THE n-ITERATED CLASP MOVE AND TORUS LINKS

STACY NICOLE MILLS

Abstract. The r-iterated clasp move creates two full twists in a torus link, while adding only r sticks to the stick number of the link. An r-iterated clasp move performed on the minimal stick representation of the $T_{r,r}$ torus links results in the minimal stick representation of the $T_{r,3r}$ for r$\geq$2, obtaining s($T_{r,3r}$)=4r. The r-iterated clasp move may also be performed in a more general set up, adding 2r-1 sticks and two full twists.

1. Introduction

A *stick link l* is a link composed of vertices connected by straight edges. The *stick number* of a link *s(l)* is the least amount of sticks necessary to create the stick link representation of l. A *torus link* $T_{r,s}$ is a link that can be arranged such that it sits on a torus, without crossing itself. The link $T_{r,s}$ crosses the meridian of the torus r times and the longitude of the torus s times. See Figure 1a and [1]. The number of components in any torus link is the greatest common divisor of r and s.[2]

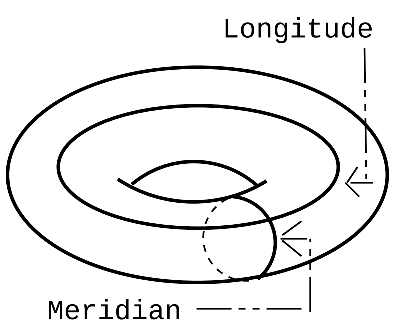


Figure 1a

Work has previously been completed on torus knots and links. Jin proved that if 2$\leq$r<s and if r does not divide s, leaving less than r components, then s($T_{r,s}$)$\leq$2s. Also, if 2$\leq$r<s<2r, then s($T_{r,s}$)=2s. If r does divide s, then there are r components. Each component must have at least three edges, and Jin composed a method in [2] for constructing the $T_{r,r}$ torus link such that it only requires three edges per component. So for r$\geq$1, s($T_{r,r}$)=3r. Figure 1b shows $T_{3,3}$ and Figure 1c shows the minimal stick representation for $T_{3,3}$. Jin goes on to prove that for r$\geq$1, s($T_{r,2r}$)=4r-1 by using r-1 quadrilaterals and one triangle as the components in the construction algorithm for the link.[2] This paper proves that s($T_{r,3r}$)=4r by adding r more sticks to $T_{r,r}$ using a geometric operation called the clasp move at one specific vertex on each component of the link. This paper also proves that by performing the *r*-iterated clasp move on a torus link with a more general stick arrangement, two full twists can be added using only 2r-1 sticks.

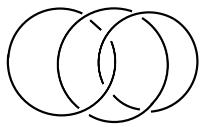


Figure 1b

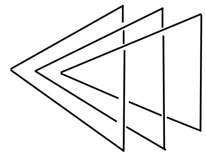


Figure 1c

2. The Clasp Move

The *clasp move* [3] is a technique that will add to a knot two crossings using one stick, creating a two tangle. To perform this move, the correct stick orientation is necessary. Three edges are needed set up as in Figure 2a. The edges E_1 and E_2 are extended so that one edge will cross under E_3, and the other edge will cross over E_3. This is shown in Figure 2b. There is a choice of how to resolve the intersection point of E_1 and E_2. For this paper, the point of intersection will always be resolved with the negative sloped edge, E_2, crossing over the edge with the positive slope, E_1. E_1 and E_2 are connected by a new vertical edge, so they clasp around E_3. The clasp move in Figure 2c is isotopic to the tangle in Figure 2d.

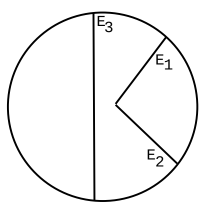


Figure 2a

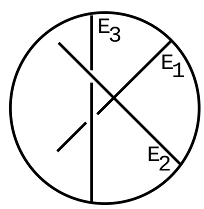


Figure 2b

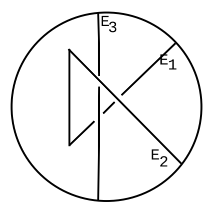


Figure 2c

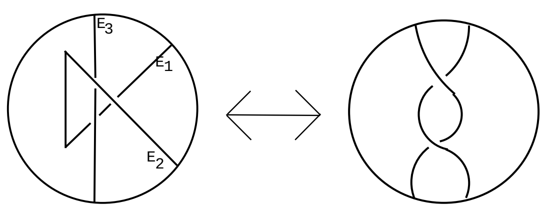
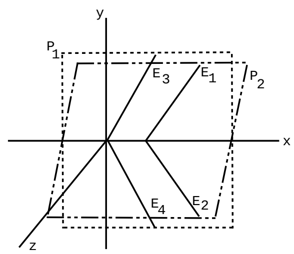


Figure 2d: Isotopic equivalence of tangles.

