tangle (b) is ∞-tangle.

(a) (b)

Figure 2. (a) 0-tangle. (b) ∞-tangle

We can form a tangle with let's say two crossings, starting with 0-tangle and twisting the string twice. We would denote this tangle by 2, as in Figure 3a. It has two crossings and the overcrossing string has a positive slope. If the overcrossing string had a negative slope we would denote this tangle by -2, as in Figure 3b.

(a) (b)

Figure 3. (a) 2 tangle. (b) -2 tangle.

We can add more integers, thus, adding more crossings, for example tangle denoted by 232. We would start with our tangle in Figure 3a. Then each time we have another integer we will reflect the tangle about it's NW-SE axis (in [3] Conway names it principal diagonal), and then add 3 more twists. We will repeat this process until we get to last integer, in our case 2. The process is illustrated in Figure 4.

(a) (b)

Figure 4. Forming 232 tangle. (a) 23 tangle. (b) 232 tangle.

All tangles formed this way are called **rational tangles**. Now if we connect the right top with the left top and right bottom with the left bottom of a tangle we will form a **rational link**. It is worth mentioning that not all tangles are formed this way. In [3] Conway describes other ways of obtaining tangles, which lead to other general links.

Another important idea which I will use later is **linking number**. In order to find a linking number, first we have to assign an orientation to the link we wish to find the linking number of. Then we look at all crossings, and identify each as +1 or -1, as it is shown in Figure 5. We add all the +1's and -1's corresponding to crossings of different components, and divide the sum by two. The absolute value of the quotient gives us the linking number. For example, Hopf link in Figure 1b has linking number 1.

(a) (b)

Figure 5. Linking number.(a) +1. (b) -1.

All the above definitions can be found in any Knot Theory book. During the REU program we[1] invented a move that will help us produce stick representations of knots.

Definition 1 *The angle α at vertex v in a stick representation of a link L created by two edges adjecent to vertex v is called a* ***clasp angle****, and* $0^\circ \leq \alpha \leq 180^\circ$.

[1]the participants of REU at CSUSB, 2001

Definition 2 *Let v be a vertex with clasp angle α, and e be an edge in a stick representation of a link L, where e is not contained within the angle α, then the* ***clasp move*** *can be performed on v and e if the plane containing v and e divides the clasp angle. (see the Figure 6)*

(a) (b)

Figure 6. (a) Clasp angle. (b) Clasp move.

The topic of this paper was motivated by a paper written by Eric Furstenberg, Jie Li, and Jodi Schneider, titled "Stick Knots", [4]. Among many different topics, they considered an upper bound for a stick number using Conway notation. In the paper they defined few ideas that I will restate. A **free vertex** is a vertex that is produced by two sticks, one of them forming a crossing with another stick, as shown in the Figure 7.

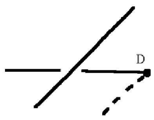


Figure 7. Free vertex D.

An **n-integer Conway knot** is a rational knot which can be expressed in Conway's notation with exactly n positive or n negative integers. These are guaranteed to produce alternating projections. Similairly I defined a **n-integer Conway link** being a rational link which can be expressed in Conway notation with exactly n positive or n negative integers.

The authors of [4] used the idea of the free vertex to prove the following lemma.

Lemma 1 *Given the projection of a stick knot K, if there is at least one free vertex, the addition of two sticks can change that crossing into three crossings.*

Using the above Lemma they found an upper bound of a stick number for 1, 2, and 3-integer Conway knots, which is:

$$\text{for } c[K] \geq 6, \text{ we have } s[K] \leq c[K] + 2 \tag{1}$$

My goal is to extend the idea of upper bound for a stick number from [4] to n-integer Conway links. I followed the ideas that were used in their proofs, and I was able to prove the upper bound of a stick number for 4-integer Conway knots. However, the proof becomes very tedious once it comes to prove it for 5, 6, up to n-integer Conway links. The reason is that the number of basic cases that will generate the other knots increases dramatically. For example, the number of basic cases for 1-integer Conway knots is just one and it is knot 7_1. For 2-integer Conway knots the number of generators increases to three, that is knots 6_1, 7_2, and 7_3. When it comes to 3-integer Conway knots the number of cases increases to six that is knots 6_2, 7_4, 7_5, 8_4, 8_6, and 9_9. What Diana[2] and I realized is that some of the cases for 3-integer knots are already generated by the ones that are included as generators. That is, knot 8_4 and 8_6 can be generated from knot 6_2 and knot 9_9 can be generated from knot 7_5. Thus, the number of cases reduces to 3. A similar reduction can be accomplished with 4-integer knots, and actually this is what I did. However, the number of cases is still large. For some reason the [4] didn't consider links. I wondered why. The reason might be that the free vertex idea was not enough to provide as efficient upper bounds. That is the link 4_1^2 can be made with 7 sticks as shown in [2], so that would mean that adding 2 sticks increases the number of crossings to 6, but also increases the number of sticks to nine. This doesn't satisfy the same inequality. However, the authors of [4], as well as [2], were not aware of a clasp move. Using the clasp move I was able to produce links 6_1^2, 6_2^2, and 6_3^2 with 8 sticks each. Now using the idea of a free vertex I was able to prove the upper bound in (1) for 1, 2, and 3-integer Conway links. The question still remains open as far as n-integer Conway knots and links.

