Stick Numbers of Links with Two Trivial Components

Alexa Mater

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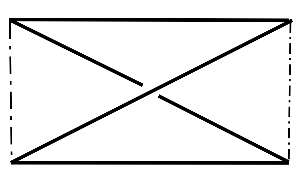
1 Introduction

A knot is an embedding of S^1 in S^3, and a link is an embedding of one or more copies of S^1 in S^3. The number of copies of S^1 is called the number of components of the link. We usually think of a link as made out of string, but we can also think of the link as made up of line segments, which we call sticks, which can connect at any angle, but cannot bend. We would like to know the minimum number of sticks required to make any given link, or alternatively, which links can be made with a given number of sticks and no fewer. We call the minimum number of sticks required to make a link the stick number of the link, denoted $s(L)$. Here we give a proof that there is only one link with $s(L) = 7$ and several lemmas which may eventually lead to a classification of the links with $s(L) = 8$.

2 Geometric Preliminaries

Any four stick component of a link is a quadrilateral. Note that, unless it lies in a plane, any such quadrilateral defines a tetrahedron by connecting each pair of nonadjacent vertices with additional edges, which we will call hinges.



A tetrahedron defined by a quadrilateral

The two canonical discs defined by a quadrilateral

Each face of the tetrahedron lies in a plane and is bounded by a triangle, consisting of two edges of the quadrilateral, plus a hinge. We define a canonical disc of the quadrilateral to be the union of the two faces adjacent to a single hinge. There are two such discs.

Suppose we color each disc defined by the quadrilateral. Then there will be two discs colored c_1 and c_2. We can also color the hinge of each disc the same color as the disc. Notice that we can color each section of each intersection arc according to the color of the face represented by each segment.

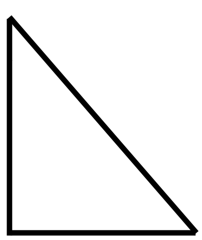
3 seven stick 2-component links

Theorem: Let L be a 2-component link. If $s(L) = 7$, them L is the 4_1^2 link.

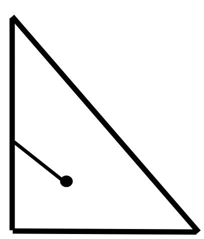
Since no knot can be made with fewer than 3 sticks, every component of a 2-compononent link must have at least 3 sticks. Therefore, every seven stick link must have one 3-stick component and one 4-stick component, i.e. a triangle and a quadrilateral (which defines a tetrahedron, as above). The triangle lies in a flat plane, and we will call portion of the plane enclosed by the triangle, the canonical disc of the triangle.

Note also that since each face of the tetrahedron lies in a plane, and the triangle lies in a plane, the triangle can only intersect each face a maximum of one time. Moreover, each intersection must form a one dimensional arc.

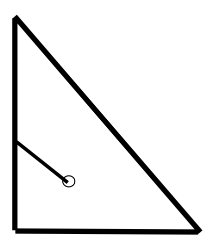
Consider the possible intersections of a triangle and one face of a tetrahedron (as described above). There are six possible types of intersections, as follows:



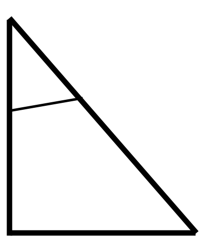
type A



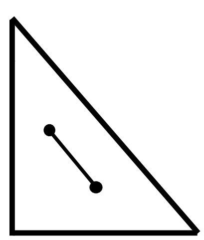
type B



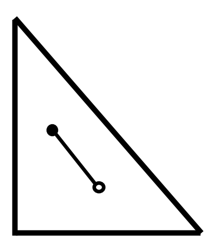
type C



type D



type E



type F

In these drawings, solid dots represent edges of the quadrilateral. Open dots represent hinges.