3. The Iterated Clasp Move

The *iterated clasp move* [3] is the clasp move performed twice on a slightly different stick set up. Two sticks and four crossings, creating a four tangle, will be added to the knot. Four edges and two vertices are needed, where the vertices lie on a line. Each angle defines a plane. Each plane contains the common line of the vertices, but the second plane is slightly rotated about that line. See Figure 3a and 3b.



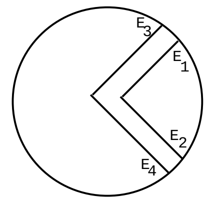
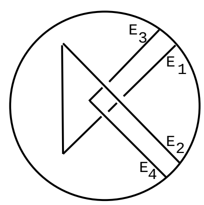


Figure 3a: Planes. Figure 3b: Initial stick set up.

Starting at the vertex formed by E_1 and E_2, perform the clasp move. E_1 and E_2 will clasp around the angle created by E_3 and E_4. Then connect E_1 and E_2 by a new edge. The new crossing created by E_1 and E_2 will be resolved as stated before, with E_2 crossing over E_1. See Figure 3c. Now the sticks are in the correct orientation to perform the clasp move using E_3, E_4, and the new edge. The new crossing created by E_3 and E_4 will be resolved with E_4 crossing over E_3. See Figure 3d.



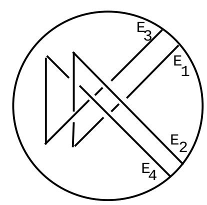
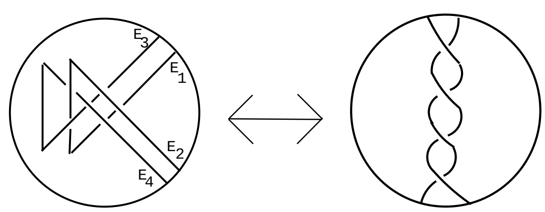


Figure 3c: One clasp move. Figure 3d: Two clasp moves.

After the second clasp move is performed, it becomes clear that a four tangle has been added to the knot or link. See Figure 3e.



Firgure 3e: Topological equivalence of the iterated clasp move.

4. The n-iterated Clasp Move

The n-iterated clasp move may be performed on many different stick arrangements. For simplicity, I will describe a specific arrangement. Let the x-axis be a line which contains n vertices, v_1,...,v_n starting at x=0 and labeling right. Let v_i=($\frac{i-1}{n}$,0,0) be the position of each vertex on the x-axis. Each of these vertices has two edges attached to it. These two edges create a plane, which we assume also contains the x-axis. Let p_1,...,p_n be the planes defined by these vertices and their adjoining edges. Let the first plane p_1 be the xy-plane with v_1=(0,0,0). The plane p_i should be rotated by an angle of $\frac{\pi}{4n}$ about the x-axis from p_{i-1}. The angle created at v_1 is translated and rotated to create v_i, so every angle with the vertex on the x-axis is congruent.

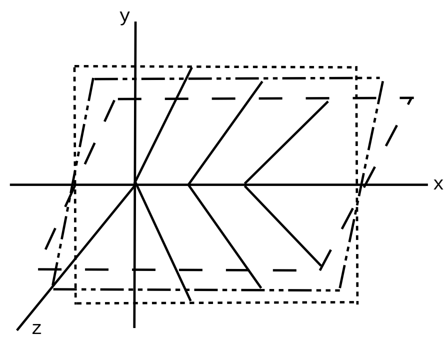


Figure 4a: Rotating angles and the planes that contain them.

