Cutwidth of a Complete Graph Embedded on an $m \times n$ Grid

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Abstract

We develop techniques for embedding complete graphs on mn vertices into a $m \times n$ grid such that minimum grid cutwidth can be obtained.

1 Introduction

A graph $G = (V, E)$ consists of a set V of vertices and a set E of edges connecting pairs of vertices. A complete graph with n vertices, denoted by K_n, is a graph in which every vertex is connected to every other vertex by a single edge. To embed a graph onto a grid, the vertices are arranged so that they form a grid and edges between vertices are routed so that they follow horizontal and vertical movements outlined by the grid formation of the vertices. The focus of this paper is the embedding of the complete graph of mn vertices onto an $m \times n$ grid. In particular, this paper is interested in finding the grid cutwidth of such a graph embedded in an $m \times n$ grid. Initially for any $m \times n$ grid, the following lower bound on the grid cutwidth can be set:

$$gcw(K_{m\times n}) \geq \begin{cases} \frac{mn^2}{4} & n \text{ even} \\ \\ \frac{m(n^2-1)}{4} & n \text{ odd.} \end{cases}$$

This lower bounded is derived from the fact that if we divide an $m \times n$ grid as evenly as possible into two parts, there will be, if n is even, $m\frac{n}{2}$ vertices on the left which we want to connect to the $m\frac{n}{2}$ vertices on the right. Thus, $\frac{m^2n^2}{4}$ edges must be run to connect the left to the right. Further, since there are m paths leading from left to right, there must be at least $\frac{mn^2}{4}$ edges running across one of these m paths. A similar process can be repeated for n odd.

To show that this lower bound can be met in general with any $m \times n$ grid, a general construction for an $m \times n$ grid will be used and shown to give the grid cutwidth previously established by the lower bound. The following are common terms that will be used in the explanation of the proof:

Definition: The cutwidth of an embedding of a graph is the maximum number of edges passing between two adjacent edges.
Definition: The cutwidth of a graph is the minmum cutwidth among all possible embeddings.
Definition: A horizontal section is a horizontal path that leads between two adjacent vertices.
Definition: Horizontal cutwidth or *hcw* is the greatest cutwidth along any horizontal section.
Definition: A vertical section is a vertical path that leads between two adjacent vertices.
Definition: Vertical cutwidth or *vcw* is the greatest cutwidth along any vertical section.
Definition: Grid cutwidth or *gcw* is the greatest cutwidth of any section of a grid.

2 Background

In 1996, Bezrukov[**?**] discusses the congestion, or grid cutwidth, of the n-cube embedded on a grid. In this paper, Bezrukov proved that the linear cutwidth of a cube in n dimensions is given by the following equation:

$$lcw(Q_n) = \begin{cases} \frac{2^{n+2}-2}{3} & \text{if } n \text{ is even} \\ \\ \frac{2^{n+1}-1}{3} & \text{for } h \text{ even.} \end{cases}$$

Bezrukov went on to prove the grid cutwidth of a cube in n dimensions was the following:

$$gcw(Q_n : P_{2^{n_1}} \times P_{2^{n_2}}) = cw(Q_{n_2}).$$

This means that, since a cube in n dimensions has 2^n vertices, an n-cube can be embedded in grids of different sizes by factoring 2^n into $2^{n_1} \times 2^{n_2}$, which can be thought of as a grid of width 2^{n_1} and length 2^{n_2}. Thus, as indicated by the equation, the grid cutwidth of a cube in n dimensions is given by the linear cutwidth of the cube in dimension n_2 ($n_2 \geq n_1$).

In 1998, Mario Rocha[**?**] wrote a paper involving the cutwidth of trees embedded in grids of various sizes. Some results that Rocha was able to prove were:

For $h \leq 7$, the cutwidth is one when $T_{2,h}$, a Binary tree of height h, is embedded into a grid with dimensions given by:

$$n \times m = \begin{cases} (2^{\frac{h+1}{2}} - 1) \times (2^{\frac{h+1}{2}} + 1) & \text{for } h \text{ odd} \\ \\ (2^{\frac{h}{2}}) \times (2^{\frac{h+2}{2}}) & \text{for } h \text{ even.} \end{cases}$$

For all h, the cutwidth is one when $T_{2,h}$ is embedded into a grid with dimensions given by:

$$n \times m = \begin{cases} (2^{\frac{h+2}{2}} - 1) \times (2^{\frac{h+2}{2}} - 1) & \text{for } h \text{ even} \\ \\ (2^{\frac{h+1}{2}} - 1) \times (2^{\frac{h+3}{2}} - 1) & \text{for } h \text{ odd.} \end{cases}$$

Also in 1998, Sara Hernandez[?] wrote a similar paper discussing the bandwidth of binary trees embedded into grids. Hernandez's main result was the establishment of the following lower bound for the bandwidth of complete binary trees embedded into grids with the smallest dimensions:

$$bw(T_h) \geq \begin{cases} ((3)2^{\frac{h}{2}} - 2)/2h & \text{when } h \text{ is even} \\ \\ (2^{\frac{h+1}{2}} - 1)/h & \text{when } h \text{ is odd.} \end{cases}$$

In 1996, Francisco Rios[?] proved that for any complete graph K_n on n vertices the linear cutwidth is equal to the following:

$$lcw(K_n) = \begin{cases} \frac{n^2}{4} & n \text{ even} \\ \\ \frac{n^2-1}{4} & n \text{ odd.} \end{cases}$$

Rios went on to show that the cyclic cutwidth of a complete graph on n vertices is:

$$ccw(K_n) = \begin{cases} \frac{lcw(K_n)+2}{2} & \frac{n}{2} \text{ even} \\ \\ \frac{lcw(K_n)+1}{2} & \frac{n}{2} \text{ odd} \\ \\ \frac{lcw(K_n)}{2} & n \text{ odd.} \end{cases}$$

Finally in 2001, Annie Wang[?] worked on the embedding of a complete graph of $2n$ vertices into a $2 \times n$ grid. Wang found the following result for the cutwidth of $K_{2\times n}$:

$$gcw(K_{2\times n}) = \begin{cases} \frac{n^2}{2} & \text{if } n \text{ is even} \\ \\ \frac{n^2-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

While all of these results vary in relevance to the embedding of complete graphs of mn vertices on to a grid, the ideas and results from all of them are used in some way throughout this paper.

3 Constructing a Complete $m \times n$ grid

3.1 Numbering

All vertices are numbered with two numbers. The first number corresponds to the row and the second number corresponds to the column. For example, a 2×3 grid would be numbered as follows:

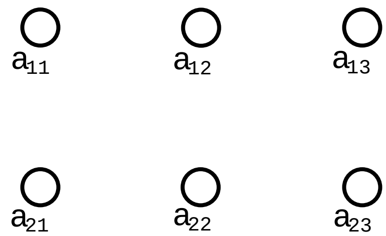

Figure 1

3.2 Connecting Vertices

Vertices within the same row or column are connected with strictly vertical or horizontal lines.

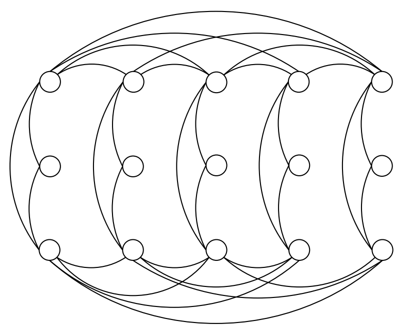
Figure 2

By convention, these lines will not be drawn in order to simplify the drawing. Vertices in different rows and columns are connected in one of two ways:

- connected with a path that travels vertically until it reaches the row of the corresponding vertex of destination and then proceeds horizontally until it reaches the vertex. (1)

- connected by a path that travels horizontally until it reaches the column of the vertex of destination and then proceeds vertically until it reaches the vertex. (2)

Both (1) and (2) are used alternately to connect a vertex to the vertices of a different row. First (1) is used then (2) is used. For example, Figure 3 shows how a_{21} would be connected to the vertices of the first row. Vertices a_{21} and a_{12} are connected with (1); Vertices a_{21} and a_{13} are connected with (2); and so on.

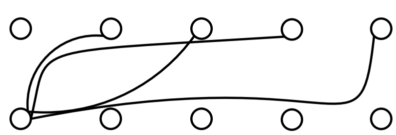
Figure 3

This alternation patterns distributes the quantity of vertical crossing edges among all vertical sections as the figure below shows by denoting the cutwidth that corresponds to each vertical and horizontal section.

