The Linear and Cyclic Cutwidth of the Complete Bipartite Graph

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Abstract

We shall consider the problem of embedding the complete bipartite graph, $K_{m,n}$, onto a linear and cyclic chassis in such a way as to minimize the cutwidth. The linear cutwidth of the complete bipartite graph is established and a partial solution to the cyclic cutwidth is presented. It is known that there is a paper in existance [3] that has established the linear cutwidth of the complete bipartite graph. However, we were unable to locate the paper and hence these results were formed independently, without the aid of this paper.

1 Introduction

A graph $G = (V, E)$ consists of a set of vertices, V, and a set of edges, E, that join pairs of vertices. Figure 2 is an example of a graph G with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 4), (2, 3), (3, 5), (4, 5), (1, 5)\}$ where edges are represented by lines and vertices by points.

A complete bipartite graph $K_{m,n}$ consists of two disjoint sets of vertices A and B such that every vertex in A is joined by an edge to each vertex in B where |A|= m and |B|= n, and no two vertices in the same set share and edge. Figure 1 is an example of the complete bipartite graph $K_{4,4}$.

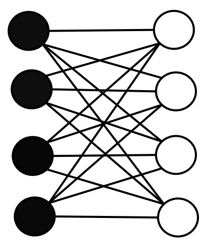

Figure 1: $K_{4,4}$

We shall now define the important terms and ideas used in this paper. For any graph G a linear embedding of a graph is simply all of the vertices of the graph embedded onto a line. Any edges that connect vertices in the non-linear embedding of G will also connect vertices in the linear embedding. An example of a linear embedding of the graph is found in Figure 2.

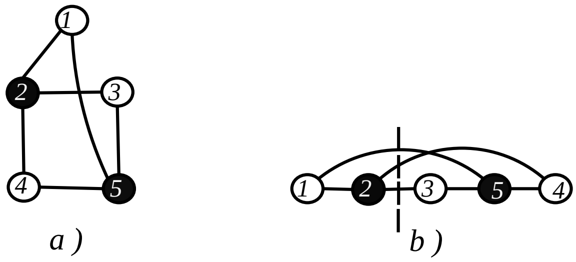

Figure 2

A region is defined to be the area between two adjacent vertices on the linear embedding of a graph (Figure 2b). The cut of a region is the number of egdes that cross the region from the left or right. For example, in Figure 2b the region between vertices 2 and 3 has a cut of 3.

A cyclic embedding of G is a graph in which all of the vertices of G are embedded onto a cycle. Any edges that connect vertices in G will also connect vertices in the cyclic embedding of G.

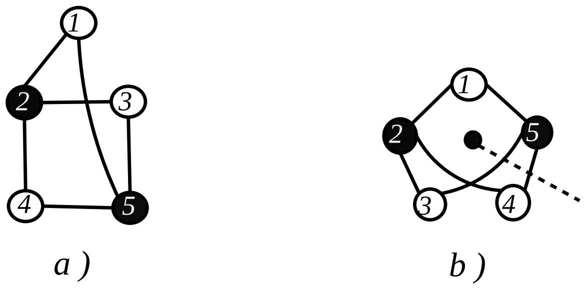

Figure 3

A region between two adjacent vertices in a cyclic embedding is defined to be the triangular area created by the two adjacent vertices and the center of the circle. The cut of a region in a cyclic embedding is the number of edges that cross over the given region. In Figure 3b the region between vertices 4 and 5 has a cut of two.

The maximum cut of a particular embedding of a graph is the largest cut that occurs on the graph. The cutwidth of the graph is the minimum of all possible maximum cuts over all possible embeddings. In Figure 2b the maximum cut of the particular embedding is three and in Figure 3b it is two.

The following shall consider the cutwidth problem for different embeddings of complete bipartite graphs. The cutwidth problem is basically the problem of searching to find optimal arrangements of vertices in different embeddings, such that the number of edges crossing any given region of the embedding is minimized. This paper will provide a proof for the linear cutwidth of a complete bipartite graph as well as a partial solution to the cyclic cutwidth of the complete bipartite graph.

2 Background

Graph theory has been applied to many different problems since it has connections to real world applications (i.e. networking, circuit layout, or code design). One specific graph theory problem related to these applications is the cutwidth problem. By discovering cutwidth one is able to optimally arrange a network or circuit such that the edges are evenly distributed across the network or circuit alleviating congestion.

