On the Lower Bounds of Ramsey Numbers of Knots

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April 7, 2003

Abstract

The Ramsey number is known for only a few specific knots and links, namely the Hopf link and the trefoil knot (although not published in periodicals). We establish the lower bound of all Ramsey numbers of any knot to be one greater than its stick number.

1 Background and Definitions

The study of Ramsey numbers of knots can be found at the intersection of knot theory and graph theory.

1.1 Knot Theory Background

A *knot* is a simple closed curve in $\Re^3$, while a *link* is a set of disjoint knots. As shown in figure 1 the unknot(a), trefoil knot(b), figure-8 knot(c), unlink(d), and Hopf link(e) are examples of inequivalent links.

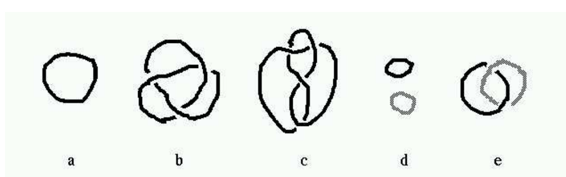
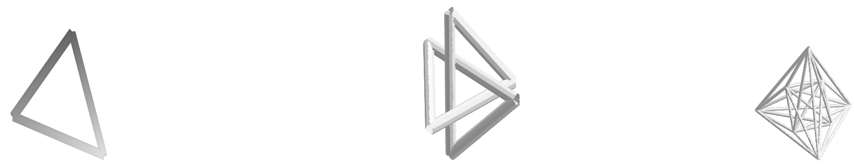


Figure 1 - Some simple knots

Stick knots are knots composed of straight line segments intersecting only two at a time. The *stick number*, $s(k)$, of a knot k, is the fewest number of sticks necessary to embed a knot in $\Re^3$. Many stick numbers for knots are known (MM). For example, $s(unknot) = 3$, $s(unlink) = 6$, $s(trefoil) = 6$, and $s(figure - 8) = 7$. Illustrations of these are in Figure 2. Also, Calvo has classified all 8-stick knots(JC).



stick unknot — stick trefoil — stick figure-8

Figure 2 - Some simple knots

1.2 Graph Theory Background

A *graph*, $G = (V, E, \delta)$, consists of a set, V, of vertices, a set, E, of edges that connects pairs of vertices, and a function, δ, that identifies the vertices incident to each edge. The *complete graph* K_n is a graph of n vertices in which there exists an edge between all vertices. A *path* is a sequence of distinct adjacent vertices. A *cycle* consists of a path with identical beginning and ending points. A *Hamiltonian cycle* of a graph is a cycle which is composed of all vertices of the graph. A *spatial embedding* of a graph is a mapping of a graph onto $\Re^3$ such that all edges are simple closed curves intersecting only at the ends of edges, while a *linear embedding* of a graph spatial embedding such that all edges are non-intersecting straight line segments.

Figure 3 - Examples of linear embeddings of K_3 and K_5

1.3 Ramsey Number Introduction

A cycle or set of disjoint cycles of a complete graph embedded in $\Re^3$ is a link. Negami (91) proved the existence of a finite integer, n, such that any linear embedding of the complete graph, K_n, (of n vertices) contains a cycle homeomorphic to the link k. (after SN). This finite integer is called the Ramsey number, $R(k)$, of a link, k. Since equivalence of links can defined in many ways, Negami also showed the existence of a finite integer, $R_+(k)$, such that any linear embedding of K_n with $n \geq R_+(k)$ contains a link ambient isotopic to k.

Conway's and Gordon's paper, "Knots and Links in Spatial Graphs," (1983) provided useful results for finding Ramsey numbers of knots and links:

- Theorem 1: Every spatial embedding of K_6 contains a nontrivial link.
- Theorem 2: Every spatial embedding of K_7 contains a nontrivial knot.

From Theorem 1, it is easily shown that $R(Hopflink) = 6$.

