The Stick Number of $T_{r,4r}$

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Abstract

This article will show that the stick number of a $T_{2,8}$ link is ten by showing that it cannot be made by nine or less sticks. Then I will show that it can be made by 10 sticks. Extending result shows that $s(T_{r,4r}) = 5r$.

1 Introduction

1.1 Torus links

In this project we are looking at the link known as $T_{2,8}$, a torus link formed by two strands and 8 crossings between them. Torus knots are knots that possesses the special property that they can be wrapped around a torus without allowing any intersections upon the surface. The pictorial depiction is easy to produce since both the components are unknots. An example of a torus know can be seen in Figure **??**.

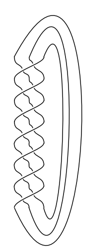

Figure 1: A sample torus knot, this one happens to be $T_{3,7}$.

1.2 Linking number

One idea that helps to classify knots easily is the idea of linking number or self linking number. The linking number of a link is a way to express how "linked the components of the link are." Often times the linking number can be thought of as the amount of times that one component of a link intersects the surface of the other component bounded in space. When counting the linking number, one has to remember that the knot is oriented and that the sign of each intersection is equal to the signs of the crossings caused by the intersection. Each time the surface is insected (assuming that it has the same sign as the rest of the breaks), two crossings are added to the link. So when dealing with torus links, we know that in a $T_{2,2k}$ one component will intersect the surface traced by the other component exactly k times. By using this idea with a mixture of geometrical and topological theorems one can analyze torus links and help with the construction of the minimum stick representation of the links. An example of a link with linking number 2 can be seen in Figure **??**.

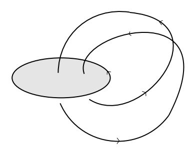

Figure 2: One component intersects the surface (shaded in) of the other comonent twice in the same direction thus the linking number of these two components is two.

1.3 Braids

The next idea that also helps in understanding torus links is the idea of braiding. A braid is an intertwining of some number of strings such that there exists a point at which the orientation fo the braid is constant (i.e. counter clockwise around the point). In space, this point is actually a line perpendicular to the plane and is refered to as the axis of the braid. A braid can be described as an $n-braid$ where n is the amount of loops going around the center. Notice that a single strand could go around the braid 4 times which would count as 4 loops not just one. Figure **??** is considered a 4−braid even though it has only two components. This is a result of four different loops going around the x, which is its axis.

A braid can be described by a group of braids generators. If you number the strands starting with the strand furthest from the axis to the closest, any crossing will occur between two adjacent strands. So we define a crossing σ_k as the positively oriented crossing between the k and $k+1$ strand. $\overline{\sigma_k}$ would be the negatively oriented crossing of the same two strands. If we look at Figure **??** again, we can see that the *braid word*, or the order of the generators that yields that braid, is $\{\sigma_1\sigma_3\sigma_1\sigma_2\overline{\sigma_1\sigma_3}\sigma_2\sigma_2\sigma_1\sigma_3\}$. As with all groups, the group of generators is not without certain identities that help us to simplify a braid word. For instance, in a 3−braid there are two sigma generators that can be used $\sigma_1$and σ_2. As it turns out in this group, the only identity that can be made is the identity that says $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Naturally as the group gets bigger

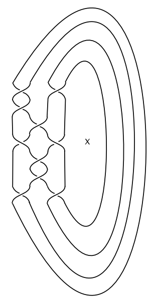

Figure 3: A two component $4 - braid$.

so does the amount of identities.

2 Why Not Nine

Since $T_{2,8}$ is a torus link we immediately know that it must be made by two polygons with sides adding up to nine. One possibility would be for one of the polygons to be a triangle and the other a hexagon. The other possibility would be for one to be a quadrilateral and the other a pentagon. I show, however, that these two different arrangements will not be able to yield a $T_{2,8}$ link.

2.1 The Triangle and the Hexagon

This part of the proof is mostly a geometric argument. If we think of the triangle as a surface and the hexagon as a curve breaking the surface we know that the hexagon has to break the surface 4 times. Since a triangle is a planar object we know that each break must be made by one stick. This means we have to use four separate sticks to make the necessary break. The result is a shape similar to Figure **??**. As one can, see the four sticks already in place need at least four more sticks to be able to connect the tops of the oriented sticks going through the surface, to the bottom of them (see Figure **??**). Since four more sticks would require at least 11 sticks, we can conclude that the $T_{2,8}$ link cannot be made with a triangle and a hexagon.