Many different people have worked with the cutwidth problem. Fransisco Rios developed a formula for the linear (lcw) and cyclic cutwidth (ccw) of a complete graph K_n. The following results were proven by Fransisco Rios [4].

For any complete graph K_n on n vertices,

$$lcw(K_n) = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \frac{n^2-1}{4}, & n \text{ odd.} \end{cases}$$

For any complete graph K_n on n vertices,

$$ccw(K_n) = \begin{cases} \frac{n^2+8}{8}, & \frac{n}{2} \text{ even} \\ \frac{n^2+4}{8}, & \frac{n}{2} \text{ odd} \\ \frac{n^2-1}{8}, & n \text{ odd.} \end{cases}$$

Heiko Schroder made progress with the cyclic cutwidth of a two-dimensional mesh P_m x P_n [5]. A mesh is a rectangular graph that has dimensions m by n. The following results are results of Schoder as amended by Dwayne Clarke [2].

For a graph G which is a P_m x P_n mesh where $m \geq n \geq 3$,

$$ccw(G) = \begin{cases} n-1, & m = n \text{ even} \\ n, & m = n, n+1, n \text{ odd or} \\ & m = n+1, n+2 \text{ and } n \text{ even} \\ n+1, & \text{otherwise.} \end{cases}$$

Joe Chavez and Rolland Trapp have completed the cyclic cutwidth problem for trees. A tree is a connected a-cyclic graph, meaning a graph with no cycles where each vertex is reachable from any other vertex. Chavez and Trapp have proven the following result about trees: if T is a tree, then $lcw(T) = ccw(T)$ [1].

There have been many advancements on the cutwidth problem in recent years. It has been proven that there is no single solution to the cutwidth problem of a general graph. So it is important to concentrate on specific cases and this is the reason we are focusing on complete bipartite graphs.

3 Linear Cutwidth of the Complete Bipartite Graph

Theorem 1: For any complete bipartite graph $K_{m,n}$,

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2}, & mn \text{ even} \\ \frac{mn+1}{2}, & mn \text{ odd.} \end{cases}$$

Proof

Let A and B be two disjoint sets of vertices, where |A|$= m$ and |B|$= n$, together composing the vertex set of $K_{m,n}$. Without loss of generality we may assume $m \leq n$.

We will now embedd the vertices of $K_{m,n}$ on a line making a linear embedding of $K_{m,n}$. For each pair of adjacent vertices in the linear embedding of $K_{m,n}$ define the cut to be the number of edges of $K_{m,n}$ that pass between the given adjacent vertices.

Let $cut(a, b)$ denote the cut in the region directly following (from the left) a vertices from A and b vertices from B. Figure 4 is a diagram of $K_{3,4}$ where the black vertices are from set A and the white are from set B.

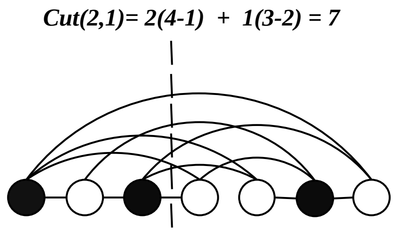

Figure 4

Explicitly,

$$cut(a, b) = a(n - b) + b(m - a)$$

We shall first minimize each individual cut along the linear embedding of $K_{m,n}$ from the left to right. The following procedure will do just this.

Let x be the number of vertices to the left of the cut(in Figure 1 $x = 3$). At each region, place $[\frac{xm}{m+n}]$ vertices from A and $x - [\frac{xm}{m+n}]$ vertices from B to the left of the cut. (Where $[x] = \lfloor x + .5 \rfloor$)

Let $y = [\frac{xm}{m+n}]$.

There are two cases to examine in order to show this process yields minimum cut for each region on the graph.

Case 1: $m < n$

For any region x where $0 < x \leq \frac{m+n}{2}$,

$$\begin{aligned} cut(y, x-y) &= y[n-(x-y)] + (x-y)(m-y) \\ &= y(n-m+2y) + x(m-2y). \end{aligned}$$

Now consider the same cut if the ratio described above is disrupted and the margin between the number of elements from each class to the left of the region is made larger.