Next, starting with the right most vertex on the x-axis, v_n, perform the clasp move. The edges should be extended until they reach x=$\frac{-1}{n}$. Then the vertical edge x=$\frac{-1}{n}$ should connect the endpoints of these two edges. Since the planes are rotating at $\frac{\pi}{4n}$, the extended portion of the negative sloping edge will cross over every positive sloping edge in its path. Following the same idea, the positive sloping edges will only cross under negative sloping edges. See Figure 4b. The rest of the vertices will follow the same procedure. The clasp move will be performed at each vertex, starting with the rightmost vertex each time. The edges adjoining at that vertex will be extended to x=($\frac{i-1}{n}$-1)=($\frac{i-1-n}{n}$). The actual clasping edge will start at the open end of the negative sloping edge and extend down, perpendicular to the x-axis, crossing over every negative sloping edge of every other component. After it crosses the x-axis at ($\frac{i-1-n}{n}$), this new edge will cross under every positive sloping edge until it meets the positive sloping edge that is part of the same component. The result is pictured in Figure 4c.

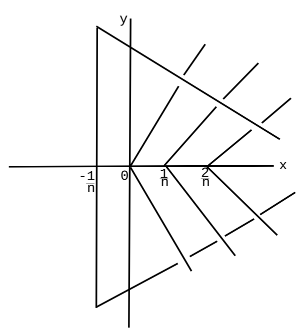


Figure 4b: First added edge after one clasp move.

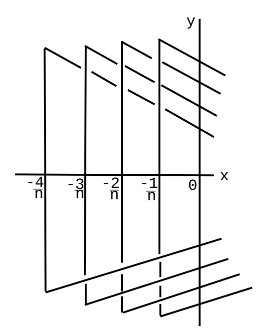


Figure 4c: Four added edges after four clasp moves.

Figure 4d is an example of the $T_{3,3}$. After three clasp moves are performed, the resulting link is the $T_{3,9}$ as shown in Figure 4e.

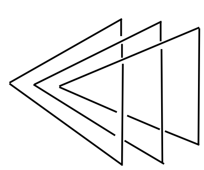


Figure 4d

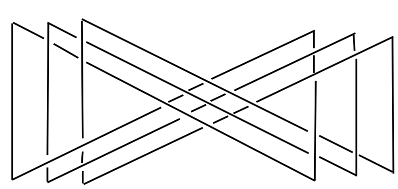


Figure 4e

5. Two Full Twists in the Band

Lemma 5.1. *The n-iterated clasp is equivalent to two full twists.*

Proof. The iterated clasp move creates two full twists with two strands. Shown below are the geometric and corresponding isomorphic representations of the beginning and ending stages of the iterated clasp move.

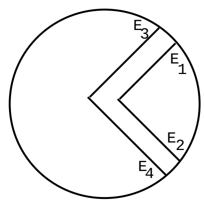


Figure 5a

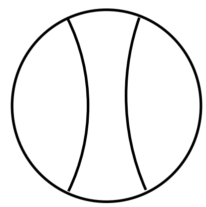


Figure 5b

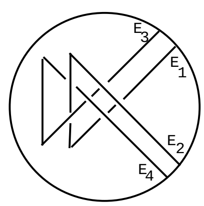


Figure 5c

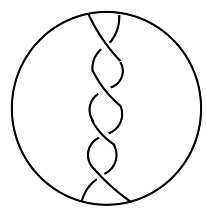


Figure 5d

There is an obvious band in Figure 5d between the two strands. Since Figure 5c and 5d are isotopic, there must be a band running between the strands in Figure 5c. Let the strand containing the edge E_3 be strand 1 and the strand containing E_1 be strand 2. To see the band in Figure 5c, imagine an arc between the strands 1 and 2. One endpoint of the arc is attached to strand 1, where the edge E_3 is labelled, and the other endpoint is attached to strand 2, where E_1 is labelled. The arc can slide along the strands down the positive sloping edges, then up the vertical edges and back down the negative sloping edges. The band flows in the space between the strands where the arc runs. See Figure 5e.

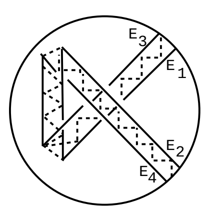


Figure 5e

The n-iterated clasp move preserves the band and creates two full twists in the band. The initial angles are composed such that the band will be preserved and two full twists will be created with any number of strands used. After the n clasp moves are performed, the strands remain in their initial order throughout the created tangle.