These patterns consider only the intersection of one face of the tetrahedron with the triangle. Notice that the complete intersection pattern, must be either a simple closed curve, or an arc with both endpoints intersecting the edges of the triangle.

Lemma 1: Let L be a seven stick, two component link. If the 3-stick component does not intersect all four faces defined by the 4-stick component, then $s(L) \leq 6$.

Proof: Suppose there is a face of the tetrahedron not intersected by the triangle. This face is bounded by two edges of the quadrilateral and one hinge. Without changing the intersection pattern, we can replace the two solid edges of nonintersected face of the tetrahedron with one solid edge along the hinge. This reduces the tetrahedron to a triangle without changing the intersection pattern of the link. Therefore, $s(L) \leq 6$. $\Box$

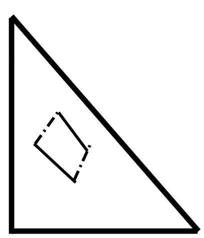
Therefore, we need only consider cases in which the triangle intersects all four faces of the tetrahedron.

Lemma 2: Any arc of intersection containing a segment as in either type E or type F can be reduced.

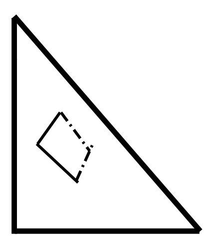
Proof: The face of the tetrahedron forming the type E or F intersection is not intersected by the boundary of the triangular component. Therfore, it is a reducing triangle. $\Box$

The possible cases are:

1. one closed curve

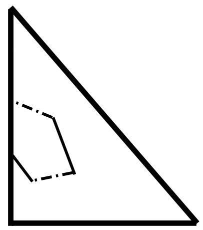


case 1a

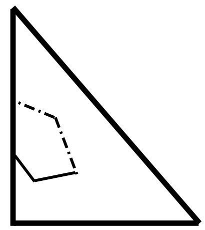


case 1b

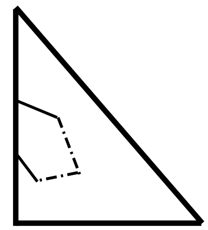
2. one four segment arc



case 2a

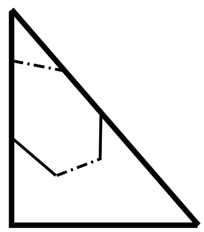


case 2b

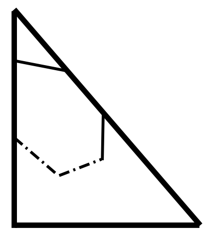


case 2c

3. one three segment arc and one one segment arc

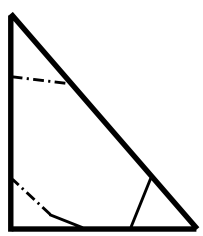


case 3a

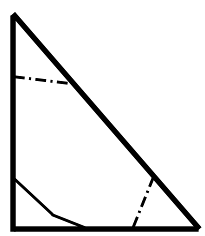


case 3b

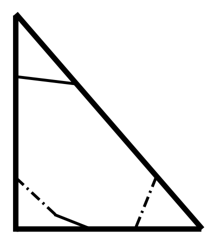
4. one two segment arc and two one segment arcs