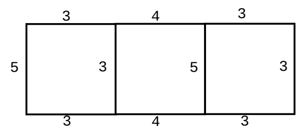

Figure 4

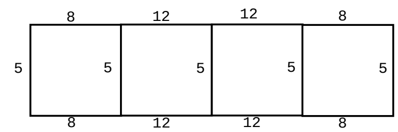

Figure 5

As it can be seen in the above figures, the vertical cutwidth of a $2 \times n$ grids where n is even alternates $n+1, n-1, n+1, \ldots, n+1, n-1$. A similar pattern can be seen for $2 \times n$ grids where n is odd. In this case the vertical cutwidth remains n. This can be shown to be generally true by the simplification of the following algorithm for finding the cutwidth of a vertical section:

When n is even and r is the column

$$vcw(K_{2\times n}) = \begin{cases} 2(\frac{n-(r-1)}{2} + \frac{r-1}{2}) + 1 & r \text{ odd} \\ 2(\frac{n-r}{2} + \frac{r}{2} - 1) + 1 & r \text{ even.} \end{cases}$$

When n is odd

$$vcw(K_{2\times n}) = \begin{cases} 2(\frac{n-(r-1)}{2} + \frac{r}{2} - 1) + 1 & r \text{ odd} \\ 2(\frac{n-r}{2} + \frac{r-1}{2}) + 1 & r \text{ even.} \end{cases}$$

These simplify to:

$$vcw(K_{2\times n}) = \begin{cases} n+1 & r \text{ odd} \\ n-1 & r \text{ even} \quad \text{When } n \text{ is even.} \end{cases}$$

$$vcw(K_{2\times n}) = \begin{cases} n & r \text{ odd} \\ n & r \text{ even} \quad \text{When } n \text{ is odd.} \end{cases}$$

This same fact about the cutwidth of $2 \times n$ grids can be extended to the cutwidth of grids with more than 2 rows. Consider a 3×3 grid. The vertical cutwidth between a_{11} and a_{21}; a_{12} and a_{22}; and a_{13} and a_{23} are all the same because n is odd and all can be thought of as being contained within the same $2\times n$ grids. For example, a_{11} and a_{21}; a_{12} and a_{22}; and a_{13} and a_{23} are contained within the $2 \times n$ grids outlined by a_{11},a_{21},a_{23},a_{13} and a_{11},a_{31},a_{23},a_{33}.

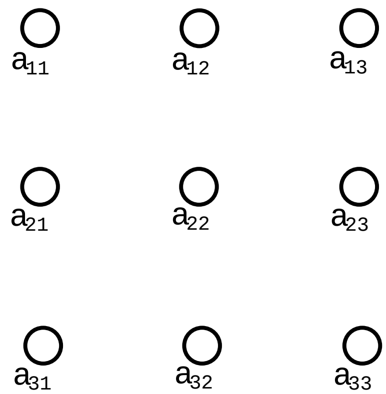

Figure 6

The same argument can be made for why the vertical cutwidth between a_{21} and a_{31}; a_{22} and a_{32}; and a_{23} and a_{33} are the same. These are contained within the $2 \times n$ grids outlined by a_{21},a_{31},a_{33},a_{23} and a_{11},a_{31}, a_{23},a_{33}. This remains true as grid size increases. This result will be used in section 5 to find the vertical cutwidth.

4 Horizontal Cutwidth

First, only the horizontal cutwidth of an $m \times n$ grid will be found for all cases except for the special case $m = n$ such that m and n are both even. In the next section the vertical cutwidth of an $m \times n$ grid will be found. Then in the final section both the horizontal and vertical cutwidths will be evaluated to yield the final results for the cutwidth of an $m \times n$ grid.

The horizontal cutwidth of a complete graph with $m \times n$ vertices embedded on an $m \times n$ grid can be divided into the following cases:

- n even
- n odd.

In each case, it will be shown that the greatest cutwidth is in the center for $2 \times n$ grids. However, this can be extended to $m \times n$ grids also because every row added contributes the same cutwidth that was originally added to the linear cutwidth of a $2 \times n$ grid.

4.1 n even

For this case we must consider the following subcases:

- $\frac{n}{2}$ even
- $\frac{n}{2}$ odd.

4.1.1 n even and $\frac{n}{2}$ even

By Rios Theorem we expect the the linear cutwidth to occur in the center of a horizontal side. Therefore, let us first find the cutwidth of this middle section for the first row of a $2 \times n$ grid.

First, we divide the graph into two equal parts with the same number of vertices on either side.

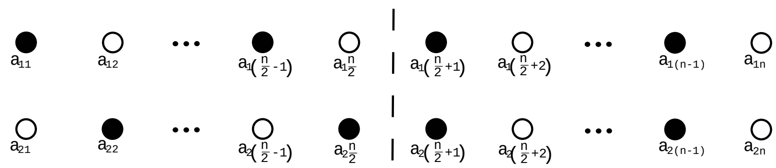

Figure 8

In the figure above, connecting the shaded vertices on the upper left to the shaded vertices on the lower right contributes to the cutwidth of this middle section. Since there are $\frac{n}{4}$ vertices on the upper left and $\frac{n}{4}$ vertices on the lower right, there will be a total of $(\frac{n}{4})(\frac{n}{4})$ or $\frac{n^2}{16}$ edges passing through this middle section. Also in the diagram, connecting the $\frac{n}{4}$ shaded vertices on the lower left to the $\frac{n}{4}$ shaded vertices on the upper right contributes to the cutwidth of this middle section. So, there will be another $\frac{n^2}{16}$ edges passing through this middle section. Thus far the cutwidth of this middle section is $\frac{n^2}{8}$, but connecting the unshaded vertices in the upper left to the unshaded vertices on the lower right and connecting the unshaded vertices in the lower left to the unshaded vertices on the upper right yields another $\frac{n^2}{8}$. Therefore the total cutwidth excluding the linear cutwidth is $\frac{n^2}{4}$. From the shading of the vertices one can also see that the cutwidth of the second row's middle section will be the same as the first row's. Thus, it is sufficient to consider only the cutwidth of the first row in trying to find the horizontal cutwidth of a $2 \times n$ grid. Now, it is necessary to check whether the middle has the greatest cutwidth of all horizontal sections. If we continue to shade our vertices in the same manner as before, the following algorithm can be derived for the cutwidth of any horizontal section, where r is the column is the cutwidth between the r column and the $r+1$ column:

$$hcw(K_{2\times n}) = \begin{cases} 4(\frac{r}{2}\frac{(n-r)}{2}) & r \text{ even} \\ (\frac{r-1}{2})(\frac{(n-r)+1}{2}) + (\frac{r+1}{2})(\frac{(n-r)+1}{2}) + (\frac{r+1}{2})(\frac{(n-r)-1}{2}) + (\frac{r-1}{2})(\frac{(n-r)-1}{2}) & r \text{ odd.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) & r \text{ even} \\ r(n-r) & r \text{ odd.} \end{cases}$$

Since $r(n-r)$ is maximized where $r = \frac{n}{2}$ and all rows are identical, the middle section found before is the horizontal cutwidth of a $2 \times n$ grid. Now if we add a row to our $2 \times n$ grid the horizontal cutwidth remains in the middle section and, is the cutwidth of the middle section of the first row. This is because every row in the grid remain identical to each other despite the addition of rows. This is because every row can be thought as being in $(m-1)$ $2 \times n$ grids for which all contribute $\frac{n^2}{4}$, the cutwidth of a $2 \times n$ grid, to the cutwidth to its middle section. For example, consider a 3×4 grid. As can be seen in Figures 9a and 9b, the first row can be thought of as being in two 2×4 grids outlined by the shaded vertices. Similarly, the second and third row can be thought of as being in two 2×4 grids as shown in Figures 9a through 9c.

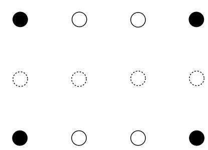

Figure 9a

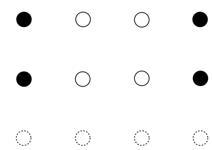

Figure 9b

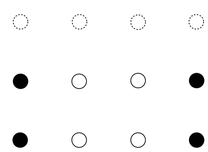

Figure 9c

Therefore, the horizontal cutwidth excluding the linear cutwidth of the row would be $(m-1)\frac{n^2}{4}$. Thus, the horizontal cutwidth of an $m \times n$ grid such that n even and $\frac{n}{2}$ even is:

$$hcw(K_{m \times n}) = \frac{mn^2}{4}.$$

4.1.2 n even and $\frac{n}{2}$ odd

By Rios Theorem we expect the the linear cutwidth to occur in the center of a horizontal side. Therefore, let us first find the cutwidth of this middle section for the first row of a $2 \times n$ grid.

First, we divide the graph into two equal part with the same number of vertices on either side.