Where $1 \leq l \leq y$,

$$\begin{aligned} cut(y-l, (x-y)+l) &= (y-l)[n-(x-y+l)] + (x-y+l)[m-(y-l)] \\ &= y(n-m+2y) + x(m-2y) - ln + 2xl - 4ly + 2l^2 + lm. \end{aligned}$$

We observe that,

$$cut(y-l, (x-y)+l) = cut(y, x-y) + l(2x - 4y + m - n + 2l).$$

We will now show that $l(2x - 4y + m - n + 2l) \geq 0$.

It is known from the above argument that $x \leq \frac{m+n}{2}$ implies:

$$\begin{aligned} x &\leq \tfrac{(m+n)(m-n)}{2(m-n)} \\ x &\leq \tfrac{m^2-n^2}{2m-2n} \\ x(2m-2n) &\geq m^2 - n^2 \quad \text{since } (2m-2n) < 0 \\ x(2m-2n) + n^2 - m^2 &\geq 0 \\ \tfrac{1}{m+n}[x(4m-2m-2n) + n^2 - m^2] &\geq 0 \quad \text{since} \quad \tfrac{1}{m+n} \geq 0 \\ \tfrac{1}{m+n}[4mx - 2mx - 2nx + n^2 - m^2] &\geq 0 \\ \tfrac{1}{m+n}[4mx - 2x(m+n) + (n-m)(n+m)] &\geq 0 \\ \tfrac{4mx}{m+n} - 2x + (n-m) &\geq 0. \end{aligned}$$

Now we know, $\frac{4mx}{m+n} - 4[\frac{mx}{m+n}] \leq 2 \leq 2l$ since, $l \geq 1$.
This implies,

$$4[\frac{mx}{m+n}] + 2l \geq \frac{4mx}{m+n}$$

Yields,

$$\begin{aligned} 4[\tfrac{mx}{m+n}] + 2l - 2x + n - m &\geq 0 \\ 4y - 2x + n - m + 2l &\geq 0. \end{aligned}$$

Therefore,

$$cut(y, x-y) \leq cut(y-l, (x-y)+l) \quad when\ m < n.$$

<u>**Case 2**</u>: $m = n$

Using the same method for vertex distribution as in case 1. Let x be the number of vertices to the left of the cut. At each region, place $[\frac{xm}{m+n}]$ vertices from A and $x - [\frac{xm}{m+n}]$vertices from B to the left of the cut. (Where $[x] = \lfloor x + .5 \rfloor$). yields,

$$cut(y, x-y) = 2y^2 + x(m-2y).$$

Now consider the same cut if the ratio described above is disrupted and the margin between the number of elements from each class to the left of the cut is made larger. Where $1 \leq l \leq y$,

$$cut(y-l, (x-y)+l) = 2y^2 + x(m-2y) + 2xl - 4ly + 2l^2.$$

We observe that, $cut(y-l, (x-y)+l) = cut(y, (x-y)) + 2xl - 4ly + 2l^2$.

We want to show that $2xl - 4ly + 2l^2 \geq 0$.

Since we know $y \leq \frac{x}{2}$ we get,

$$\begin{aligned} 4y &\leq 2x \\ 2x - 4y &\geq 0 \\ 2x - 4y + 2l &\geq 0 \\ 2xl - 4yl + 2l^2 &\geq 0. \end{aligned}$$

Therefore, $cut(y, x-y) \leq cut(y-l, (x-y)+l)$ when $m = n$.

Therefore using the above mentioned method for vertex distribution will yield the minimum cut for each region of the linear embedding of $K_{m,n}$

Having established the minimum cut for each region on the graph, we may now determine the linear cutwidth of $K_{m,n}$ by finding the largest cut on the graph. The largest cut in the linear embedding of $K_{m,n}$ occurs at the middle region of the graph (i.e. the region directly following the $\frac{m+n}{2}$ vertex if $m+n$ even or the $\frac{m+n-1}{2}$ vertex if $m+n$ odd). To show that this is true we should examine four seperate cases.

<u>**Case 1**</u>: Both m and n are even

Using the above mentioned method for vertex distribution yields,

$$cut(\frac{m}{2}, \frac{n}{2}) = \frac{mn}{2}.$$

We now consider any other cut on the graph to the the left of the middle region. Let q be the number of vertices we shift from the middle region in set A and p the number in set B, where $0 \leq q \leq \frac{m}{2}$ and $0 \leq p \leq \frac{n}{2}$.

