Star Move and Construction of a 10 Stick 11 Crossing Knot

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Abstract

A geometric construction called the star move is introduced. Given the correct setup, the star move can be used to add 2 sticks and 3 crossings to a stick knot. In particular, the star move may be preformed on an 8 stick 8 crossing stick knot to obtain a 10 stick 11 crossing stick knot. Previously, the smallest knot known to have fewer sticks than crossings was an 11 stick 12 crossing knot [1].

1 Introduction

A knot is defined as a closed curve in space which does not intersect itself. Any knot may be represented using a series of straight lines or sticks connected at vertices instead of a smooth curve. A vertex of a stick knot is external if extending both sticks forming the vertex does not interfere with the rest of the knot. For example, a stick representation of the trefoil requires six sticks (figure 1.1) and all vertices in the figure are external.

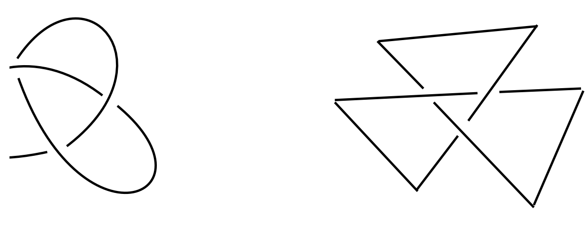


Figure 1.1

One area of research in knot theory involves finding representations of knots with the least number of sticks. Any geometric construction which adds more crossings than sticks improves the upper bound for the number of sticks needed to construct certain types of knots.

One way to identify a knot from any diagram is to use the Dowker code of the diagram. To find the Dowker code of the knot, the crossings of the knot are numbered, going all the way around the knot so that each crossing has two numbers associated with it, an odd number and an even number (figure 1.2). Beginning with the crossing at 1 and assuming it is correct, the crossings are marked with an x which would be incorrect if the knot diagram were alternating.

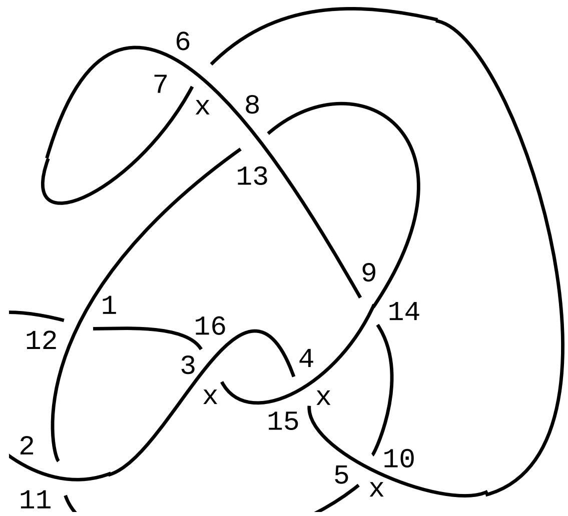


Figure 1.2

Next a table is made with the odd numbers in sequence across the top and the even numbers corresponding to them below. The even numbers of the incorrect crossings become negative.

1	3	5	7	9	11	13
10	-14	-8	12	2	6	-4

For an knot with n crossings there should be n numbers in each row of the table. The sequence of n even numbers can be input into knotscape to identify the knot. Entering the Dowker code of the knot in figure 1.2 in knotscape, it is identified as knot 5 a 1, a 5 crossing alternating knot indexed 1 in the knotscape knot table.

2 Star Move

2.1 Initial Position Requirements

Let K be stick representation of a knot. A star move may be performed on K to create a new knot, N, only if the starting position requirements for the star move exist in K.

Lemma 1 In order to perform the star move there must be two external vertices of K positioned as in figure 2.1. If vectors are placed along some of the sticks as shown in figure 2.2 and proj $_{rxs}$ t > 0, or in other words, the vector t is directed upwards from the plane formed by r and s, then the star move may be performed.