2.2 The Quadrilateral and the Pentagon

The fact that the quadralateral can outline more than one plane makes the arguement against it more complex. The first thing one has to notice is that the quadrilateral can never be oriented perfectly flat once the link has been made. If this were to happen an arguement link there was in section **??** could be used to prove its inefficiency. As a result one can conclude that the quadrilateral must be bent if there will be a way to make a 10 stick, 8 crossing knot (see Figure **??**).

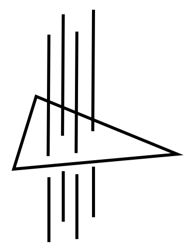

Figure 4: Four different sticks are required to pierce the surface formed by the triangle four times.

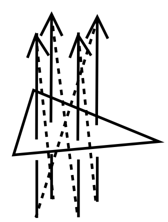

Figure 5: Four more sticks are needed to connect the four sticks (dotted ones) together.

2.2.1 Two in Each Face

After determining that the quadrilateral must be bent we analyze the ways linking with the pentagon can occur. First of all, we must think of how the pentagon must intersect the surface topologically traced by the quadrilateral. Clearly if all four sticks pierce the same plane the figure cannot be made as in the discussion of the triangle and the hexagon. This means at least one line must pierce through the second plane as seen in Figure **??**. However, there is a problem with this set up. Assuming that the stick through the upright plane is connected directly to two of the other sticks, this still requires at least two more sticks to connect the three sticks through the bottom plane. Thus a total of 10 sticks must be used to make $T_{2,8}$. This leaves the only other possibility and when there is two sticks breaking each triangle as seen in Figure **??**.

2.2.2 Proof that $T_{2,8}$ is Not the Knot Produced

The first thing to notice about the setup in Figure **??** is that the connection between the sticks must alternate from top to bottom and vise versa. This is seen in the proof of why three cannot be in the same face. If the two sticks through the same plane are connected, an extra stick is required. This condition would force the sticks through the other plane causing an additional stick to be necessary. This requires the maker to add an extra stick to connect the both

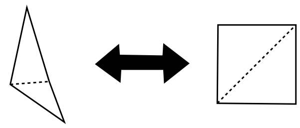

Figure 6: The quadrilateral cannot be flat but instead can be thought of as two triangles defining two planes if you fold the quadrilateral over the diagonal.

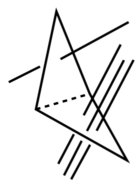

Figure 7: One stick through one three through the other

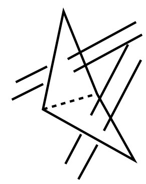

Figure 8: Two sticks go through each plane

pairs of sticks totalling to the addition of two sticks (one for each consecutive pair) and using a total of ten sticks instead of nine. The only knot universe that will work is the one depicted in Figures **??** and **??**.

The only way that a pentagon can cross a quadrilateral 8 times.

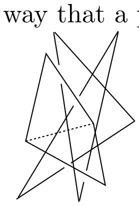

Figure 9: Three dimensional

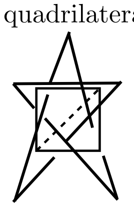

Figure 10: Two dimensional

At this point we can use a linear transformation on the whole link, so that the quadrilateral can be aligned with the x, y, and z axis as shown in Figure **??**.

This allows us to look down the z-axis and view the whole knot from above, causing the quadrilateral to appear to be a triangle and the pentagon to look like a star. Since the pentagon has a star shape it adds 5 self crossings in addition to the 8 crossings between the pentagon and the quadrilateral . The other key aspect of the knot diagram is the fact that it seems to be a $3-$braid. Since $T_{2,8}$ is a two braid, either the link has to be reducible to a $2-$braid or it is not $T_{2,8}$. The only way that a $T_{2,8}$ can appear to be a $3-$braid is for it to appear as is does in Figure **??**.