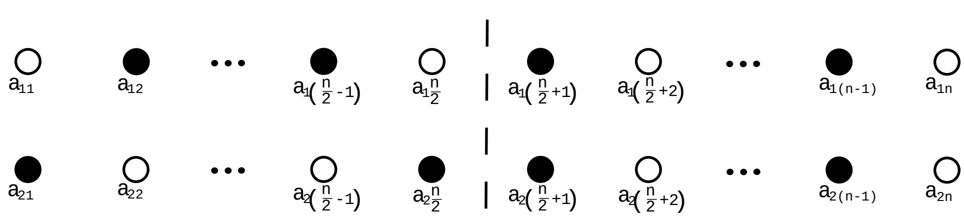

Figure 10

In Figure 10, connecting the shaded vertices on the upper left to the shaded vertices on the lower right contributes to the cutwidth of this middle section.

Since there are $\frac{\frac{n}{2}-1}{2}$ vertices on the upper left and $\frac{\frac{n}{2}+1}{2}$ vertices on the lower right, there will be a total of $(\frac{\frac{n}{2}-1}{2})(\frac{\frac{n}{2}+1}{2})$ or $\frac{n^2-4}{16}$ edges passing through this middle section. Also in the figure, connecting the $\frac{\frac{n}{2}+1}{2}$ shaded vertices on the lower left to the $\frac{\frac{n}{2}+1}{2}$ shaded vertices on the upper right contributes to the cutwidth of this middle section. So, there will be another $\frac{n^2+4n+4}{16}$ edges passing through this middle section. Thus far the cutwidth of this middle section is $\frac{n^2+2n}{8}$, but connecting the unshaded vertices in the upper left to the unshaded vertices on the lower right and connecting the unshaded vertices in the lower left to the unshaded vertices on the upper right yields another $\frac{n^2-2n}{8}$. Therefore the total cutwidth excluding the linear cutwidth is $\frac{n^2}{4}$. From the shading of the vertices one can also see that the cutwidth of the second row's middle section will be the same as the first row's. Thus, it is sufficient to consider only the cutwidth of the first row in trying to find the horizontal cutwidth of a $2\times n$ grid. Now, it is necessary to check whether the middle has the greatest cutwidth of all horizontal sections. If we continue to shade our vertices in the same manner as before the following algorithm can be derived for the cutwidth of any horizontal section, where r is the column to the left of the horizontal section that one wants the cutwidth of:

$$hcw(K_{2\times n}) = \begin{cases} 4(\frac{r}{2}\frac{(n-r)}{2}) & r \text{ even} \\ (\frac{r-1}{2})(\frac{(n-r)+1}{2}) + (\frac{r+1}{2})(\frac{(n-r)+1}{2}) + (\frac{r+1}{2})(\frac{(n-r)-1}{2}) + (\frac{r-1}{2})(\frac{(n-r)-1}{2}) & r \text{ odd.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) & r \text{ even} \\ r(n-r) & r \text{ odd.} \end{cases}$$

Since $r(n-r)$ is maximized where $r = \frac{n}{2}$ and all rows are identical, the result for the cutwidth of the middle section found before is the horizontal cutwidth of a $2\times n$ grid. Now if we add a row to our $2\times n$ grid the horizontal cutwidth remains in the middle section and, is the cutwidth of the middle section of the first row. This is because every row in the grid remain identical to each other despite the addition of rows. This is because every row can be thought as being in $(m-1)$ $2\times n$ grids for which all contribute $\frac{n^2}{4}$, the cutwidth of a $2\times n$ grid, to the cutwidth to its middle section.Therefore, the horizontal cutwidth excluding the linear cutwidth of the row would be $(m-1)\frac{n^2}{4}$. Thus, the horizontal cutwidth of an $m\times n$ grid such that n even and $\frac{n}{2}$ even is:

$$hcw(K_{m\times n}) = \frac{mn^2}{4}.$$

4.2 n odd

Similar to the previous case, we are going to concentrate on the cutwidth of the middle section, $a_{1(\frac{n-1}{2})}$ to $a_{1(\frac{n-1}{2}+1)}$. However, this case must be further divided into the following cases:

- $\frac{n+1}{2}$ odd
- $\frac{n+1}{2}$ even.

4.2.1 n odd and $\frac{n+1}{2}$ odd

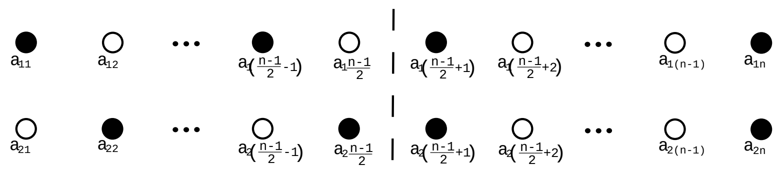

Figure 11

In Figure 11 above, connecting the $\frac{n-1}{4}$ shaded vertices on the $\frac{\frac{n+1}{2}+1}{2}$ upper left to the shaded vertices on the lower right contributes to the cutwidth of this middle section. Thus, there will be a total of $\frac{n-1}{4}\frac{\frac{n+1}{2}+1}{2}$ or $\frac{(n-1)(n+3)}{16}$ edges passing through this middle section. Connecting the $\frac{n-1}{4}$ shaded vertices on the lower left to the $\frac{\frac{n+1}{2}+1}{2}$ shaded vertices on the upper right contributes to the cutwidth of $\frac{(n-1)(n+3)}{16}$ to the middle section. Thus far the cutwidth of this middle section is $\frac{(n-1)(n+3)}{8}$. Now, connecting the $\frac{n-1}{4}$ unshaded vertices in the upper left to the $\frac{\frac{n+1}{2}-1}{2}$ unshaded vertices on the lower right yields an additional $\frac{(n-1)^2}{16}$. Then connecting the $\frac{n-1}{4}$ unshaded vertices in the lower left to the $\frac{\frac{n+1}{2}-1}{2}$ unshaded vertices on the upper right also yields another $\frac{(n-1)^2}{16}$. Therefore the total cutwidth excluding the linear cutwidth is $\frac{n^2-1}{4}$. Again we must consider whether the middle has the greatest cutwidth of all horizontal sections. If we continue to shade our vertices in the same manner as before the following algorithm can be derived for the cutwidth of any horizontal section, where r is the column:

$$hcw(K_{2\times n}) = \begin{cases} 2(\frac{r-1}{2}\frac{(n-r)}{2}) & r \text{ even} \\ \\ 2(\frac{r}{2}\frac{(n-r)+1}{2}) + (\frac{r}{2}\frac{(n-r)-1}{2}) & r \text{ odd.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) & r \text{ even} \\ \\ r(n-r) & r \text{ odd.} \end{cases}$$

The maximum of $r(n-r)$ occurs where $r = \frac{n-1}{2}$ and since all rows are identical, the the result found before is the horizontal cutwidth of a $2 \times n$ grid. Again the addition of rows to our $2 \times n$ grid doesn't move the location of the horizontal cutwidth. Every row added can be thought of as contributing another

$\frac{n^2-1}{4}$ to the cutwidth of the middle section. Therefore, the horizontal cutwidth of an $m \times n$ grid such that n even and $\frac{n+1}{2}$ is even is:

$$hcw(K_{m\times n}) = \frac{m(n^2-1)}{4}.$$

4.2.2 n odd and $\frac{n+1}{2}$ even

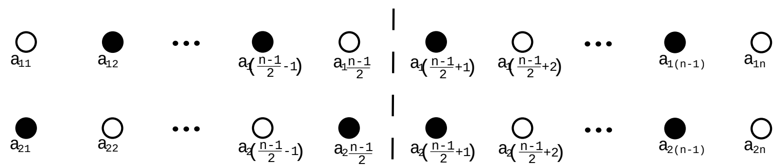

Figure 12

In the diagram above, connecting the $\frac{\frac{n-1}{2}-1}{2}$ shaded vertices on the upper left to the $\frac{n+1}{4}$ shaded vertices on the lower right contributes to the cutwidth of this middle section. Then connecting the $\frac{\frac{n-1}{2}+1}{2}$ shaded vertices on the lower left to the $\frac{n+1}{4}$ shaded vertices on the upper right contributes $\frac{(n+1)^2}{16}$ to the cutwidth of this middle section. Thus far the cutwidth of this middle section is $\frac{n^2-1}{8}$. Now connecting the unshaded vertices in the upper left to the unshaded vertices on the lower right yields an additional $\frac{(n+1)^2}{16}$. Then connecting the unshaded vertices in the lower left to the unshaded vertices on the upper right also yields another $\frac{(n-1)(n+3)}{16}$. Therefore the total cutwidth excluding the linear cutwidth is $\frac{n^2-1}{4}$. Now we must consider whether the middle has the greatest cutwidth of all horizontal sections. If we continue to shade our vertices in the same manner as before the following algorithm can be derived for the cutwidth of any horizontal section, where r is the column:

$$hcw(K_{2\times n}) = \begin{cases} 2(\frac{r-1}{2}\frac{(n-r)}{2} + \frac{r+1}{2}\frac{n-r}{2}) & r \text{ even} \\ \\ 2(\frac{r}{2}\frac{(n-r)+1}{2}) + (\frac{r}{2}\frac{(n-r)-1}{2}) & r \text{ odd.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) & r \text{ even} \\ \\ r(n-r) & r \text{ odd.} \end{cases}$$