case 4a

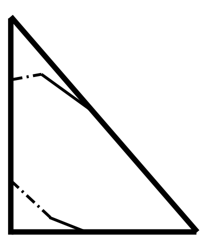


case 4b

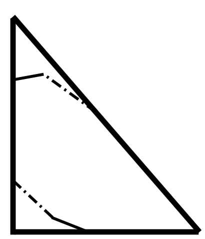


case 4c

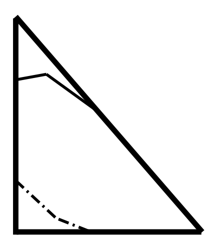
5. two two segment arcs



case 5a



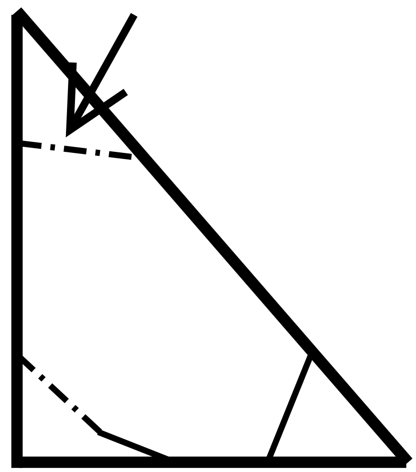
case 5b



case 5c

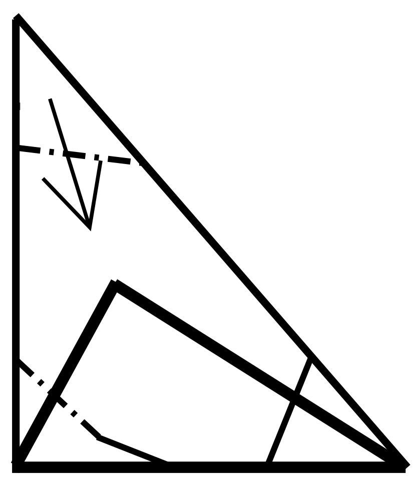
By lemma 2, every pattern in cases 1, 2, and 3 can be reduced by at least one segment, reducing the pattern to one with fewer than four segments. Therefore every one of these cases can be reduced to a link with $s(L) < 7$.

In case 4a, there is one arc consisting of only a single segment, as illustrated



a one segment arc

Since this intersection does not contain any solid edges, we can continuously transform the triangle to eliminate the intersection, without changing the link type:

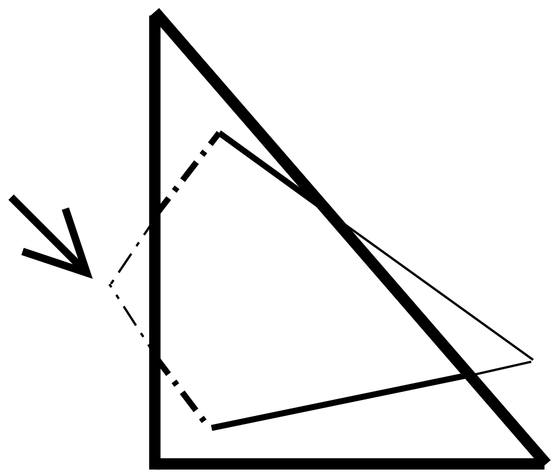


continuously deforming the triangle

But, the resulting intersection pattern has only three segments, therefore, it cannot be a seven stick link.

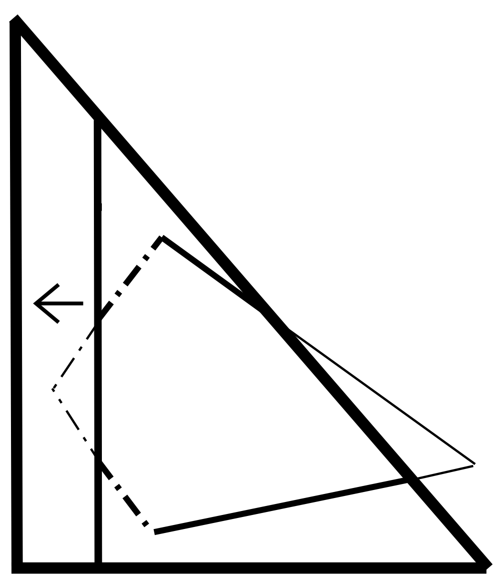
We can perform a similar deformation in case 3b and case 4, thereby reducing these links to links requiring fewer than seven sticks.