Then,

$$\begin{aligned} cut(\tfrac{m}{2}-q, \tfrac{n}{2}-p) &= (\tfrac{m}{2}-q)(n-\tfrac{n}{2}+p)+(\tfrac{n}{2}-p)(m-\tfrac{m}{2}+q) \\ &= \tfrac{mn}{2}-2qp. \end{aligned}$$

Relating the two equations gives,

$$cut(\frac{m}{2}-q, \frac{n}{2}-p) = cut(\frac{m}{2}, \frac{n}{2}) - 2qp.$$

Therefore, $cut(\frac{m}{2}-q, \frac{n}{2}-p) \leq cut(\frac{m}{2}, \frac{n}{2})$ since $2pq \geq 0$.

Therefore, if both m and n are even the cutwidth of $K_{m,n}$ is $\frac{mn}{2}$.

<u>**Case 2**</u>: m odd and n even
Using the same method for vertex distribution as before yields,

$$cut(\frac{m-1}{2}, \frac{n}{2}) = \frac{mn}{2}.$$

We consider the other cuts on the graph:

$$\begin{aligned} cut(\tfrac{m-1}{2}-q, \tfrac{n}{2}-p) &= \tfrac{mn}{2}-p(2q+1) \\ &= cut(\tfrac{m-1}{2}, \tfrac{n}{2})-p(2q+1). \end{aligned}$$

We have, $cut(\frac{m-1}{2}-q, \frac{n}{2}-p) \leq cut(\frac{m-1}{2}, \frac{n}{2})$ since $p(2q+1) \geq 0$.

Therefore, if m is odd and n is even the cutwidth of $K_{m,n}$ is $\frac{mn}{2}$.

<u>**Case 3**</u>: m even and n odd
Calculating the cut of the middle region yields,

$$cut(\frac{m}{2}, \frac{n-1}{2}) = \frac{mn}{2}.$$

Using the same method as in case 1, consider any other cut on the graph to the the left of the middle region. Let q be the number of vertices we shift from the middle region in set A and p the number in set B. Where, $0 \leq q \leq \frac{m}{2}$ and $0 \leq p \leq \frac{n}{2}$. Then,

$$\begin{aligned} cut(\tfrac{m}{2}-q, \tfrac{n-1}{2}-p) &= \tfrac{mn}{2}-q(2p+1) \\ &= cut(\tfrac{m}{2}, \tfrac{n-1}{2})-q(2p+1). \end{aligned}$$

We have, $cut(\frac{m}{2}-q, \frac{n-1}{2}-p) \leq cut(\frac{m}{2}, \frac{n-1}{2})$ since $q(2p+1) \geq 0$.

Therefore, if m is even and n is odd the cutwidth of $K_{m,n}$ is $\frac{mn}{2}$.

Case 4: m odd and n odd
Calculating the cut of the middle region gives,

$$cut(\frac{m+1}{2}, \frac{n-1}{2}) = \frac{mn+1}{2}$$

Using the same method as in Case 1 we consider other cuts on the graph:

$$\begin{aligned} cut(\tfrac{m+1}{2} - q, \tfrac{n-1}{2} - p) &= \tfrac{mn+1}{2} + p - q - 2qp \\ &= cut(\tfrac{m+1}{2}, \tfrac{n-1}{2}) + p - q - 2qp. \end{aligned}$$

Note: If one of p or q is zero and the other is not, it must be p that is zero. This is due to the fact that the vertices were arranged in such a way that there is always a vertex from A directly to the left of the middle region. This ensures that $p - q - 2pq \geq 0$.

We have, $cut(\frac{m+1}{2} - q, \frac{n-1}{2} - p) \leq cut(\frac{m+1}{2}, \frac{n-1}{2})$ since $p - q - 2pq \geq 0$.

So if m and n are both odd the cutwidth of $K_{m,n}$ is $\frac{mn+1}{2}$.