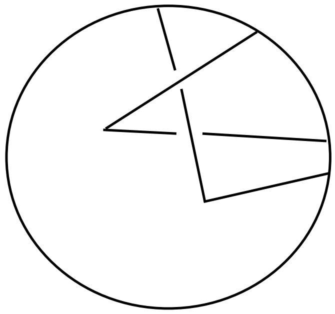


Figure 2.1

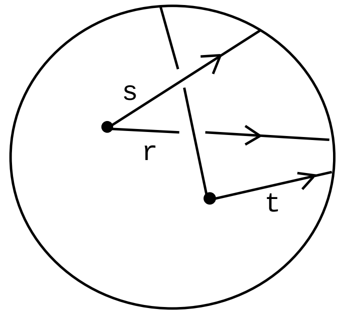


Figure 2.2

2.2 Steps of Star Move

The first step of the star move is to extend the sticks forming the two external vertices outward as in figure 2.3. This creates two double points where two sticks intersect. The double points are resolved so that the crossings are alternating with the crossings in the original setup as in figure 2.4.

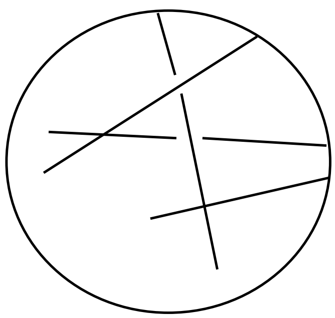


Figure 2.3

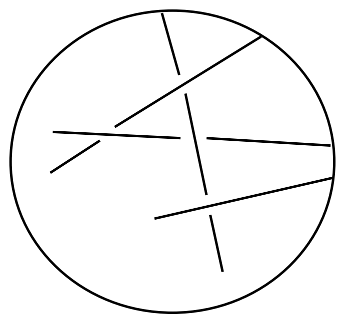
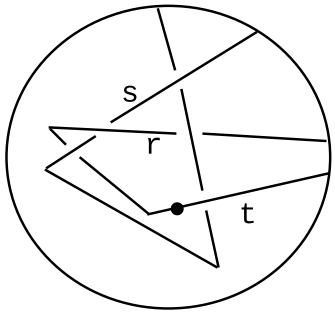


Figure 2.4

The next step is to connect the two overcrossing sticks with a new stick and connect the two undercrossing sticks with a new stick. This creates another crossing as in figure 2.5. This crossing keeps the tangle alternating since from the original setup, stick t in the figure can be extended to penetrate the plane of lines r and s at some point so that it ends beneath the plane. Then the stick connecting sticks t and r will pass beneath stick s as in the figure. The end result of the star move is figure 2.6.



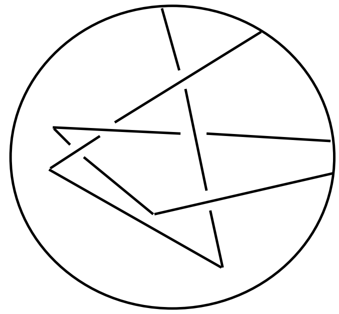
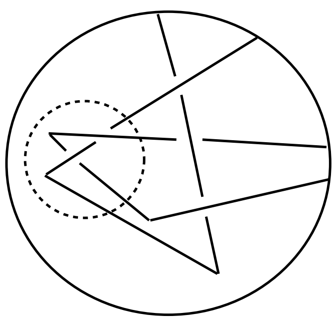


Figure 2.5 Figure 2.6

This adds 2 sticks and 3 crossings. The star move may also be repeated.

Lemma 2 The restrictions for the original setup of the star move force the 2 new external vertices in the dotted circle in figure 2.7 to meet the same restrictions.

Proof Again, vectors can be placed along the sticks from the external vertices. By the construction of the star move, u is in the plane of r and s (see figure 2.2), proj $_{rxs}$ w < 0 and proj $_{rxs}$ v > 0. Therefore, proj $_{wxu}$ v > 0 and the star move may be repeated using these two vertices.