- Theorem 3: The Ramsey number of the hopf link is 6.

Since all linear embeddings of a complete graph are spatial embeddings, every linear embedding of K_6 contains a nontrivial link. The hopf link is the only nontrivial link with stick number less than or equal to 6. Hence, every linear embedding of K_6 contains the hopf link. Thus, the Ramsey number of the hopf link is 6. An example of a linear embedding of K_6, and the hopf link it contains, is illustrated in Figure 4.

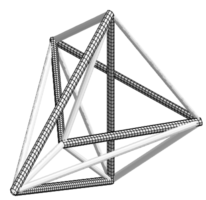


Figure 4 - A linear embedding of K_6 containing the Hopf link

Similar results are not easy to obtain from Theorem 2, since both the trefoil and figure-8 are knots with stick numbers less than or equal to seven. However, results about the bounds of the Ramsey numbers of the trefoil and figure-8 can be made. By examining a particular linear embedding of K_7, it can be shown that any Hamiltonian cycle, a 7-cycle, is equivalent to a 6-cycle. This would mean that not *every* linear embedding of K_7 will contain knots with stick number seven, the only one being the figure-8. Hence the Ramsey number of the figure-8 must be greater than seven. In this same embedding, the only knotted cycles are right-handed trefoils. Thus $R_+(trefoil)$ is also greater than seven.

- Theorem 4: $R(k) > 7$, where k is the figure 8 knot.
- Theorem 5: $R_+(k) > 7$, where k is the left-handed or right-handed trefoil knot.

A similar result can be made about K_8. The only knotted cycles in a specific linear embedding of K_8 are the trefoil and figure-8. Thus, as with K_7, knots with stick number eight will have Ramsey number greater than eight.

- Theorem 6: $R(k) > 8$, where k is any knot with stick number 8.

Clearly the Ramsey number of a knot is always greater than or equal to its stick number($R(k) \geq s(k)$). Conjunctively, Theorem 4 and Theorem 6 state that $R(k) > s(k)$ for knots with s(k)=7,8. We also show this to be true for all knots; the Ramsey number of a knot is always greater than its stick number.

- Main Theorem: $R(k) > s(k)$ for all knots.

We accomplish this using a function that produces a linear embedding of K_n, in which all Hamiltonian cycles of length n and knot type k are reducible to length $n-1$, whike retaining the knot type.

Our techniques significantly use the following lemma (after Calvo's Reduction Lemma):

- Triangle Reduction Lemma: Let abc be a path from a cycle C of a graph embedded in $\Re^3$. The triangle created by abc is reducible to the line segment ac if it is not intersected by another edge in C. (Illustrated in Figure 5.)

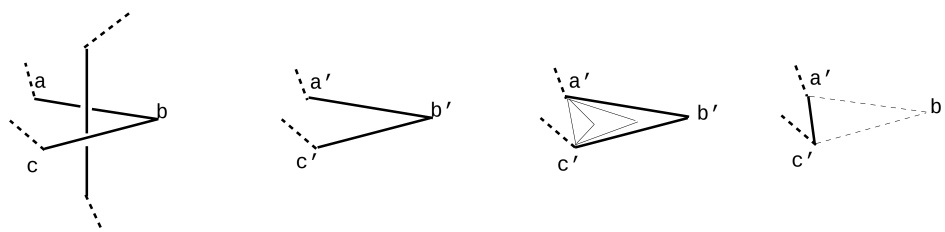


Figure 5 - Example of Non-reducing triangle $\triangle ABC$ and reducing triangle $\triangle A'B'C'$