Therefore, For any complete bipartite graph $K_{m,n}$ where $m \leq n$

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2}, & mn \text{ even} \\ \frac{mn+1}{2}, & mn \text{ odd.} \end{cases}$$

4 A lower bound for Cyclic Cutwidth

For each pair of adjacent vertices in a cyclic embedding G, define the *cut* of a region to be the number of edges that cross the triangular region created by the two adjacent vertices and the center of the circle. The maximum cut of a particular embedding of a graph is the largest cut that occurs on the graph. The cutwidth of the graph is the minimum of all possible maximum cuts over all possible embeddings.

Theorem 2: For any graph G,

$$ccw(G) \geq \frac{lcw(G)}{2}.$$

Proof

Let y be the linear cutwidth of G. Consider any cyclic embedding of G with cyclic cutwidth x. Assume that $x < \frac{y}{2}$.

We shall number the vertices of the cyclic embedding clockwise from a_1 to a_n beginning at a region where the cutwidth x occurs. We also number the cuts of the graph such that the cut to the right and adjacent to a_i will be α_i (Figure 5).

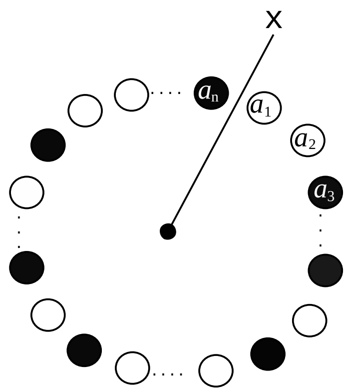

Figure 5

We now transform the cyclic embedding of G into a linear embedding. To do this we shall arrange each of the vertices, a_i, in order on a linear embedding, connecting all of the vertices that were connected in the cyclic embedding (Figure 6).

Figure 6

Let α_i be the cut on the cyclic embedding such that when transformed into a linear graph a maximum cut occurs at the cut α_i. Assume l is the number that the cut α_i increases by in the linear embedding. So the maximum linear cut is $\alpha_i + l$. We know

$$\begin{aligned} \alpha_i + l &\leq \alpha_i + x && \text{since } l \leq x \\ &\leq 2x && \text{since } a_i \leq x \\ &< y && \text{by hypothesis.} \end{aligned}$$

Therefore, $\alpha_i + l < y$.

A contradiction arises since if $lcw(G) = y$ then the largest linear cut on this embedding $\alpha_i + l \geq y$ by definition of linear cutwidth. So $x \geq \frac{y}{2}$. Therefore, a lower bound for the cyclic cutwidth of G is half of the linear cutwidth of G.

So for any graph G,

$$ccw(G) \geq \frac{lcw(G)}{2}.$$

Note: Since this fact is proven for any graph G it is also true for $K_{m,n}$, the complete bipartite graph.

5 Cyclic Cutwidth of the Complete Bipartite Graph

We have the following partial results about the cyclic cutwidth of the complete bipartite graph.

Theorem 3:

$$ccw(K_{m,n}) = \begin{cases} \frac{mn}{4}, & m, n \text{ both even} \\ \frac{mn+3}{4}, & \text{if } m = n \text{ odd.} \end{cases}$$

Proof

Let A and B be two disjoint sets of vertices, where |A|= m and |B|= n, together composing $K_{m,n}$. Without loss of generality we may assume that $m \leq n$.

<u>**Case 1**</u>: m, n both even

It has been established by Theorem 2 that $\frac{lcw(K_{m,n})}{2}$ (otherwise written $\frac{mn}{4}$) is a lower bound for $ccw(K_{m,n})$. So to prove that $ccw(K_{m,n}) = \frac{mn}{4}$, it is sufficient to show that it is always possible to cyclically embed $K_{m,n}$ with a maximum cut of $\frac{mn}{4}$ when m and n are even.

We shall now arrange the vertices of $K_{m,n}$ in such a way that each cut will be $\frac{mn}{4}$. Divide the cyclic embedding into four quadrants: I, II, III, IV. Let quadrants II and IV of the graph each be composed of $\frac{m}{2}$ vertices from A and I and II each of $\frac{n}{2}$ vertices from B (Figure 7).