Since the maximum of $r(n-r)$ is where $r = \frac{n-1}{2}$ and all rows are identical, the result found before is the horizontal cutwidth of a $2 \times n$ grid. Again the addition rows to our $2 \times n$ grid doesn't change. Since every row can be thought as being in $(m-1)$ $2 \times n$ grids for which all contribute $\frac{n^2-1}{4}$ to the cutwidth of its middle section, the horizontal cutwidth excluding the linear cutwidth of the row would be $(m-1)\frac{n^2-1}{4}$. Thus, the horizontal cutwidth of an $m \times n$ grid such that n even and $\frac{n+1}{2}$ is odd is:

$$hcw(K_{m\times n}) = \frac{m(n^2-1)}{4}.$$

4.3 Conclusion

The horizontal cutwidth of a grid is:

$$hcw(K_{m\times n}) = \begin{cases} \frac{mn^2}{4} & n \text{ even} \\ \frac{m(n^2-1)}{4} & n \text{ odd.} \end{cases}$$

5 Vertical Cutwidth

In finding the vertical cutwidth of a complete graph on $m \times n$ vertices embedded in an $m \times n$ grid let us consider the following cases:

- m even and n even $m \neq n$
- m even and n odd
- m odd and n even
- m odd and n odd.

5.1 m even and n even and $m \neq n$

For this case consider a 4×6 grid. By Rios Theorem it is known that the linear cutwidth of a vertical side of the grid will occur in the middle section and it will be $\frac{m^2}{4}$. Therefore, the final cutwidth of the middle will be $\frac{m^2}{4} + x$ such that x is the quantity of lines that pass across this middle section from the points either directly above or below the middle. The value of x can be found by summing the amount of edges needed to connect the vertices above and below the vertical section to every other column on the opposing side of the grid.

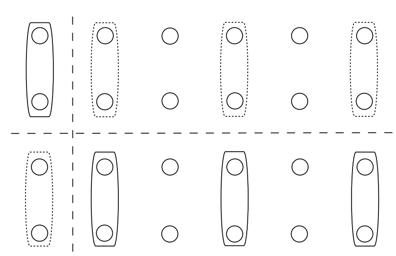

Figure 13

Thus, there are $\frac{m}{2}$ vertices above the middle section in the first column to which we want to connect the $\frac{m}{2}\frac{n}{2}$ vertices below the middle section. Therefore, there must be $\frac{m}{2}\frac{m}{2}\frac{n}{2}$ edges passing across this middle section. The same process is repeated to connect the $\frac{m}{2}$ vertices below the middle section in the first column to the $\frac{m}{2}\frac{n}{2}$ vertices above. This yields another $\frac{m^2 n}{8}$ thereby creating a cutwidth, excluding the linear cutwidth, of $\frac{m^2 n}{4}$. However, $\frac{m^2 n}{4}$ is just the cutwidth of the middle. The question of whether this is the vertical cutwidth along any vertical side. Therefore, it is necessary to show that all cutwidths above or below the middle are less than or equal to it, because it has already been shown that the vertical sections remain the same or alternate with the greatest value first. By Rios Theorem we know that the linear cutwidth decreases as one proceeds from the middle. Thus, it is not necessary to consider the linear cutwidth if we can show that the number added to the linear cutwidth also decreases as one proceeds from the middle. Further, since construction of the grid is symmetric about the center of the grid it is only necessary to show that the cutwidth decrease as one proceeds up from the middle (the cutwidth below will mirror the cutwidth above).

Now, if we consider the cutwidth of the section just above the middle, we know that within this column there are $\frac{m}{2} - 1$ vertices above the section and $\frac{m}{2} + 1$ vertices below the section. Connecting the vertices above and below the section to every other row of vertices on the other side is what contributes to the cutwidth on the section. So, connecting the $\frac{n}{2} - 1$ vertices above to every other row of vertices below yields $(\frac{m}{2} - 1)(\frac{m}{2} + 1)(\frac{n}{2})$ or $\frac{(m^2-4)n}{8}$.

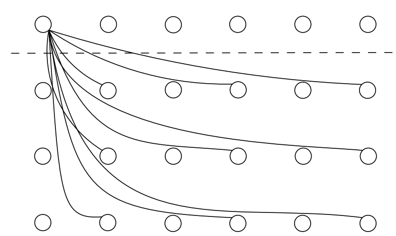

Figure 14

Then a similar process is done for the vertices below the section. The $\frac{m}{2} + 1$ vertices below the section are connect to every other row of vertices above the section, which also yields $(\frac{m}{2} - 1)(\frac{m}{2} + 1)(\frac{n}{2})$ or $\frac{(m^2-4)n}{8}$. Therefore, making the cutwidth of the section, excluding the linear cutwidth, $\frac{(m^2-4)n}{4}$.

Now we know that the vertical cutwidth of the middle vertical section is $\frac{m^2 n}{4}$ and the vertical cutwidth of the section just above it is $\frac{(m^2-4)n}{4}$. The $\frac{(m^2-4)n}{4}$ is clearly less than $\frac{m^2 n}{4}$ because $(m^2 - 4) < m^2$. Further, as one proceeds up the 4 in $(m^2 - 4)$ will continually increase. Thereby, decreasing the vertical cutwidth as one proceeds up. Therefore, the vertical cutwidth is achieved in the middle section, the cutwidth of an $m \times n$ grid such that m and n are even and not equal is:

$$vcw(K_{m\times n}) = \frac{m^2(n+1)}{4}$$

5.2 m even and n odd

For this case consider a 4×5 grid. Again Rios Theorem tells us that the linear cutwidth of the vertical section of the grid will occur in the middle and will be $\frac{m^2}{4}$. The contribution of the rest of the grid to this cutwidth can be found in a similar manner as before. There are $\frac{m}{2}$ vertices above the middle in the first column to which we want to connect $\frac{m}{2}\frac{n-1}{2}$ vertices below the middle. Therefore, there must be $\frac{m}{2}\frac{m}{2}\frac{n-1}{2}$ edges passing through the middle section.

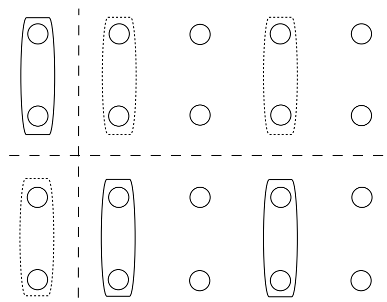

Figure 15

The same process is repeated for the vertices below the middle. The $\frac{m}{2}$ vertices below the middle in the first column are connected to the $\frac{m}{2}\frac{n-1}{2}$ vertices above the vertical. These two cutwidths summed together give a final cutwidth, excluding the linear cutwidth, of $\frac{m^2(n-1)}{4}$.

Again it is necessary to show that is the vertical cutwidth occurs in the middle. Similar to the previous case, the cutwidth of the section just above the middle is found to be $2\frac{m+2}{2}\frac{m-2}{2}\frac{n-1}{2}$ or $\frac{m^2-4)(n-1)}{4}$.

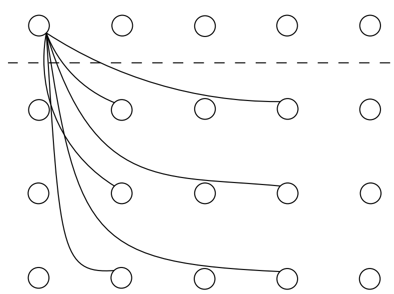

Figure 16

Now we know that the vertical cutwidth of the middle vertical section is $\frac{m^2(n-1)}{4}$ and the vertical cutwidth of the section just above it is $\frac{(m^2-4)(n-1)}{4}$. The $\frac{(m^2-4)(n-1)}{4}$ is clearly less than $\frac{m^2(n-1)}{4}$ because $(m^2-4) < m^2$. Further, as one proceeds up the 4 in $(m^2 - 4)$ will continually increase. Thereby, decreasing the vertical cutwidth as one proceeds up. Therefore, since the vertical cutwidth is achieved in the middle section, the cutwidth of an $m \times n$ grid such that m is even and n is odd:

$$vcw(K_{m\times n}) = \frac{m^2 n}{4}.$$

5.3 m odd and n even

For this case consider a 3×4 grid. By Rios Theorem the linear cutwidth of the vertical section of the grid will occur in the middle section and will be $\frac{m^2-1}{4}$. The contribution of the rest of the grid to this cutwidth can be found in a similar manner as before. Since there are $\frac{m-1}{2}$ vertices above the middle in the first column to which we want to connect $\frac{m+1}{2}\frac{n}{2}$ vertices below the middle there must be $\frac{m-1}{2}\frac{m+1}{2}\frac{n}{2}$ edges passing through the middle section.