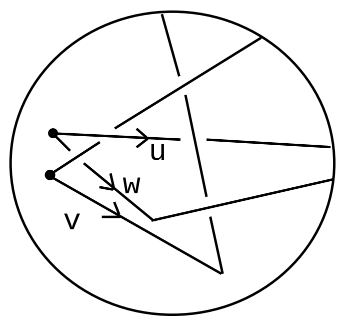


Figure 2.7 Figure 2.8

2.3 Naming of the Star Move

This move was named the star move because it was initially developed as in figure 2.9. However, the representation in 2.6 proved more convenient since the tangle is simplified to be alternating with fewer crossings.

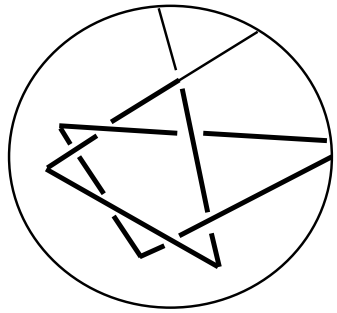


Figure 2.9

2.4 Effects of Star Move to Knot

The two external vertices needed to perform the star move form a 2-tangle shown in figure 2.10. Performing the star move forms the tangle shown in figure 2.11.

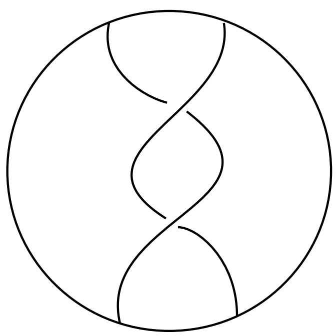


Figure 2.10

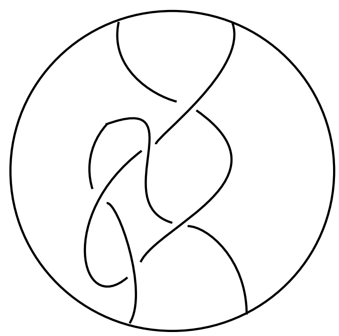


Figure 2.11

The portion of the tangle in the dotted ellipse is added to the 2-tangle. As a result of the star move a 3-tangle is added onto the side of the original 2-tangle (figure 2.12).

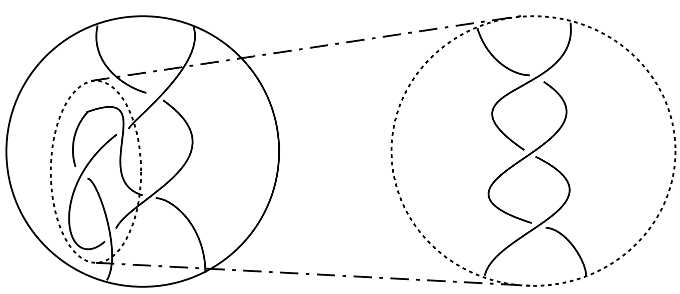


Figure 2.12

The effect of the star move on the knot depends on the orientation of the two strands of the starting tangle. There are two possibilities for this orientation. First, the two strands may be oriented in the same direction as in figure 2.13. The dotted line represents the rest of the knot outside the tangle. Starting at 1 and following the orientation of the knot cycles through 1, 2, 3, 4 and back to 1. Figure 2.14 shows the effect on the knot after the star move. Now, beginning at 1 the new knot cycles through 1, 3, 2, 4 and back to 1. Again, the tangle is shown in solid lines while the rest of the knot is represented by the dotted line. The dotted lines do not change as a result of the move, only the tangle is changed.