2 Proofs

2.1 Proof of Theorem 4 $R(k) > 7$, where k is the figure-8 knot

Let ε_7 (Figure 6) be a linear embedding of K_7 in $\Re^3$ defined be the following set of labelled and classified vertices:

$$A : (0, \tfrac{1}{3}\sqrt{3}, \tfrac{2}{3}\sqrt{6}) \Big\} \textbf{axial vertex}$$

$$\left.\begin{array}{l} 1 : (1,0,0) \\ 2 : (-1,0,0) \\ 3 : (0,\sqrt{3},0) \end{array}\right\} \textbf{corner vertices}$$

$$\left.\begin{array}{l} X : (\frac{-1}{10}, \frac{17}{30}\sqrt{3}, \frac{1}{3}\sqrt{6}) \\ Y : (\frac{2}{5}, \frac{4}{15}\sqrt{3}, \frac{1}{3}\sqrt{6}) \\ Z : (\frac{-3}{10}, \frac{1}{6}\sqrt{3}, \frac{1}{3}\sqrt{6}) \end{array}\right\} \textbf{middle vertices}$$

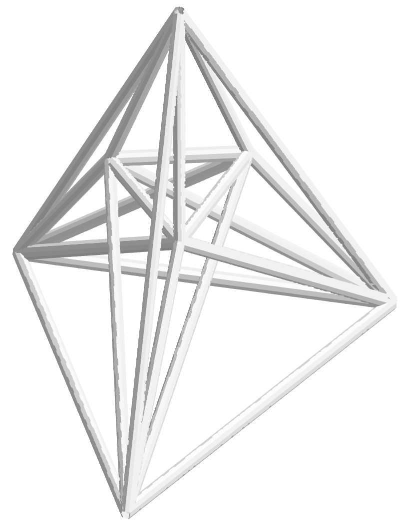


Figure 6 - Incomplete ε_7 and complete ε_7

Note that ε_7 is rotationally symmetric about the axis of vertex A and the middle of triangle 123. For a seven stick knot to be contained in ε_7, the equivalent cycle, C, must be Hamiltonian. We shall attempt to contruct a Hamiltonian cycle with non-reducing triangles, by building outwards from A.

First note that a reducing triangle is formed if A' s incident edges are both middle vertices or both corner vertices. So A must be adjacent to a middle and a corner vertex. Let's fix corner vertex 1 to A. Now the only middle vertice that A can be adjacent to is X. All other middle vertices would result in a reducing triangle. Now we have $XA1 \in C$. Now we build C from 1.

Clearly 1 is not adjacent to a corner vertex, so it must be adjacent to a middle vertice. The only two choices, $XA1Z$ and $XA1Y$, contain reducing triangles. By symmetry, all constructions of Hamiltonian cycles in ε_7 are reducible to 6-cycles. Hence the figure-8 is not contained in ε; $R(figure - 8) > 7$.

2.2 Proof of Theorem 5 $R_+(k) > 7$, where k is the left-handed or right-handed trefoil knot

It is a routine excercise to verify that the only knotted cycles in ε_7 are right-handed trefoils. When all reducible triangles are reduced in the knotted cycles, the corresponding cycle is $1X2Y3Z$ (Figure 7), the six stick right -handed trefoil. Obviously, the mirror image of ε_7 contains only left-handed trefoils. Hence, $R_+(k) > 7$, where k is the right-handed or left-handed trefoil.

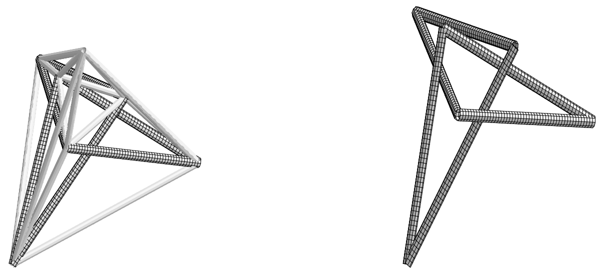


Figure 7 - ε_7 and its reduced right-handed trefoil

2.3 Proof of Theorem 6 $R(k) > 8$, where k is any knot with $s(k) = 8$

Let ε_8 (Figure 8) be a linear embedding of K_8 in $\Re^3$ defined by the following set of labelled and classified vertices:

$$\left.\begin{array}{l} A : (0, \frac{1}{3}\sqrt{3}, \frac{2}{3}\sqrt{6}) \\ B : (0, \sqrt{3}, 0) \\ C : (-1, 0, 0) \\ D : (1, 0, 0) \end{array}\right\} \textbf{corner vertices}$$

$$\left.\begin{array}{l} 1 : (\frac{-7}{20}, \frac{9}{20}\sqrt{3}, 0) \\ 2 : (0, \frac{3}{30}\sqrt{3}, \frac{1}{15}\sqrt{6}) \\ 3 : (\frac{9}{20}, \frac{1}{4}\sqrt{3}, \frac{3}{10}\sqrt{6}) \\ 4 : (\frac{-1}{10}, \frac{3}{5}\sqrt{3}, \frac{3}{10}\sqrt{6} \end{array}\right\} \textbf{middle vertices}$$

Unlike ε_7, ε_8 contains non-reducing Hamiltonian cycles. We show that there are precisely 5 and that they are either trefoils or the figure-eight. Let C be a non-reducing Hamiltonian cycle in ε_8. We can elimiate many reducing Hamiltonian cycles from our examination of ε by establishing the following lemmas, which are clear upon visual examination of :

- Lemma 1: Three adjacent corner vertices can't exist in C.
- Lemma 2: Three adjacent middle vertices can't exist in C.

Although the middle vertices lie upon planes of the tetrahedron formed from the corner vertices, we allow triangle reductions when it is only a middle vertex that intersect the surface bounded by the potentially reducing triangle.

2.4 Proof of Main Theorem $R(k) > s(k)$

Let ε_n be the linear embedding of K_n defined by the following function for the position(ρ) in $\Re^3$ of vertex i in a complete graph with n vertices:

$$\rho(i, n) = [cos(\frac{i * \pi}{n}), sin(\frac{i * \pi}{n}), \frac{i * pi}{n}]$$

The vertices of the embedding lie on a half-circle of the helix parameterized by the following equations:

$$\left.\begin{array}{l} x(t) = sin\ t \\ y(t) = cos\ t \\ z(t) = t \end{array}\right\} t : [0, \pi]$$

INSERT ILLUSTRATIONS OF epsilonn

Given Hamiltonian cycle C in ε_n, let k denote the knot type represented by C. We will show that the stick number of k is less than or equal to $n-1$. This will be accomplished by showing that under a specific manipulation of C , a topologically equivalent cycle with one less edge can be created.

Since C is Hamiltonian, it contains all vertices of ε_n. We wish to focus on the nth vertex and those adjacent to it, say p and q. The following three step algorithm constructs an $n-1$-cycle with knot type k from C:

1. Begin by removing vertex n and its incident edges pn and qn. This reduces the number of edges by two.

2. Next, lengthen the edges still incident with p and q, in effect, pushing p and q off the half circle helix. Extension of these edges is sufficient when a new edge, pq, can be added such that it doesn't interfere with C.

3. Reconstruct the knot by adding new edge pq. This increases the number of edges by one.

In cases where the extending edges are parallel, an ϵ-movement of p (or q) along the half-circle helix will make the extending edges nonparallel. This will allow construction of the new edge pq without upsetting the integrity of C, as long as ϵ is less than the distance between p (or q) and it's closest vertex .

Note that the net effect of this algorithm reduces the number of edges by one, yet still retains the knot type of C.

3 Acknowledgements

Many thanks to Rolland Trapp and Joseph Chavez for their patience and wisdom in their direction of the Research Experience for Undergraduates (REU) at California State University San Bernardino (CSUSB)in the Summer of 2002. Thanks to Maple for helping to develop figures for this paper and thanks to Bradley Marchan (California State University Bakersfield) for helping to develop my understanding of Maple. Acknowledgements are also deserved by CSUSB and the National Science Foundation (NSF-REU Grant DMS-01394226) for their financial support of the REU program.

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