In case 5a, there is a hinge lying outside the triangle, as follows:



A hinge outside the triangle

Again, we can continuously deform this triangle to reduce case 5a to case 2c:



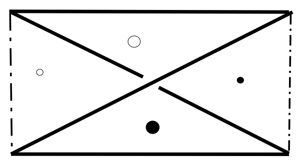
Reducing case 5a to case 2c

In case 5c, we can perform a deformation similar to that in case 3a, since there is an arc of intersection containing only a hinge and no solid edges. This reduced case 5c to an intersection with only two segments.

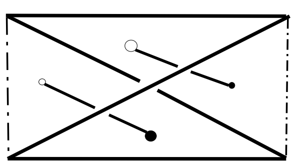
This leaves only case 5b, which cannot be reduced.

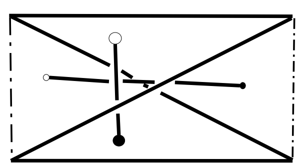
Theorem: Case 5b is exactly the intersection pattern of the 4_1^2 link.

Proof: In order to show that case 5b gives the 4_1^2 link, we consider how the triangle intersects the quadrilateral, rather than how the quadrilateral intersects the triangle. By lemma 1, the triangle intersects each face of the tetrahedron at least once. Notice that each endpoint of the intersection arc corresponds to an intersection of the triangle with a face of the tetrahedron. There are exactly four endpoints, therefore the triangle intersects the tetrahedron four times, i.e. once on each face. (Open circles represent intersections with the c_1 disc and filled circles with c_2, large circles represent intersections with the front two faces and small circles with the back two faces.)

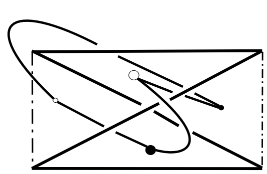


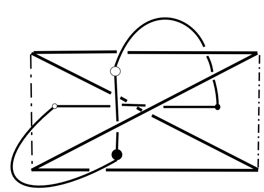
We will first consider the edges that connect the points of intersection on the inside of the tetrehedron. Notice that the two c_1 intersections cannot be connected, because in the intersection pattern we are considering, the triangle's intersections alternate between c_1 faces and c_2 faces. So the front c_1 intersection must be connection to one of the two c_2 intersections. (These are the same up to symmetry). Then the other c_1 intersection must be connected to the other c_2 intersection.



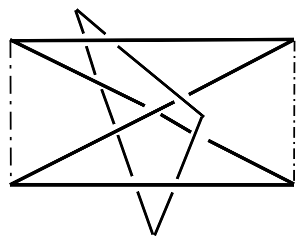


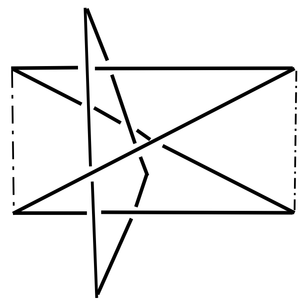
Each of these options determines completely how the vertices are connected on the outside. These are shown below.





Neither of these diagrams is a stick link, and it remains to show that each of these patterns can be constructed with a quadrilateral and a triangle, which is true:





Therefore, every two component link with $s(L) = 7$ is the 4_1^2 link. $\square$

4 Some Results Regarding Eight Stick Links

Since every unknot requires at least three sticks to make, no component of a stick link can have fewer than three sticks. Therefore, an eight stick link must consist of either two four-stick components, or one three-stick and one five-stick component. Here, we consider links with two four-stick components.

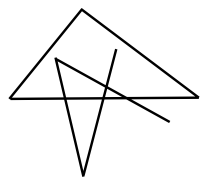
It is already known that all of the five and six crossing two component links have stick number 8. Here we will begin work towards a proof that these are all of the links with stick number 8.

Consider a two dimensional projection of a stick link. It is always possible to make such a diagram with $s(L) - 1$ line segments, by taking the projection looking down one edge of the stick link. This edge is then reduced to a point, which we will call the projection point. We will call such a projection a reducing projection. In the case of an eight stick link with two four-stick components, such a diagram will then consist of a triangle with a projection point and a quadrilateral.