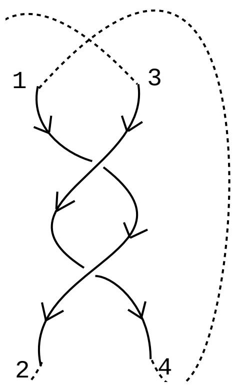
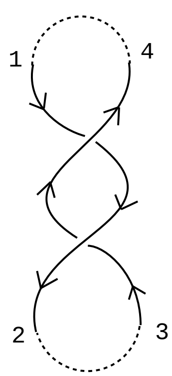


Figure 2.13 Figure 2.14

The second possibility is that the two strands in the original tangle are oriented in opposite directions as in figure 2.15. Starting at 1 in the original tangle the knot cycles through 1, 2, 3, 4 and back to 1. Figure 2.16 shows the effect on the knot after the star move. Now, beginning at 1 the cycle goes from 1 to 4 and back to 1. Beginning at 2 the cycle goes from 2 to 3 and back to 2. Since there are two distinct cycles, the knot has been transformed into a link with 2 components.



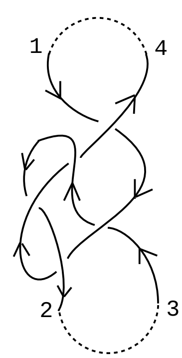


Figure 2.15 Figure 2.16

When the strands in the starting 2-tangle are oriented in the same direction, the strands in the 3-tangle added by the star move are also oriented in the same direction. Thus performing the star move again on the added portion does not add any components to the knot.

When the strands in the starting 2-tangle are oriented in opposite directions, the strands in the 3-tangle added by the star move are also oriented in opposite directions. Performing the star move once formed a 2 component link. Performing the star move again reconnects the 2 components to form a 1 component knot again. Thus, performing the star move an odd number of times starting with a 2-tangle with strands oriented in opposite directions forms a 2 component link. Performing the star move an even number of times forms a 1 component knot.

3 Constructing a 10 Stick 11 Crossing Knot

To construct the 10 stick 11 crossing knot, we begin with the knot 8_{19}, an 8 crossing knot which can be made with 8 sticks. A smooth curve representation of knot 8_{19} is shown in figure 3.1. The stick representation of knot 8_{19} is shown in figure 3.2.

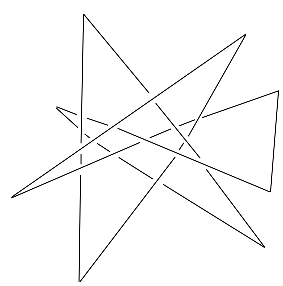
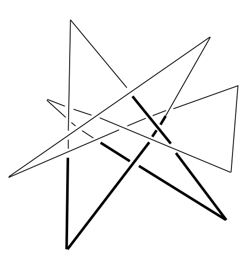


Figure 3.1 Figure 3.2

The star move can be performed on the two highlighted bottom vertices in figure 3.3 to obtain the knot in figure 3.4. Since the star move adds 2 sticks and 3 crossing to the original 8 stick 8 crossing knot, we now have a 10 stick 11 crossing knot.



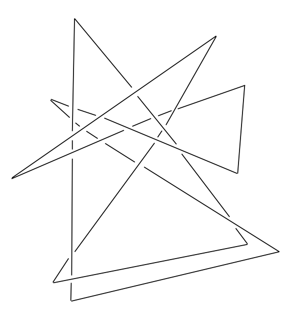


Figure 3.3 Figure 3.4

The stick knot can be plotted with the vertex coordinates
(3.200000, 0.000000, 2.400000)
(-0.300000, -2.817136, 1.476684)
(-1.006262, 4.419703, 1.476684)
(1.600000, -1.600000, -1.800000)
(-0.500000, -3.000000, -2.300000)
(2.420866, 0.497467, 1.677199)
(-2.418078, 0.515212, -1.674017)
(0.498950, -2.429479, 1.666458)
(0.513813, 2.426803, -1.664600)
(-2.427540, -0.506284, 1.664913).

To verify that this knot actually has 11 crossings, the Dowker code of the knot is found and input into knotscape. A smooth diagram of the knot is made and the crossings are numbered and marked appropriately (figure 3.5).