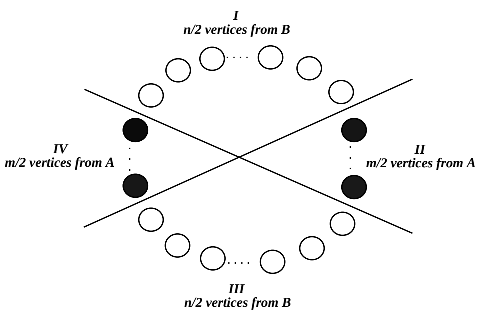

Figure 7

We now develop a method to calculate each cut of this embedding. Without loss of generality we will define the initial cut to be the cut between quadrants I and II. It is obvious any region that lies between two adjacent quadrants will have a cut of $(\frac{n}{2})(\frac{m}{2}) = \frac{mn}{4}$ since there are $\frac{n}{2}$ vertices on one side of the region, whose edges will contribute to the cut, adjacent to $\frac{m}{2}$ on the other.

Let l be an integer representing the number of vertices (going clockwise) away from the initial cut where the cut is being examined. Let $0 \leq l \leq \frac{m}{2}$.

Since the graph is symmetric the same process could be done if the initial cut was between the third and fourth quadrants. We note that any vertex loss from A in quadrant II in the movement from the initial cut is replaced by a vertex from A in quadrant III. This is intuitively why each cut is $\frac{mn}{4}$ since there are still $\frac{m}{2}$ vertices from A adjacent to $\frac{n}{2}$ vertices from B whose edges cross the region we are examining. Explicitly we have,

$$(\frac{m}{2} - l)(\frac{n}{2}) + l(\frac{n}{2}) = \frac{mn}{4}.$$

Therefore, each region on this cyclic embedding of $K_{m,n}$ will have a cut of $\frac{mn}{4}$. Now since $\frac{mn}{4}$ is a lower bound for $ccw(k_{m,n})$ we have proven for m and n even that,

$$ccw(K_{m,n}) = \frac{mn}{4}.$$

<u>Case 2</u>: $m = n$ and m,n odd

We shall establish a lower bound for the cyclic cutwidth of $K_{m,n}$ when $m = n$ and m and n are odd. Since $m = n$, without loss of generality will will consider $K_{n,n}$ where n is odd. Using Theorem 2 we know $ccw(K_{n,n}) \geq \frac{lcw(K_{n,n})}{2}$. Using Theorem 1 where n is odd implies,

$$ccw(K_{n,n}) \geq \frac{n^2+1}{4}.$$

We now note that $\frac{n^2+1}{4}$ is not an integer since $n^2 + 1$ is not divisible by 4. Let $n = 2k + 1$ where k is a positive integer. Consider,

$$\begin{aligned} \tfrac{n^2+1}{4} &= \tfrac{(2k+1)^2+1}{4} \\ &= \tfrac{4k^2+4k+2}{4} \\ &= k^2 + k + \tfrac{1}{2}. \end{aligned}$$

Clearly this is not an integer. So the lower bound can be increased to the nearest integer greater than $\frac{n^2+1}{4}$. It is clear by examining the above written equations that this number is $\frac{n^2+3}{4}$. So we have now established that,

$$ccw(K_{n,n}) \geq \frac{n^2+3}{4}.$$

To prove that this lower bound is in fact the cutwidth of $K_{n,n}$ it is sufficient to show that it is always possible for $K_{n,n}$ to have a maximum cut equal to $\frac{n^2+3}{4}$.

In order to do this we must first arrange the vertices of $K_{n,n}$ in an alternating pattern such that no vertex will be directly next to another vertex of the same class (Figure 8). We know this is possible since in $K_{n,n}$ both A and B have exactly n elements.

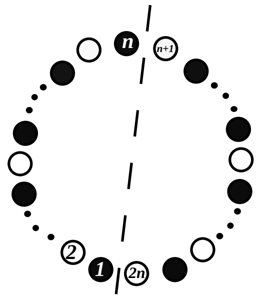

Figure 8

To show that this arrangement always yields a maximum cut of $\frac{n^2+3}{2}$ we first number the vertices from 1 to $2n$ (Figure 8). Divide the graph into two equal halves with a line going through the region between 1 and $2n$ and the region between n and $n+1$. Let there be $\frac{n+1}{2}$ vertices from B (black) and $\frac{n-1}{2}$ vertices from A (white) in the half of the circle containing vertex 1. So in the other half of the circle containing vertex $2n$ there are $\frac{n-1}{2}$ vertices from B and $\frac{n+1}{2}$ vertices from A.