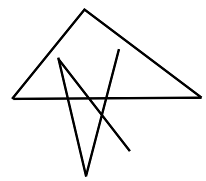
Each edge of the quadrilateral can cross the triangle no more than two times. Moreover, the quadrilateral can have no more than one self-crossing. Therefore, the entire diagram can have no more that nine crossings. It follows immediately that no linking with a crossing number greater than 9 can have a stick number of 8.

We consider first diagrams with seven crossings. Every seven crossing link is alternating. Therefore, we need only show that no seven crossing reducing projection can be alternating to show that every such projection reduces to a link of six or fewer crossings.

Lemma (Triple lemma): Every reducing projection containing a "triple", i.e., one of the following two patterns:



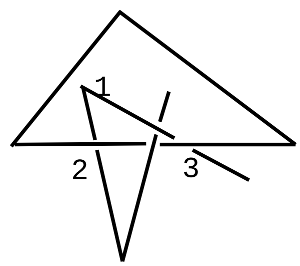
Triplet 1



Triplet 2

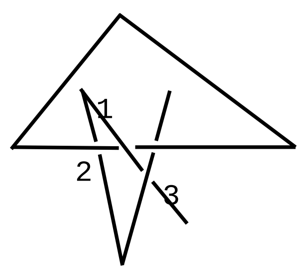
is non-alternating.

Proof: Consider an alternating diagram of Triple 1:



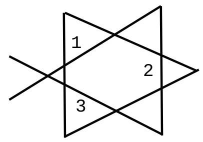
Notice that edges 1 and 2 define a plane. In order to cross over the edge of the triangle, edge 3 must be above the plane, since edges 1 and 2 both cross under the triangle. Edge 3 must be under the plane in order to cross under edge 1, this means that edge three intersects the plane defined by edges 1 and 2, which is impossible, since one edge of a quadrilateral cannot intersect the plane formed by two other edges.

Similarly, consider an alternating diagram of Triple 2:

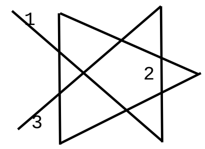


In this case, edges 2 and 3 define a plane intersected by edge 1, which is again impossible. Therefore, every reducing projection containing a triple is non-alternating. □

Lemma (Star lemma): Every reducing projection containing a "star", i.e., one of the following two patterns:



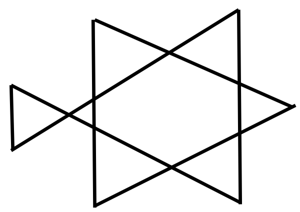
Star 1



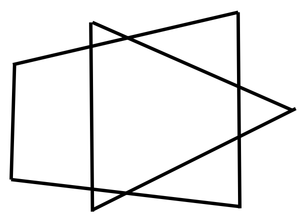
Star 2

can be reduced to a projection with fewer than 7 crossings.

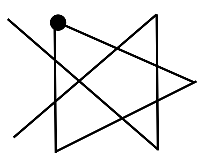
Proof: First consider Star 1. There is only one place to put the fourth edge of the quadrilateral, namely between edges 1 and 3, as follows:

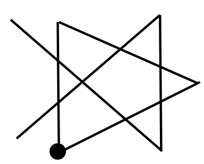


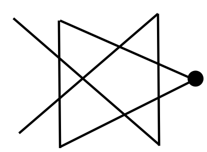
But this arrangement allows a Reidemeister 1 move on the self crossing in the quadrilateral, reducing the projection to a six crossing projection.



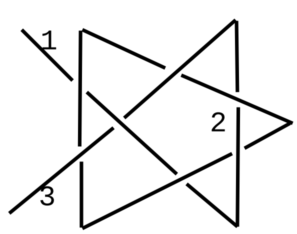
Now consider Star 2. Notice that this situation has three subcases, based on the location of the projection point.