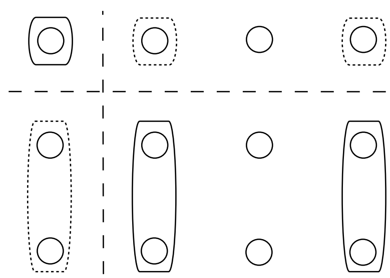

Figure 17

The same process is repeated for the vertices below the middle in the first column. The $\frac{m+1}{2}$ vertices below the middle are connected to the $\frac{m-1}{2}\frac{n}{2}$ vertices above the vertical. These two cutwidths summed together give a final cutwidth, excluding the linear cutwidth, of $\frac{(m^2-1)n}{4}$. Again it is necessary to show that is the greatest cutwidth along the vertical side. Similar to the previous case, the cutwidth of the section just above the middle is found to be $2\frac{m-3}{2}\frac{m+3}{2}\frac{n}{2}$ or $\frac{m^2-9)n}{4}$.

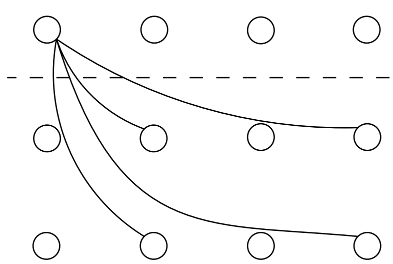

Figure 18

Now we know that the vertical cutwidth of the middle vertical section is $\frac{(m^2-1)n}{4}$ and the vertical cutwidth of the section just above it is $\frac{(m^2-9)n}{4}$. The

$\frac{(m^2-9)n}{4}$ is clearly less than $\frac{(m^2-1)n}{4}$ because $(m^2-9) < (m^2-1)$. Further, as one proceeds up the 9 in (m^2-9) will continually increase, thereby, decreasing the vertical cutwidth as one proceeds up. Therefore since the greatest cutwidth is achieved in the middle section and the cutwidth of all vertical sides are the same, the cutwidth of an $m \times n$ grid such that m is even and n is odd:

$$vcw(K_{m\times n}) = \frac{(m^2-1)(n+1)}{4}.$$

5.4 m odd and n odd

For this case consider a 3×5 grid. By Rios Theorem the linear cutwidth of the vertical section of the grid will occur in the middle and will be $\frac{m^2-1}{4}$. The contribution of the rest of the grid to this cutwidth can be found in a similar manner as before. Since there are $\frac{m-1}{2}$ vertices above the middle in the first column to which we want to connect $\frac{m+1}{2}\frac{n-1}{2}$ vertices below the middle there must be $\frac{m-1}{2}\frac{m+1}{2}\frac{n-1}{2}$ edges passing through the middle section.

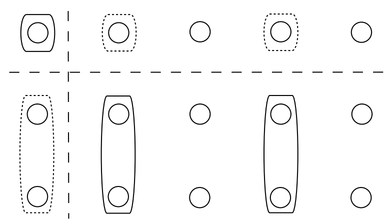

Figure 19

The same process is repeated for the vertices below the middle in the first column. The $\frac{m+1}{2}$ vertices below the middle are connected to the $\frac{m-1}{2}\frac{n-1}{2}$ vertices above the section. These two cutwidths summed together give a final cutwidth, excluding the linear cutwidth, of $\frac{(m^2-1)(n-1)}{4}$. Again it is necessary to show that is the vertical cutwidth. Similar to the previous case, the cutwidth of the section just above the middle is found to be $2\frac{m-3}{2}\frac{m+3}{2}\frac{n-1}{2}$ or $\frac{m^2-9)(n-1)}{4}$.

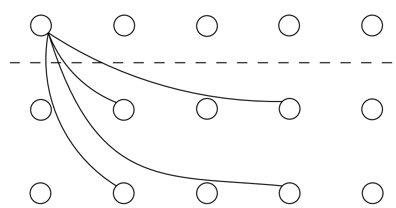

Figure 20

Now we know that the vertical cutwidth of the middle vertical section is $\frac{(m^2-1)(n-1)}{4}$ and the cutwidth of the section just above it is $\frac{(m^2-9)(n-1)}{4}$. The $\frac{(m^2-9)(n-1)}{4}$ is clearly less than $\frac{(m^2-1)(n-1)}{4}$ because $(m^2-9) < (m^2-1)$. Further, as one proceeds up the 9 in (m^2-9) will continually increase. Thereby, decreasing the vertical cutwidth as one proceeds up. Therefore since the greatest cutwidth is achieved in the middle section and the cutwidths of all vertical sections are the same, the vertical cutwidth of an $m \times n$ grid such that m is even and n is odd:

$$vcw(K_{m\times n}) = \frac{(m^2-1)n}{4}.$$

5.5 Conclusion

The vertical cutwidth of a grid is:

$$vcw(K_{m\times n}) = \begin{cases} \frac{m^2(n+1)}{4} & m \text{ and } n \text{ even, } m \neq n \\ \frac{m^2 n}{4} & m \text{ even and } n \text{ odd} \\ \frac{(m^2-1)(n+1)}{4} & m \text{ odd and } n \text{ even} \\ \frac{(m^2-1)n}{4} & m \text{ odd and } n \text{ odd.} \end{cases}$$

6 m and n Even and $m = n$

For this particular case the normal manner of constructing the grid will not yield the smallest cutwidth for the grid. To construct the grid with the smallest cutwidth, the first row is connected to the second with the standard (1) then (2) pattern as used before. However, the second row is connected to the first by (2) then (1) in alternation. For example, in a 3×4 grid the first two rows would be connected as shown in Figure 21.

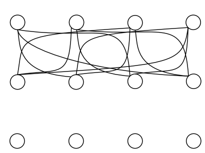

Figure 21

The first and third rows would be connected as shown in Figure 22.

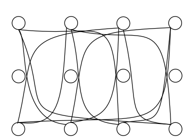

Figure 22

With this technique of construction m and n even with $m = n$ can be divided into the following two cases:

- m and n even and $m = n$ and m multiple of four
- m and n even and $m \neq n$ and m not a multiple of four.

6.1 m and n even and $m = n$ and m multiple of four

Like previous cases, we are first going to consider a $2 \times n$ grid and then the $2 \times n$ grid can be extended to the $m \times n$ grid. First, we are going to develop an algorithm for finding the cutwidth of the first row.

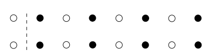

Figure 23a

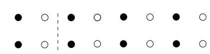

Figure 23b

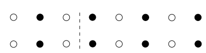

Figure 23c

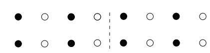

Figure 23d

From the Figures 23a-23d above, the following algorithm can be derived:

$$hcw(K_{2\times n}) = \begin{cases} 2(\frac{r-1}{2})(\frac{n-r+1}{2}) + 2(\frac{r+1}{2})(\frac{n-r-1}{2}) & r \text{ odd} \\ 4(\frac{r}{2})(\frac{n-r}{2}) & r \text{ even.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) - 1 & r \text{ odd} \\ r(n-r) & r \text{ even.} \end{cases}$$

The equation $r(n-r)$ is maximized when $r = \frac{n}{2}$. However, $\frac{n}{2}$ is even. So, for our odd portion of the equation the closest odd integer would be $\frac{n+2}{2}$ or $\frac{n-2}{2}$. Thus, the greatest that the even portion could be is $\frac{n^2}{4}$ and the greatest that the odd portion could be is $\frac{n^2}{4} - 2$. Therefore, the cutwidth of the first row is $\frac{n^2}{4}$.