If one looks at Figure **??** and attempts to write the braid word as a product of braid generators, one will find that (assuming, without loss of generality, that

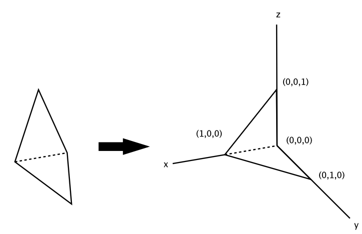

Figure 11: A linear transformation moving the quadrilateral to the center of the axis

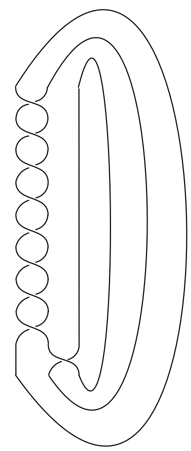

Figure 12: This is a $T_{2,8}$ link represented by a $3 - braid$

all the crossings between the components are positive since if they were negative they would just be a rotated or reflected version of one of the all positive crossing sets) the codeword will be $\{\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_2\}$ or $\{\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1\overline{\sigma_2}\}$ (which is the one in Figure **??**) when written in its simplest terms. That means that if the 9 stick knot universe is going to be $T_{2,8}$ it will have to be simplified to one of these two code words. This tells us that at most one of self crossings can remain after the braid identities are applied. Furthermore, if you observe the crossing order seen in Figure **??**, you will notice that all the crossings can be topologically moved in such a way that it appears like they do in Figure **??**. There is only one set of crossings that would not allow for this change to be preformed. This topic is discussed a bit later.

In most cases the crossings can be moved into the Figure **??**, shape so we should be able to write the braid word for this shape, and it should simplify into the braid word of the $T_{2,8}$. So, once again assuming without loss of generality that the crossings between the two components of the links are all positive and

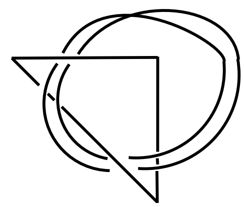

Figure 13: A way the different stick representatives can be thought of after being moved topologically

not assigning a specific value to the self crossing of the pentagon we find that the braid word for Figure **??** is $\{\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\}$ or $\{\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2\overline{\sigma_1}\}$, which when simplified come out to be $\{\sigma_1\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\}$ and $\{\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2\overline{\sigma_1}\}$ respectively. Clearly these two cannot be simplified any more and thus are not $T_{2,8}$.

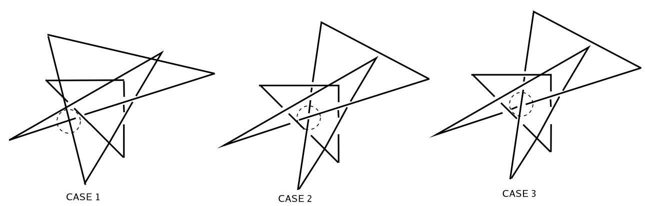

Figure 14: The different cases can easily be seen here

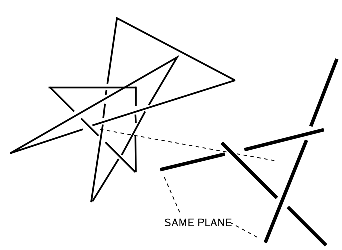

Figure 15: As you can see these two sticks have one end in the same plane in the same plane yet weave around another stick.

The remaining cases are the ones in which Figure **??** cannot be reached. These cases are caused by the fact that one of the crossings cannot be moved outside the loop by standard reidemeister moves. If you look at this specific crossing in Figure **??**, you can see that the two sticks have endpoints in the same plane and end up crossing another stick and then each other in a way that clearly cannot be done by only those three sticks. Therefore, this stick diagram cannot actually occur in space with only 9 sticks, the stick number of $T_{2,8} > 9$ otherwise $T_{2,8} = 10$ $\square$.

3 Stick Number of $T_{r,4r}$

With the stick number of $T_{2,8}$ established to be 10, the question arises: is there a generic way to write the stick number of $T_{r,4r}$? Initially it seems that the stick number should be 5r from the given example. To prove this, we have to look at a generic construction of $T_{r,4r}$ by building upon the known minimum stick representation of $T_{2,8}$. One way to make $T_{2,8}$ with 10 sticks is to connect two pentagons as can be seen below in Figure **??**. This 10 stick representation of $T_{2,8}$ can be made by adding two points along a line containing each vertex as can be seen in Figure **??**. If you label each pair of points as one and two in an alternating fashion and connect each 1 to the 2's adjacent to it in teh core you will yield $T_{2,9}$ by simply changing one of the line sets so that instead of $1 \rightarrow 2$ you can make $1 \rightarrow 1$ so that takes out one crossing ending in $T_{2,8}$.