Point 1


Point 2

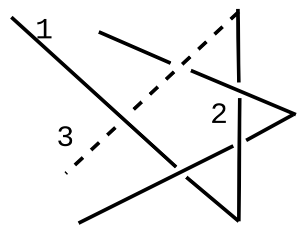

Point 3

Consider an alternating diagram of a type 2 star.

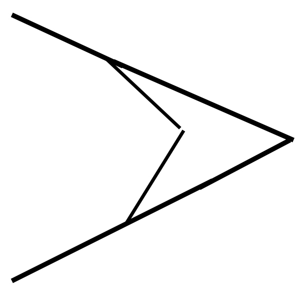


An alternating type 2 star

In cases 1 and 2, the two right hand edges of the triangle form a plane. Below is a diagram of the intersection of these 2 edges of the triangle with edges 1 and 2 of the quadrilateral.



Consider the intersection pattern of the above case on the two edges of the triangle shown above.

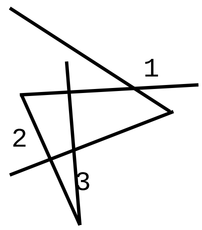


But this pattern is impossible, since it contains two segments contributed by the same face of tetrahedron defined by the quadrilateral. Therefore, cases 1 and 2 cannot be alternating.

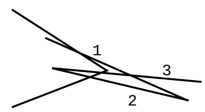
Now consider case 3. In this case, the left and bottom edges of the triangle form a plane. At both the top and bottom vertices of the left edge of the triangle, this edge must be above the plane defined by edges 1 and 2 of the quadrilateral, implying that the whole edge must be above this plane. However, this edge crosses under edge 3 of the quadrilateral, which itself crosses under edge 1, therefore the left edge of the triangle must pass under the plane defined by edges 1 and 2, which is a contradiction.

Therefore, a type 2 star cannot be alternating, which implies that it is reducible. □

Lemma (Corner Lemma): Every reducing projection containing a "corner", i.e., one of the following two patterns, where the two edges of the triangle lie in a single plane (i.e. the projection point is not between them):



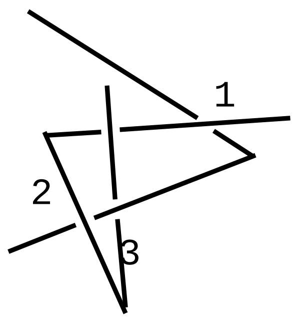
Corner 1

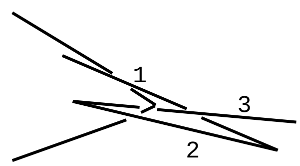


Corner 2

is nonalternating.

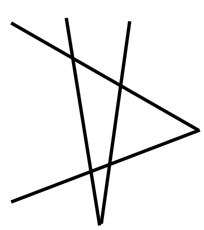
Proof: Consider alternating diagrams of the two corner projections shown above.





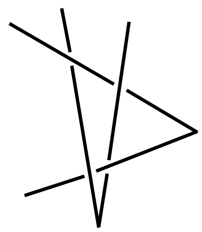
In both cases, edges 1 and 2 lie in a plane, and the plane defined by the two edges of the triangle lies below this plane. Edge 3 must lie under the plane of edges 1 and 2, since it crosses under the triangle, but it must be above this plane to cross over edge 1, implying that edge 3 intersects the plane defined by edges 1 and 2. But edge 3 is adjacent to edge 2, so it cannot intersect this plane. Therefore, any projection containing a corner is nonalternating. □

Lemma (Weave lemma): Every reducing projection containing a "weave", pictured below, where the two edges of the triangle lie in a single plane (i.e. the reducing point is not between them):

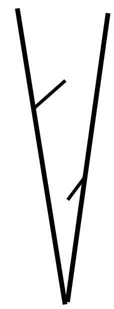

a Weave

is nonalternating.