A similar process is repeated for the second row, which yields:

$$hcw(K_{2\times n}) = \begin{cases} 2(\frac{r-1}{2})(\frac{n-r-1}{2}) + 2(\frac{r+1}{2})(\frac{n-r+1}{2}) & r \text{ odd} \\ 4(\frac{r}{2})(\frac{n-r}{2}) & r \text{ even.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) + 1 & r \text{ odd} \\ r(n-r) & r \text{ even.} \end{cases}$$

The equation $r(n-r)$ is maximized when $r = \frac{n}{2}$. However, $\frac{n}{2}$ is even. So again, for our odd portion of the equation the closest odd integer would be $\frac{n+2}{2}$ or $\frac{n-2}{2}$. Thus, the greatest that the even portion could be is $\frac{n^2}{4}$ and the greatest that the odd portion could be is $\frac{n^2}{4}$. Therefore, the greatest cutwidth on the second row is $\frac{n^2}{4}$.

Since the cutwidth of both the first and second rows is $\frac{n^2}{4}$, the horizontal cutwidth of $K_{2\times n}$ is $\frac{n^2}{4}$. This result for $K_{2\times n}$ can now be applied to $K_{m\times n}$ by multiplying $\frac{n^2}{4}$ by $(m-1)$, the number of $2\times n$ grids that any row in a $m\times n$ is contained within. Thus, with the addition of the linear cutwidth, the horizontal cutwidth of a $K_{m\times n}$ is:

$$hcw(K_{m\times n}) = \frac{mn^2}{4}.$$

Now we are going to discover the vertical cutwidth of an $m\times n$ grid by first considering the vertical cutwidth of a $2\times n$ grid. If we analyze the Figures 24a-24d below, we see than the connecting of the vertices between the dashed lines to the shaded vertices contributes to the cutwidth of the vertical section between the dashed lines.

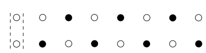
Figure 24a

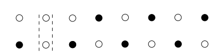
Figure 24b

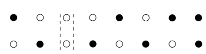
Figure 24c

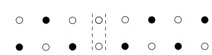
Figure 24d

From this the following algorithm can be derived for the cutwidth each vertical section:

$$vcw(K_{m\times n}) = \begin{cases} 2(\frac{r-1}{2}) + (\frac{n-r-1}{2}) + (\frac{n-r+1}{2}) & r \text{ odd} \\ \\ (\frac{r-2}{2}) + (\frac{r}{2}) + 2(\frac{n-r}{2}) & r \text{ even.} \end{cases}$$

This simplifies to:

$$vcw(K_{m\times n}) = \begin{cases} n-1 & r \text{ odd} \\ \\ n-1 & r \text{ even.} \end{cases}$$

Therefore, since the vertical cutwidth stays the same for all vertical sections, the vertical cutwidth of a $2\times n$ grid, excluding linear cutwidth, is $n-1$. This can now be used to find the vertical cutwidth of an $m\times n$ grid by multiplying $n-1$ by $\frac{m^2}{4}$, the number of $2\times n$ grids that the middle vertical section is contained within. This yields the following for the vertical cutwidth of a $m\times n$ grid, including linear cutwidth:

$$vcw(K_{m\times n}) = \frac{mn^2}{4}.$$

Thus, since the vertical and horizontal cutwidths are the same, the grid cutwidth of a $m\times n$ grid where $m=n$, mn is a multiple of four, and both even is the following:

$$gcw(K_{m\times n}) = \frac{mn^2}{4}.$$

6.2 m and n even and $m = n$ and m not a multiple of four

As with previous cases, we are first going to consider a $2 \times n$ grid and then the $2 \times n$ grid can be extended to the $m \times n$ grid. First, we are going to develop an algorithm for finding the cutwidth of the first row.

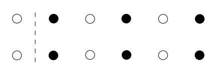

Figure 25a

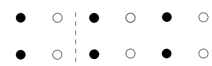

Figure 25b

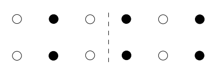

Figure 25c

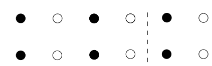

Figure 25d

From Figures 25a-25c above, the following algorithm can be derived:

$$hcw(K_{2\times n}) = \begin{cases} 2(\frac{r-1}{2})(\frac{n-r+1}{2}) + 2(\frac{r+1}{2})(\frac{n-r-1}{2}) & r \text{ odd} \\ \\ 4(\frac{r}{2})(\frac{n-r}{2}) & r \text{ even.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) - 1 & r \text{ odd} \\ \\ r(n-r) & r \text{ even.} \end{cases}$$

The equation $r(n-r)$ is maximized when $r = \frac{n}{2}$. However, $\frac{n}{2}$ is odd. So, for our even portion of the equation the closest even integer would be $\frac{n+2}{2}$ or $\frac{n-2}{2}$. Thus, the greatest that the even portion could be is $\frac{n^2}{4} - 1$ and the greatest that the odd portion could be is $\frac{n^2}{4} - 1$. Therefore, the cutwidth of the first row is $\frac{n^2}{4} - 1$ and can be thought of as being located in the middle.

A similar process is repeated for the second row, which yields:

$$hcw(K_{2\times n}) = \begin{cases} 2(\frac{r-1}{2})(\frac{n-r-1}{2}) + 2(\frac{r+1}{2})(\frac{n-r+1}{2}) & r \text{ odd} \\ \\ 4(\frac{r}{2})(\frac{n-r}{2}) & r \text{ even.} \end{cases}$$

This simplifies to:

$$hcw(K_{2\times n}) = \begin{cases} r(n-r) + 1 & r \text{ odd} \\ \\ r(n-r) & r \text{ even.} \end{cases}$$

The equation $r(n-r)$ is maximized when $r = \frac{n}{2}$. However, $\frac{n}{2}$ is odd. So again, for our even portion of the equation the closest even integer would be $\frac{n+2}{2}$ or $\frac{n-2}{2}$. Thus, the greatest that the even portion could be is $\frac{n^2}{4} - 1$ and the greatest that the odd portion could be is $\frac{n^2}{4} + 1$. Therefore, the greatest cutwidth is on the second row and is $\frac{n^2}{4} + 1$.

Since the cutwidth of the first is less than the second, $\frac{n^2}{4}+1$ is the horizontal cutwidth of $K_{2\times n}$. This result for $K_{2\times n}$ can now be applied to $K_{m\times n}$ by multiplying $\frac{n^2}{4}$ by $(m-1)$, the number of $2\times n$ grids that any row in a $m\times n$ is contained within. Thus, with the addition of the linear cutwidth, the horizontal cutwidth of a $K_{m\times n}$ is:

$$hcw(K_{m\times n}) = \frac{m(n^2+4)}{4}.$$

Now we are going to discover the vertical cutwidth of an $m \times n$ grid by first considering the vertical cutwidth of a $2\times n$ grid. If we analyze the figures below, we see that the connecting of the vertices between the dashed lines to the shaded vertices contributes to the cutwidth of the vertical section between the dashed lines.

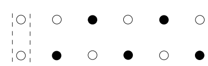

Figure 26a

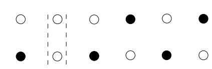

Figure 26b

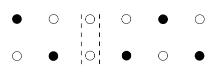

Figure 26c

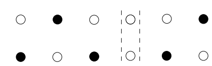

Figure 26d

From this the following algorithm can be derived for the cutwidth of each vertical section:

$$vcw(K_{m\times n}) = \begin{cases} 2(\frac{r-1}{2}) + (\frac{n-r-1}{2}) + (\frac{n-r+1}{2}) & r \text{ odd} \\ \\ (\frac{r-2}{2}) + (\frac{r}{2}) + 2(\frac{n-r}{2}) & r \text{ even.} \end{cases}$$

This simplifies to:

$$vcw(K_{m\times n}) = \begin{cases} n-1 & r \text{ odd} \\ \\ n-1 & r \text{ even.} \end{cases}$$

Therefore, since the vertical cutwidth stays the same for all vertical sections, the vertical cutwidth of a $2\times n$ grid, excluding linear cutwidth, is $n-1$. This can now be used to find the vertical cutwidth of an $m\times n$ grid by multiplying $n-1$ by $\frac{m^2}{4}$, the number of $2\times n$ grids that the middle vertical section is contained within. This yields the following for the vertical cutwidth of a $m\times n$ grid, including linear cutwidth:

$$vcw(K_{m\times n}) = \frac{mn^2}{4}.$$

Thus, since the horizontal cutwidth is greater than the vertical, the grid cutwidth for this grid is $\frac{m(n^2+4)}{4}$. However, $\frac{m(n^2+4)}{4}$ is much greater than the expected $\frac{mn^2}{4}$ by the lower bound. Thus, this arrangement of vertices in not optimal.