Finding the cut of the region between vertices 1 and $2n$ will help to determine the cut of the other regions on the graph. We will connect pairs of vertices with the shortest edge possible. As we count the edges crossing the region we will not yet consider diameters (i.e. edges from a vertex i to $i+n$) since these edges require a choice regarding which way to send it around the center of the circle.

Vertex 1 connects with a total of $\frac{n-1}{2}$ non-diameter vertices that will contribute to the cut of the given region. It is clear that the edge from 1 to $n+1$ will be a diameter and for the time being we should not be concerned with such edges.

We now move to vertex 3, which contributes a total number of $\frac{n-3}{2}$ non-diameter edges to the cut of the region. The reason vertex 3 contributes one less then vertex 1 is since we are connecting with the shortest path one edge will not have a shortest path over the region we are considering. Namely, the edge from 3 to $n+1$ will not contribute to the cut of the region between 1 and $2n$.

This pattern continues as we move farther from vertex 1 until there is only one egde contributed by a vertex. So we know that the vertices from B on this half of the cycle will contribute,

$$[\frac{n-1}{2} + \frac{n-3}{2} + \frac{n-5}{2} + ... + 1].$$

And now we determine how many edges the vertices from A in the half of the graph containing vertex 1 will contribute to the cut of the region between vertices 1 and $2n$. The edge from vertex 2 to vertex $n+2$ will be a diameter and we shall not consider that at this moment. We are left with a total of $\frac{n-3}{2}$ edges crossing the region. A similar argument works for the remaining vertices of A, yielding a contribution of

$$[\frac{n-3}{2} + \frac{n-5}{2} + \frac{n-7}{2} + ... + 1].$$

Totalling the two contributions to the cut of the region yields,

$$\begin{aligned} \tfrac{n-1}{2} + 2[(\tfrac{n-3}{2}) + (\tfrac{n-5}{2}) + ... + 1] &= \tfrac{n-1}{2} + 2(\tfrac{(\frac{n-3}{2})(\frac{n-1}{2})}{2}) \\ &= \tfrac{n-1}{2} + \tfrac{n^2-4n+3}{4} \\ &= \tfrac{n^2-2n+1}{4}. \end{aligned}$$

Therefore, the region has a cut of $\frac{n^2-2n+1}{4}$ without considering the diameters of the graph. It should be noted that each region on the graph will have a cut of $\frac{n^2-2n+1}{4}$ when diameters are not considered. Since if you shift say to the region between 1 and 2, the only change will be that there are $\frac{n+1}{2}$ vertices from A (white) and $\frac{n-1}{2}$ vertices from B (black) in the half of the circle containing 1. The total contribution to the cut of the region will remain the same.

Diameters have been ignored until this point. Every $K_{n,n}$ where n is odd will have exactly n diameters since the arrangement alternates the vertices in such a way that each vertex i will be in a different class from the vertex $i + n$.

Victor Sciotino has proven that the largest contribution of the n diagonals to a region's cut will be $\frac{n+1}{2}$ [6]. So the contribution of the diagonals to the cut of any given region will either be $\frac{n+1}{2}$ or $\frac{n-1}{2}$. In proving this result the method used to get these values is alternating the direction of the diameters (i.e. if one diameter is sent to one side of the center of the circle, then the diameter adjacent to it is sent to the other side). Adding the larger contribution to our previous result yields,

$$\frac{n^2 - 2n + 1}{4} + \frac{n+1}{2} = \frac{n^3 + 3}{4}.$$

This arrangement will yield a maximum cut of $\frac{n^2+3}{4}$. Therefore, since we have established a lower bound of $\frac{n^2+3}{4}$ for $ccw(K_{n,n})$ and proven that it is always possible to obtain a maximum cut of $\frac{n^2+3}{4}$ we can conclude that,

$$ccw(K_{n,n}) = \frac{n^2 + 3}{4}.$$

Now substituting $m = n$ into the above equation where m and n are odd yields,

$$ccw(K_{m,n}) = \frac{mn + 3}{4}.$$

6 Conclusion

In this paper we have proven a complete solution for the linear cutwidth of the complete bipartite graph. A partial solution to the cyclic cutwidth is included. Further research could be done on the cyclic cutwidth problem for the complete bipartite graph.

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