Proof: Consider an alternating diagram of Weave, and the intersection pattern generated on the pictured face of the quadrilateral by such a diagram.



an alternating weave



the intersection pattern on the quadrilateral

As the intersection pattern shows, the triangle intersects one face of the tetrahedron defined by the quadrilateral two times, which is impossible. Therefore, every intersection pattern containing a weave is nonalternating. □

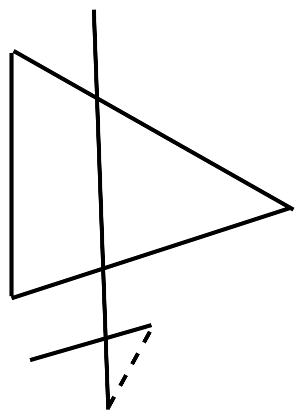
Theorem: Every seven crossing, seven stick reducing projection is nonalternating.

Proof: As noted above, each edge of the quadrilateral can cross the triangle no more than twice. In order to cross the triangle twice, a line segment must begin and end outside the triangle. Similarly, any line that crosses the triangle only once must have one endpoint outside the triangle and one inside the triangle. Therefore, since a quadrilateral is a closed curve, every single crossing segment must be adjacent to another single crossing edge. Since this is the case, every reducing projection of two quadrilaterals must contain an even number of crossings between the two components, implying that every seven crossing projection consists of six crossings between the components plus one self crossing in the quadrilateral.

Moreover, since no edge of the quadrilateral can cross the triangle more that twice, either three edges of the quadrilateral cross the triangle twice and the other does not cross at all, or two edges cross twice and two cross once. Since, as noted above, the two single crossing edges must be adjacent. We will denote these two cases as 2-2-2-0 and 2-2-1-1 respectively.

4.1 Case 1: 2-2-2-0

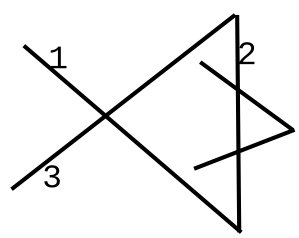
First, notice that the zero-crossing edge of the quadrilateral cannot be involved in the self-crossing. If it were, then there would be two endpoints on the same side of the triangle, which would have to be connected by another zero-crossing edge.



The zero-crossing edge cannot be in the self-crossing

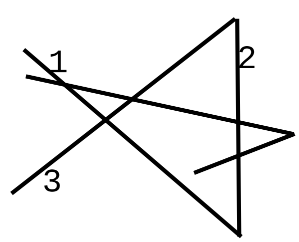
Therefore, the crossing has to occur between edges 1 and 3, since it must occur between nonadjacent edges, and edge 2 is nonadjacent only to the zero-crossing edge.

Edge 2 must be crossed by two edges of the triangle, which obviously must be adjacent. These two edges must connect on one side of edge 2. Clearly, this must be on the "outside", of the quadrilateral since otherwise the third edge of the triangle would not be able to intersect the quadrilateral.



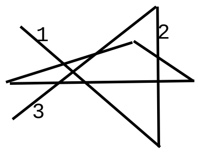
The edges intersecting side 2 must connect outside the quadrilateral

The top edge of the triangle may intersect 1, 2, or 3 edges of the quadrilateral. If it intersects 3 edges, then we have a pattern like this:

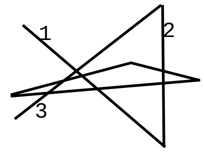


which by the Triple lemma is nonalternating.