Now lets see if it is possible to obtain the optimal case. To do this, let us first sum the quantity of edges that pass through the middle sections of the bottom portion of the grid. Since the bottom middle section was found to be $\frac{(m-1)(n^2+4)}{4}$ and every middle section is two less than the section that proceed it, the sum of the quantity of edges that pass through the middle sections of the bottom portion of the grid would look as follows:

$$\left[\frac{(m-1)(n^2+4)}{4}\right]+\left[\frac{(m-1)(n^2+4)}{4}-2\right]+\left[\frac{(m-1)(n^2+4)}{4}-4\right]+\ldots+\left[\frac{(m-1)(n^2+4)}{4}-(2\frac{n}{2}-2)\right]$$

$$\begin{aligned} &= \tfrac{m}{2}\tfrac{(m-1)(n^2+4)}{4} - \textstyle\sum_{i=1}^{\frac{n}{2}}(2i-2) \\ &= \tfrac{(m^2-m)(n^2+4)}{8} - \tfrac{n^2-2n}{4}. \end{aligned}$$

Ideally, the cut of all horizontal middle sections would be $\frac{(m-1)n^2}{4}$. So, let us now find the quantity of edges we have in excess of having $\frac{(m-1)n^2}{4}$ edges per row for these middle sections in the bottom portion of the grid. This can be done by subtracting $\frac{m}{2}\frac{(m-1)n^2}{4}$ edges from the total amount of edges, which yields $\frac{n}{2}\frac{n}{2}-n$.

Note that $\frac{n}{2}\frac{n}{2}$ is odd, because $\frac{n}{2}$ is odd, and n is even. Therefore, $\frac{n}{2}\frac{n}{2}-n$ is odd. Since the amount of vertices that need to be moved to the upper portion of the graph is odd it is impossible to make all horizontal sections $\frac{(m-1)n^2}{4}$. This is because across the middle vertical sections the cut is $\frac{(m-1)n^2}{4}$. Thus, since every edge rerouted so that is doesn't run across a middle horizontal section in the lower half increases one vertical section by one and decreases another by one. Thus, another edge must be changed in order to restore all middle vertical sections to $\frac{(m-1)n^2}{4}$. Therefore, if we are going to retain the $\frac{(m-1)n^2}{4}$ along the middle vertical section, we must change an even number of edges. Thus, it is impossible to obtain a grid cutwidth of $\frac{mn^2}{4}$.

Since it is impossible to obtain a grid cutwidth of $\frac{mn^2}{4}$, the next possible chose to obtain is $\frac{mn^2}{4}+1$ or $\frac{(m-1)n^2}{4}+1$ without the linear cutwidth. From before, we know that the sum of the quantity of edges that pass through the middle sections of the bottom half of the grid is $\frac{(m^2-m)(n^2+4)}{8} - \frac{n^2-2n}{4}$. Now we can subtract $(\frac{n}{2})(\frac{(m-1)n^2}{4}+1)$ from this total to find the amount of edges that need to be rerouted to the upper portion of the graph. This yield $\frac{n}{2}\frac{n-2}{2}$, which is always even. Therefore, it should be possible to construct a grid with grid cutwidth $\frac{mn^2}{4}+1$.

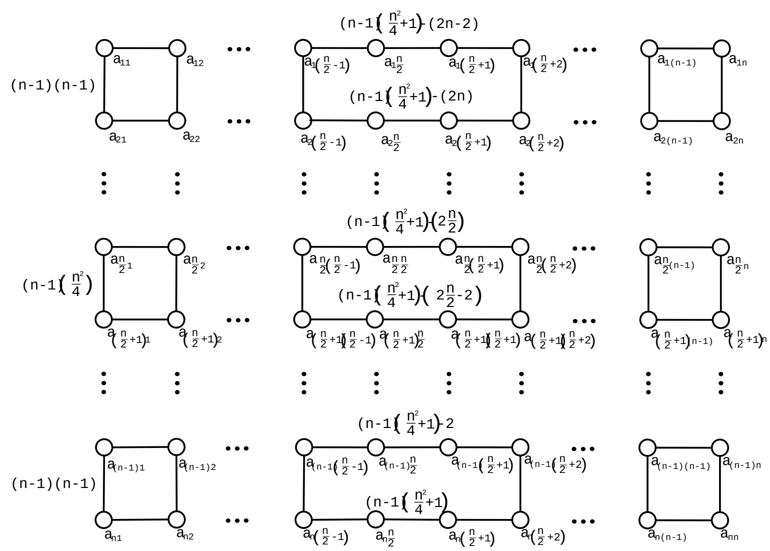

Figure 27

The figure above, shows the cutwidths in various sections of our grid thus far. Now, we are going to attempt to alter the grid in order to get a grid cutwidth of $\frac{mn^2}{4}+1$. To demonstrate the changes a 6×6 grid will be used.

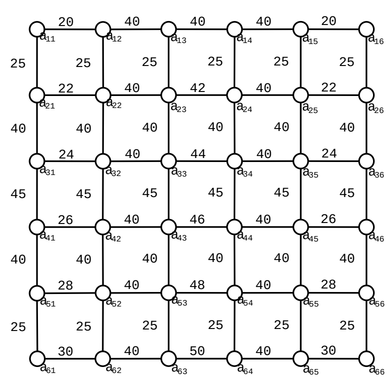

Figure 28a

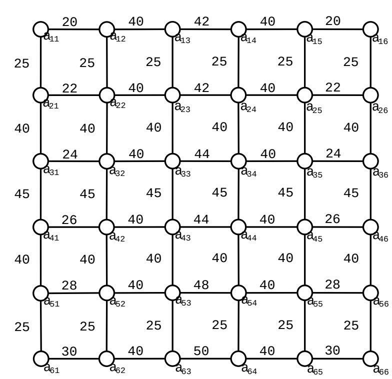

Figure 28b

Figure 28a above, shows the cutwidths of all sections of the grid. First, we are going to decrease the cutwidth of the section between a_{43} and a_{44} by 6, which is the total number of edges over being able to make all middle horizontal sections 46. In general, we are going to decrease the cutwidth of the section between $a_{(\frac{n}{2}+1)(\frac{n}{2})}$ and $a_{(\frac{n}{2}+1)(\frac{n}{2}+1)}$ by $\frac{n(n-2)}{4}$. Further, we are going to do this without altering the cutwidths of any of the vertical sections. To demonstrate how this is going to be done consider the edge connecting a_{43} to a_{14}. If we reroute this edge to run a_{43} to a_{13} to a_{14}, we decrease a_{43} to a_{44} and a_{44} to a_{14} by a cutwidth of one and increase a_{43} to a_{13} and a_{13} to a_{14} by a cutwidth of one. However, if we counteract this change by rerouting the edge connection a_{44} to

a_{13} to run a_{44} to a_{14} to a_{13}, we decrease a_{44} to a_{43} and a_{43} to a_{13} by a cutwidth of one and increase a_{44} to a_{14} and a_{14} to a_{13} by a cutwidth of one. Thus, the final result is an increase of two to the cutwidth between a_{13} to a_{14} and a decrease of two to the cutwidth between a_{43} to a_{44}, as can be seen in Figure 28b. In general, this can be repeated $(\frac{m}{2})(\frac{\frac{n}{2}+1}{2})(\frac{\frac{n}{2}+1}{2})+(\frac{m}{2})(\frac{\frac{n}{2}+1}{2})(\frac{\frac{n}{2}+1}{2})$ times, as can be generalized by the shading in Figure 29.

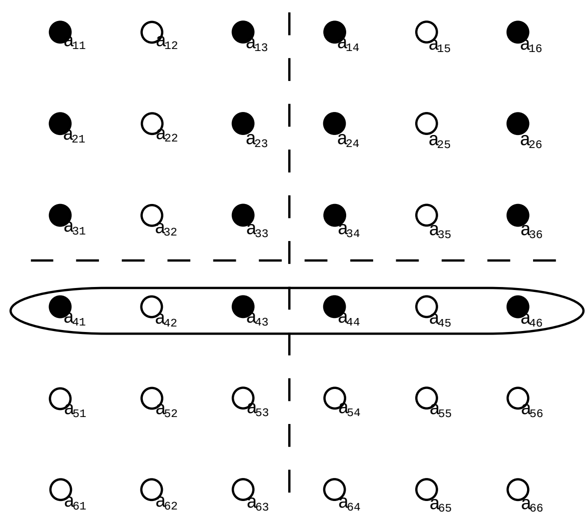

Figure 29

However, while there are many edges that can be rerouted to decrease the cutwidth between $a_{(\frac{n}{2}+1)(\frac{n}{2})}$ and $a_{(\frac{n}{2}+1)(\frac{n}{2}+1)}$, the upper portion of the grid is only able to accept a small portion of these edges. In fact, using the sum from before we find that there is $\frac{(m-1)(n^2+4)}{8}+\frac{2n-3n^2}{4}$ edge running across the middle sections of the upper portion of the graph. If we then subtract out $\frac{m}{2}\frac{(m-1)n^2+4}{4}$ and use the fact that $m=n$, we find that the upper portion of the grid can accept $\frac{n(n+2)}{4}$ vertices. Previously it was determined that we wanted to reroute $\frac{n(n-2)}{4}$ edges to the upper portion of the graph. Thus, since $\frac{n(n+2)}{4}$ is always greater than $\frac{n(n-2)}{4}$, we will always be able to reroute $\frac{n(n-2)}{4}$ edges. Thus, we can say that the lower portion of the grid would look as illustrated in Figure 30 or more concretely as Figure 31.