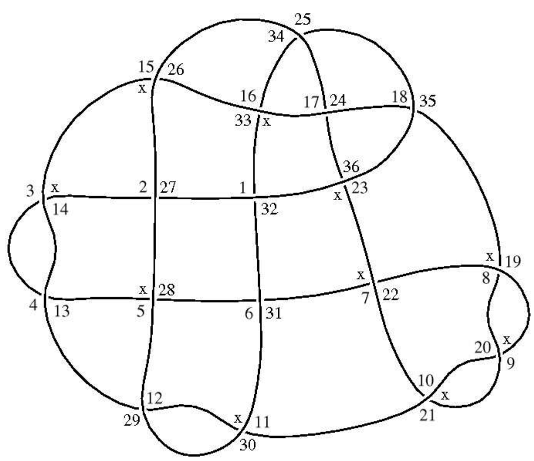


Figure 3.5

1	3	5	7	9	11	13	15	17	19	21	23	25
32	−14	−28	−22	−20	−30	4	−26	24	−8	−10	−36	34

27	29	31	33	35
2	12	6	−16	18

Knotscape identifies this knot as 11n70, an 11 crossing non-alternating knot indexed 70 in the knotscape knot table so using the star move on an 8 stick 8 crossing knot, a 10 stick 11 crossing knot has been constructed.

4 Upperbound For Stick Number of Knots

Since we now have a geometric move which adds 2 sticks and 3 crossings to certain knots and can be iterated, by starting with a knot with a small number of sticks which admits the star move, we can obtain an upperbound for the stick number of certain knots which approaches 2/3 the crossing number of the knot. The smallest knot which admits a star move is the trefoil. A smooth representation and stick representation which admits the star move are shown in figure 4.1. This knot has 6 sticks and 3 crossings.

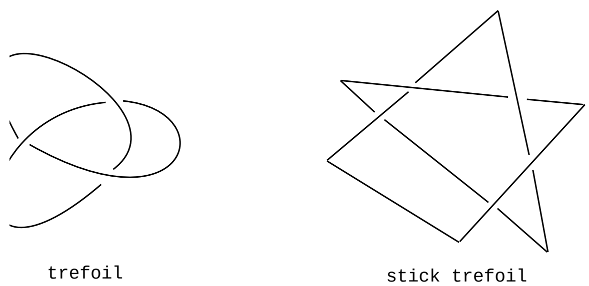


Figure 4.1

By performing the star move on the two left vertices of the stick trefoil in figure 4.1, the knot in figure 4.2 can be constructed which has 8 sticks and 6 crossings.

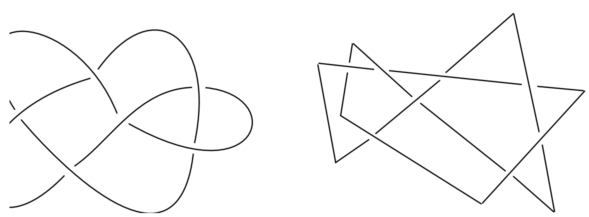


Figure 4.2

Again, the star move can be performed on two of the new vertices to construct the knot in figure 4.3 which has 10 sticks and 9 crossings.

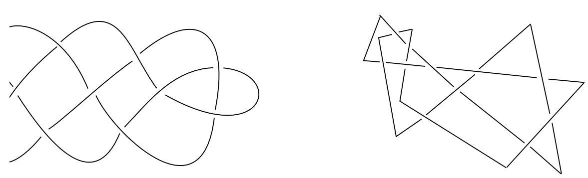


Figure 4.3

Let s(K) be the stick number of a knot K, and c(K) be the crossing number of K. Then for knots created by repeatedly performing the star move on a base knot, $s(K) \leq \frac{2}{3}c(K) + a$, where a is some constant. For the type of knots in the figure above, built from the trefoil as a base knot, $s(K) \leq \frac{2}{3}c(K) + 4$.

5 Acknowledgements

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References

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