Let the exterior strands be the strands that start off passing through v_1 and v_n. Let the interior strands be those that start off passing through every other vertex from v_2 to v_{n-1}. Again to visualize the band, the arc's endpoints will lie on the exterior strands. Instead of being a perfectly smooth arc, now the arc will be composed of line segments from v_{i-1} to v_i, where there are i=2,...n strands. The order of the strands does not change throughout the tangle after the iterated clasp move, so the arc can slide along the tangle without breaking. The exterior strands form the boundaries for the band. Added strands will never intersect any other strand or pierce the band. Thus, the band will remain unaffected.

To generalize pictorially, let the center strand in Figure 5e in section 4 represent n-2 strands. The band may be thought of as a twisted piece of ribbon with the ends connected. The strands are drawn on the ribbon in parallel lines. When the ribbon is twisted, the lines stay skewed, and their order never changes. □

6. Clasp Move on a $T_{r,r}$

Theorem 6.1. *If $r \geq 2$, $s(T_{r,3r})=4r$.*

Proof. An r-iterated clasp move will be performed on a minimum stick link $T_{r,r}$ to result in $T_{r,3r}$. A slightly modified version of the construction Jin uses to create $T_{r,r}$ [2] is required for the proof and is included for convenience.

Let (0,0,0), (1,1,0), and (1,-1,0) be the vertices of the triangle L_1. For each i=2,...,r the triangle of L_i should be rotated $\frac{\pi}{4r}$ around the x-axis from L_{i-1}. Then L_i should be translated $\frac{1}{r}$ units in the positive direction on the x-axis from L_{i-1}. The union of all L_i triangles creates the (r,r) link. Thus, there will be r vertices on the x-axis at a distance of $\frac{1}{r}$ units apart. The r-iterated clasp move will be performed using these vertices and their adjacent edges.

Starting with the vertex on the far right v_r and continuing left, perform the clasp move on each vertex as described in Section 4. By Lemma 5.1, this will add two full twists to $T_{r,r}$ to create $T_{r,3r}$ with the addition of only r more sticks.

Now if s($T_{r,3r}$)<4r, then at least one component must be a triangle. The linking number for a triangle and a quadrilateral would need to be three, which is the common linking number of any two components of the $T_{r,3r}$. However, the linking number for a triangle and a quadrilateral is at most two. Therefore, the stick number of $T_{r,3r}$ is equal to 4r. □

7. Generalized Iterated Clasp

In Section 2, the stick set up for the clasp move began with two angles and two vertices. It is possible to start with only two edges and two vertices. A few adjustments to the link can create the stick orientation as previously described. In this section, I will describe the more general setting where the n-iterated clasp move may be performed in torus links. With the correct stick set up, only 2r-1 sticks are needed to add two full twists.

Theorem 7.1. *The iterated clasp move may be performed in a more generalized set up on a stick representation of a link, adding two full twists to the band and only 2r-1 sticks.*

Proof. The plane p_0 is defined by two edges m_1 and m_2, connected at the vertex v_0. The value of the angle α created at v_0 is less than π. Define the line k in p_0 such that it bisects the angle α. See Figure 7a. Define the plane p_k as the plane perpendicular to k. Now let e_1,..., e_n be rotating edges that pierce the plane p_0. Let a_i and b_i be the two endpoints on each edge e_i such that a_i lies above the plane p_0 and b_i lies below p_0, with i= 1,..., n. Let each edge e_i intersect p_0 at the point v_i. Beginning with e_1, project each edge onto p_k. The projection of the edges from e_1 to e_n must rotate in one direction, so that e_{i+1} is rotated more than e_i. However, the angle of rotation does not need to be constant.

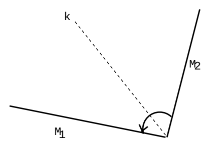


Figure 7a

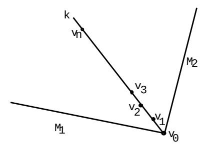


Figure 7b

Now, let each edge e_i be split into two, one edge from a_i to v_i $\overline{a_iv_i}$ which is the edge above p_0, and another edge from v_i to b_i $\overline{v_ib_i}$ which is the edge below p_0. This will add r-1 sticks to the stick number. Starting with v_1, slide each v_i along the plane p_0 to the line k until it is an epsilon distance from v_{i-1} as in Figure 7b. The new angles created by $\overline{a_iv_i}$ and $\overline{v_ib_i}$ at each v_i define planes, p_i. The plane p_i also contains the edge e_i, because the endpoints a_i and b_i did not move. Since the projection of each edge e_i onto p_k is rotating, then the new projection of the angle created by $\overline{a_iv_i}$ and $\overline{v_ib_i}$ is also rotating. This rotation of angles creates a conducive environment for the iterated clasp move to be performed, using the vertices on line k.