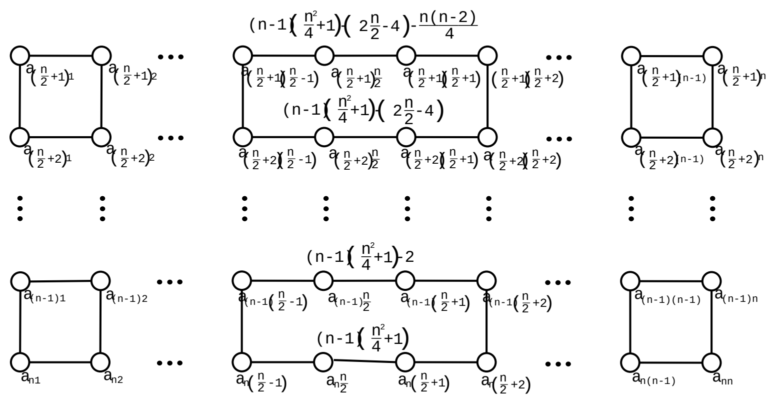

Figure 30

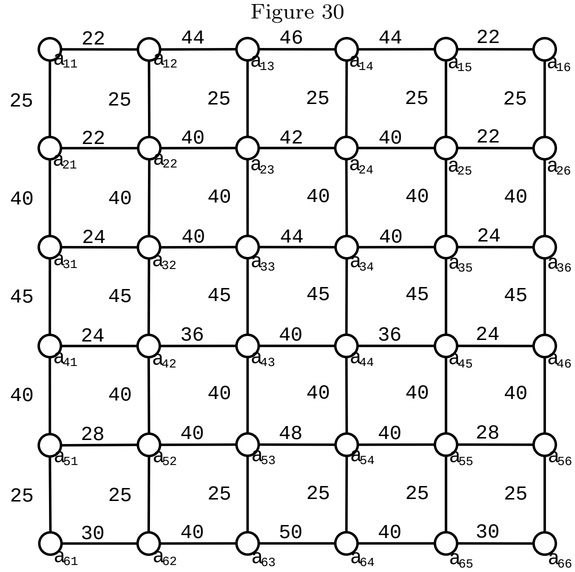

Figure 31

Now edges for the middle section of the lower portion of the graph can be shifted up a row to make all rows $\frac{(m-1)n^2}{4}+1$. For example, considering the 6×6 grid, in order to make the section a_{43} to a_{44} equal 46, six edges will be rerouted to that pass through the middle section of the row below. For example, the edge connecting a_{53} to a_{14} will be rerouted to a_{53} to a_{43} to a_{44} to a_{14}. To counteract the changes made in the cutwidth and further increase the cutwidth between a_{43} and a_{44}, the edge connecting a_{54} to a_{13} will be rerouted to a_{54} to a_{44} to a_{43} to a_{13}. This process will continue until the cutwidth for the middle section is $\frac{(m-1)n^2}{4}+1$. Then this process is repeated for the next row until the bottom of the grid is reached. Once completed we have a grid of cutwidth $\frac{(m-1)n^2}{4}+1$. Thus, the grid cutwidth is:

$$gcw(k_{m\times n}) = \frac{mn^2}{4}+1.$$

6.3 Conclusion

The grid cutwidth of a $m \times n$ grid where $m = n$ and m and n even is:

$$gcw(K_{m\times n}) = \begin{cases} \frac{mn^2}{4} & m \text{ multiple of } 4 \\ \\ \frac{mn^2}{4}+1 & m \text{ not a multiple of } 4. \end{cases}$$

7 Conclusions

7.1 Results

The horizontal cutwidth for an $m \times n$ grid is known to be:

$$hcw(K_{m\times n}) = \begin{cases} \frac{mn^2}{4} & n \text{ even} \\ \\ \frac{m(n^2-1)}{4} & n \text{ odd.} \end{cases}$$

The vertical cutwidth for an $m \times n$ grid is known to be:

$$vcw(K_{m\times n}) = \begin{cases} \frac{m^2(n+1)}{4} & m \text{ and } n \text{ even, } m \neq n \\ \\ \frac{m^2 n}{4} & m \text{ even and } n \text{ odd} \\ \\ \frac{(m^2-1)(n+1)}{4} & m \text{ odd and } n \text{ even} \\ \\ \frac{(m^2-1)n}{4} & m \text{ odd and } n \text{ odd.} \end{cases}$$

In order to prove the the cutwidth of a grid is

$$gcw(K_{m\times n}) = \begin{cases} \frac{mn^2}{4} & n \text{ even} \\ \\ \frac{m(n^2-1)}{4} & n \text{ odd,} \end{cases}$$

we must prove the following inequalities:

- $\frac{mn^2}{4} \geq \frac{m^2(n+1)}{4}$
- $\frac{mn^2}{4} \geq \frac{(m^2-1)(n+1)}{4}$
- $\frac{m(n^2-1)}{4} \geq \frac{m^2 n}{4}$
- $\frac{m(n^2-1)}{4} \geq \frac{(m^2-1)n}{4}$.

The first inequality can be simplified as follows:

$$\begin{aligned} \frac{mn^2}{4} &\geq \frac{m^2(n+1)}{4} \\ mn^2 &\geq m^2(n+1) \\ n^2 &\geq m(n+1) \\ n^2 - mn &\geq m \\ n(n-m) &\geq m. \end{aligned}$$

With the stipulation that $n > m$, $n(n-m)$ is clearly greater than m. The next inequality would simplify as follows:

$$\begin{array}{rcl} \frac{mn^2}{4} & \geq & \frac{(m^2-1)(n+1)}{4} \\ mn^2 & \geq & (m^2-1)(n+1) \\ mn^2 & \geq & m^2n + m^2 - n - 1 \\ 1 & \geq & m^2n + m^2 - n - mn^2 \\ 1 & \geq & mn(m-n) + m^2 - n \\ 1 & \geq & m(n(m-n)+m) - n. \end{array}$$

Since $n > m$, $m(n(m-n)+m) - n$ will always be less than 1. Therefore, the inequality is true. Simplification of the next inequality is:

$$\begin{array}{rcl} \frac{m(n^2-1)}{4} & \geq & \frac{m^2n}{4} \\ m(n^2-1) & \geq & m^2n \\ (n^2-1) & \geq & mn \\ n - \frac{1}{n} & \geq & m. \end{array}$$

Since $n > m$ and m and n are integer, $n - \frac{1}{n}$ will always be greater than m. This is because the closest that m and n could be is 1 apart and n minus a fraction less than 1 will always be greater than m, which is $n-1$. Finally, the last inequality would simplify as follows:

$$\begin{array}{rcl} \frac{m(n^2-1)}{4} & \geq & \frac{(m^2-1)n}{4} \\ m(n^2-1) & \geq & (m^2-1)n \\ mn^2 - m & \geq & m^2n - n \\ mn^2 + n & \geq & m^2n + m \\ n(mn+1) & \geq & m(mn+1) \\ n & \geq & m. \end{array}$$

Since all four inequalities are true the following theorem can be made with the inclusion of $m = n$ and m and n even:

Theorem: The cutwidth for any embedding of a complete graph on an $m \times n$ grid such that $m \leq n$ is:

$$gcw(K_{m \times n}) = \begin{cases} \frac{mn^2}{4} & n \text{ even and } m \neq n \\ \frac{mn^2}{4} & n \text{ even and } m = n \text{ and } m \text{ multiple of four} \\ \frac{mn^2}{4} + 1 & n \text{ even and } m = n \text{ and } m \text{ not a multiple of four} \\ \frac{m(n^2-1)}{4} & n \text{ odd.} \end{cases}$$

7.2 Open Questions

After completion of complete graphs embedded on grids, work was begun on complete bipartite graphs embedded on grids. While we were able to complete some results on grid cutwidth, much work has yet to be done.

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