If the top edge crosses only one edge of the quadrilateral (i.e. edge 2), then the entire projection must be one of the following:

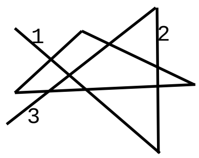


case A

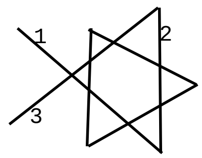


case B

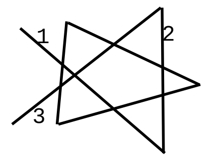
If the top edge crosses two edges of the quadrilateral (i.e. edges 1 and 3), the the following projections are possible:



case C



case D



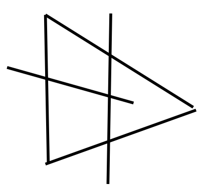
case E

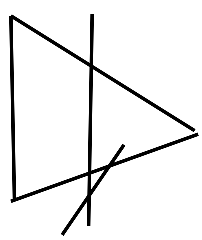
In cases A, B and C, the triple lemma applies so these must be nonalternating, in cases D and E the star lemma applies, so these are also nonalternating.

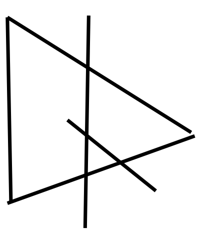
Therefore, every 7 stick reduced projection of type 2-2-2-0 is nonalternating and is therefore reducible to a link of crossing number 6 or less.

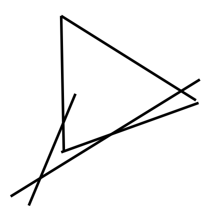
4.2 Case 2: 2-2-1-1

Since the self crossing in the quadrilateral must occur between two nonadjacent edges, it must occur between one single crossing edge and one double crossing edge. The edges involved in the self crossing can either cross the same or different edges of the triangle. Moreover, the self crossing can occur either inside or outside the triangle. This gives four possible cases for the two quadrilateral edges involved in the self crossing:

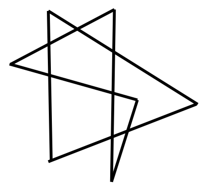

case 1


case 2

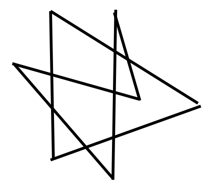

case 3


case 4

In case 1, there are two possible ways to connect the remaining two edges of the diagram, as follows:



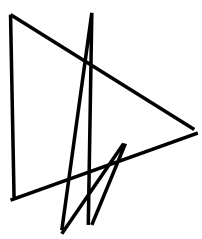
subcase 1a



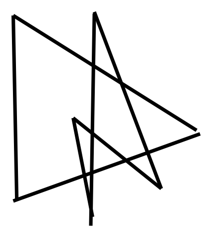
subcase 1b

however these are the same up to symmetry.

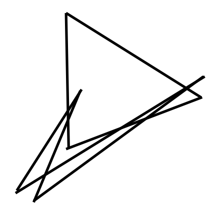
In the other three cases, there is one way to connect the remaining edges in each case:



case 2 completed



case 3 completed



case 4 completed

Subcase 1a contains a star pattern if the reducing point is at the lower corner or the righthand corner and subcase 1b contains a star patter if the reducing point is at the upper corner or the righthand corner. Otherwise, each subcase contains a corner pattern. In either case, it must be nonalternating.

Cases 2 and 3 each contain a triple pattern and are therefore nonalternating.

Case 4 contains a weave unless the reducing point is at the righthand corner of the triangle. In that case it contains a corner pattern. Therefore, case 4 is nonalternating.

We conclude that every seven crossing reducing projection with seven stick is nonalternating, and is therefore reducible to fewer than seven crossings. □

It remains to show that eight and nine crossing reducing projections of the above type reduce to links of six or fewer crossings, and also that eight stick links consisting of a triangle and a pentagon cannot have crossing number greater than 6.

5 Acknowledgements

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[2] Colin C. Adams, Bevin M. Brennan, Deborah L. Greilsheimer, and Alexander K. Woo. Stick Numbers and Composition of Knots and Links, *J. of Knot Theory and its Ramifications, 6.* 1996, 149-161.