We will assume that $\overline{a_nv_n}$ is rotated the furthest clockwise when projected into p_k. If the projected edges are rotating counterclockwise, the crossings later defined would simply be reversed. The projection of the edges onto p_k will determine how the crossings are resolved. Starting at v_n, extend the edges $\overline{a_nv_n}$ and $\overline{v_nb_n}$ past v_n. Since $\overrightarrow{v_nb_n}$ is rotated at the greatest angle, it will cross under every edge before it pierces p_0 at v_n; then it will cross over every edge in its path. Similarly, starting at a_n, $\overrightarrow{a_nv_n}$ will cross over every edge until it reaches v_n, since it is rotated at the greatest angle. It then extends below p_0, so the extended portion of $\overrightarrow{a_nv_n}$ will cross under every edge beginning with the edge $\overrightarrow{v_nb_n}$, connecting then to c_n. For the remaining clasp moves, the edge $\overrightarrow{a_iv_i}$ is rotated further clockwise than $\overrightarrow{a_mv_m}$, where m<i, so $\overrightarrow{a_iv_i}$ will cross under every other edge once it pierces the plane. Similarly, the edge $\overrightarrow{v_ib_i}$ is rotated further clockwise than $\overrightarrow{v_mb_m}$, where m<i, so $\overrightarrow{v_ib_i}$ will cross over every other edge once it pierces the plane. The subscripts of the clasping edges c_i will correspond to the subscripts of the edges connected to them. Each c_i will connect $\overrightarrow{a_iv_i}$ to $\overrightarrow{v_ib_i}$ as before. See Figure 7c. The r clasping edges will add r to the stick number. In all, (r-1)+r=2r-1 sticks will be added to the stick number of the original torus link. Following Lemma 5.1, the r-iterated clasp move will add two full twists to the band.

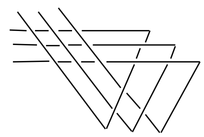


Figure 7c

□

Corollary 1. *The upperbound for the stick number after n-iterated clasp moves are performed on* $T_{r,r}$ *is* $s(T_{r,(2n+1)r}) \leq 4r+(2r+1)(n-1)$ *for* $n \in \mathbb{N}$ *and* $r>1$.

To perform the n-iterated clasp move, there must be n vertices in a line k. The edges attached to each vertex must be rotating in one direction when projected onto the plane p_k perpendicular to k. These edges are then extended, clasped together, and the crossings at each vertex resolved. However, before the crossings are resolved, clasping edges lie on the same rotating planes as the extended edges they clasp together. If each crossing is resolved in the same way, the clasping edge and the adjacent edge that was chosen to cross over at the intersection point create planes that rotate in the same manner that the original planes did.

The stick arrangement after the n-iterated clasp move is performed fulfills the requirements for another n-iterated clasp move to be performed. It is the same set up as first described in Section 7 before the clasp move is performed. Following the method in Section 7, the n-iterated clasp move may be again performed, adding 2n-1 sticks and two full twists. Thus, if one iterated clasp move can be performed as described, then many more iterated clasp moves can be performed using the clasping edges just added. Since 2n-1 sticks and two full twists will be added each time, $s(T_{r,(2n+1)r}) \leq 4r+(2r+1)(n-1)$ for $n \in \mathbb{N}$ and $r>1$.

8. Acknowledgements

Thank you Dr. Joseph Chavez and Dr. Rolland Trapp for your support and dedication to this REU program. Thank you Dr. Trapp, my mentor, for working very hard with me during this program and guiding me through this project. This has been a great learning experience. This work was completed during the 2003 REU in Mathematics at California State University in San Bernardino, California, jointly sponsored by CSUSB and NSF-REU Grant DMS-0139426.

References

[1] C. Adams, *The Knot Book*, W. H. Freeman Co., New York, 1994.

[2] G. T. Jin, *Polygon Indices and Superbridge Indices of Torus Knots and Links*, J. Knot Theory and its Ramifications, Vol. 6, No. 2 (1997) pp. 281-289.

[3] A. O'Connor, B. Podlesny, N. Soriano, R. Trapp, D. Wall. *Clasp Moves and Stick Number*, 2002. Preprint.