3 Two-component links with s[L]≤8

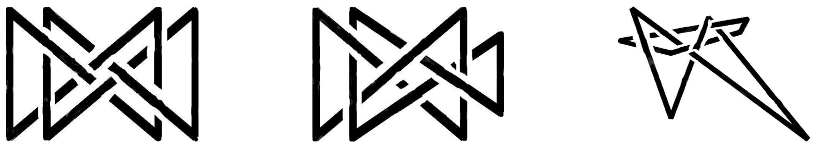
Not knowing how to get started on general proof of upper bound for a stick number for knots, I decided to take it one step at a time, and see why authors of [4] neglected links. According to Theorem 2.1 in [2] the only links that can be made with eight or fewer sticks are links 0_1^2, 2_1^2, 4_1^2, and 5_1^2. The next link in the table of knots and links in [1] is link 6_1^2, and since it cannot be made with eight sticks the inequality $s[K] \leq c[K] + 2$ would not be satisfied. However, the invention of clasp move shows that there are two more two-component links that have stick representation equal to eight, and the idea of the linking number gave me a hint that link 6_3^2 might have also eight sticks. If we start with a Hopf link which has a stick number of 6 then two clasp moves will produce either link 6_1^2, 6_2^2, or 6_3^2 depending on how we choose our over and under crossings. The procedure is presented in the Figure 6 on the next page.

Theorem 1 *Links 6_1^2, 6_2^2, and 6_3^2 have stick number equal to eight.*

Proof: Assume that 6_1^2, 6_2^2, and 6_3^2 can be made with 7 sticks. That means that one of the components must be a triangle and another must be a quadrilateral. However, none of the combinations of 6_1^2 or 6_2^2 will produce a link with 6 alternating crossings and linking number equal to three. When we project into

[2]Diana Wall participated with me in REU 2001

one of the sticks of the quadrilateral from 6^2_3, the projection will give us two triangles. We can't have two triangles with six alternating crossings. Therefore, the stick number must be eight since it is shown in the Figure 8 below.

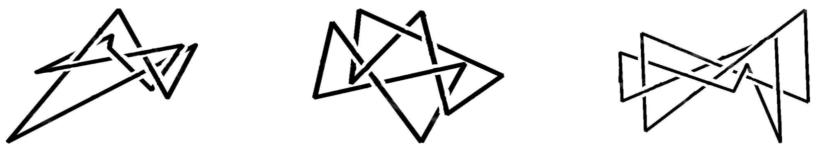


(a) (b) (c)

Figure 8. Two-component links with their Conway's notation.
(a) 6^2_1, 6. (b) 6^2_2, 33. (c) 6^2_3, 222.

4 Upper bound of stick number for two-component links

The following three theorems were motivated by [4]. A base case is a link from which other links can be generated. For example, if we take a look at one-integer Conway link 6^2_1, its Conway notation is 6. Looking at Figure 8a we can choose any of the vertices to be a free vertex, and by Lemma1 the addition of two sticks will increase the crossing number by two, therefore, producing link 8^2_1 with Conway notation of 8. The results from previous section are very helpful, especially the invention of link 6^2_3 with 8 sticks. One and two-integer Conway links have one base case each, 6^2_1 and 6^2_2 respectively. When it comes to three-integer Conway links, the number of base cases increases to four, and they are: 6^2_3, 7^2_1, 7^2_3, and 8^2_4, which can be made with 8, 9, 9, and 10 sticks respectively, as shown in Figures 8c, and 9.



(a) (b) (c)

Figure 9. Two-component links with their Conway's notation.
(a) 7^2_1, 412. (b) 7^2_3, 232. (c) 8^2_4, 323.

Theorem 2 *The upper bound for one-integer Conway links with* $c[K] \geq 6$ *is* $s[K] \leq c[K] + 2$.

Proof: It was shown in Figure 8a that one-integer link 6^2_1 can be made with eight sticks, and this will be our base case. Any of the vertices can be a free vertex depending on a projection. So if we choose a free vertex then using Lemma1 we can transform that link into link 8^2_1. By repeating this procedure we can generate all one-integer links.

Theorem 3 *The upper bound for two-integer Conway links with* $c[K] \geq 6$ *is* $s[K] \leq c[K] + 2$.

Proof: In Figure 8b we have shown that link 6^2_2 can be constructed with eight sticks. Again this will be our base case. A free vertex can be chosen and by Lemma1 we can generate link 8^2_2 with 10 sticks. Therefore all the two-integer Conway links can be genereated this way.

Theorem 4 *The upper bound for three-integer Conway links with* $c[K] \geq 6$ *is* $s[K] \leq c[K] + 2$.

Proof: In Figure 8c and we have shown that link 6^2_3 can be constructed with eight sticks. In Figure 9 we have shown our other three base cases which can be made with 9, 9, and 10 sticks respectively. Each of the base cases has a free vertex, and each of our base cases will generate other links. For example, link 6^2_3 will generate either link 8^2_3 or 8^2_6, depending where we chose our free vertex to be. We can generate all three-integer Conway links this way.

The question about the upper bound for any link remains still open. It seems that the upper bound that I used in Theorems 2, 3, and 4 might be the answer. But even better question is, can this upper bound be improved.

During the REU program we were using a DrawKnot program, written by Dr. Rolland Trapp on Maple, to draw links in stick representation. The program intakes the vertices of the links and outputs pictures such as Figure 8.

5 References

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