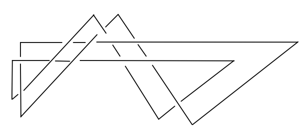

Figure 16: This 10 stick rep of $T_{2,8}$

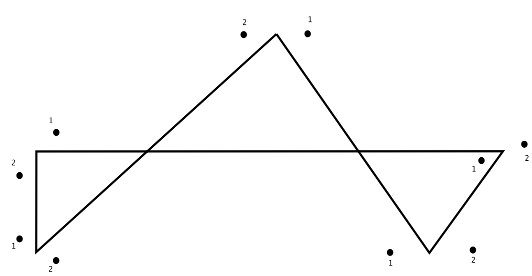

Figure 17: If you lable each consecutive point as 1 or 2 and then connect the ups to the downs along the core line you will get $T_{2,9}$. A small twist fixes the problem and makes it $T_{2,8}$

3.1 5n is an Upper Bound

If this construction is repeated with r points added along the line (numbering each from 1,2,3,...,r with the point), and the points are connected such a way that $\{1, 2, 3, ..., r\} \rightarrow \{r, ..., 3, 2, 1\}$. Similar to the construction of the $T_{2,8}$ one of the three sides has to be a $\{1, 2, 3, ..., r\} \rightarrow \{1, 2, 3, ..., r\}$. By reducing the space between the dots this construction can be done for an infinitely large r. This construction will yield a $T_{r,4r}$ with $5r$ sticks. This gives us an upper bound for $s(T_{r,4r})$. An example of this construction when $r = 3$ can be seen in Figure **??**.

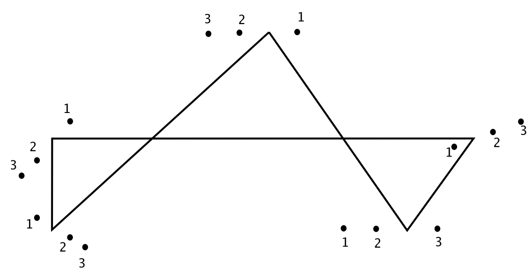

Figure 18: Here is a r=3 construction

3.2 Proof 5n is a Lower Bound

There are several ways to make $T_{r,4r}$ with less than $5n$ sticks. Notice that the construction described above adds a pentagon from each $T_{r-1,4(r-1)}$ to $T_{r,4r}$. One way would be if there was a point where a quadralateral or triangle could be added to the construction described above at some point to go from $T_{r-1,4(r-1)}$ to $T_{r,4r}$. This cannot happen because in a $T_{r,4r}$ any two components can form a $T_{2,8}$. If either a triangle or a quadralateral could be added and form $T_{r,4r}$, there would be two components of the link that formed a $T_{2,8}$ with less than 10 sticks. Since we know the $s(T_{2,8}) = 10$ we know that this cannot happen. This would be true of all 5-5 (5 sticks in the first component, 5 sticks in the 2nd component) stick constructions so no 5-5 stick constructions can improve on this number.
That means the only way that a lower value can be found for $T_{r,4r}$ would be if there was a 6-4 or 7-3 construction that would yield a smaller quantity that 5n. We can immediately throw out the 7-3 constructions because, as can be seen in section **??**, at least 11 sticks are necessary to make $T_{2,8}$ with a triangle as a component. That leaves only the 6-4 construction to consider. The 6-4 construction will not work because of a argument similar to why the 5-5 construction does not work. For a 6-4 construction a hexagon and a quadralateral form $T_{2,8}$. Quadralaterals cannot be added since that would mean that two quadralaterals would be forming a $T_{2,8}$. Pentagons cannot be added since that would be implying that a pentagon and a quadralateral are making a $T_{2,8}$. So that means that a hexagon must be added each time yielding $6r - 2$ sticks for a $T_{r,4r}$. $6r - 2 > 5r$ so there is not a way to make $T_{r,4r}$ with less than $5r$ sticks, therefore 5r is also a lower bound. Hence, $s(T_{r,4r}) = 5r$. $\square$

Conclusion

There are several new conjectures that can be drawn from the results above. First of all using the previously found results that $s(T_{r,r}) = 3r$ and $s(T_{r,2r}) = 4r - 1$ [**?**] also that $s(T_{r,3r}) = 4r$ [**?**], a pattern seems to emerge. First of all $s(T_{r,kr})$ appears to be $\lceil \frac{s(T_{2,2k})}{2} \rceil (r-2) + s(T_{2,2k})$. You can arrive at this conclusion logically from the idea that $\lceil \frac{s(T_{2,2k})}{2} \rceil$ represents the component made of more sticks when the two component link is made so that the two components as close to the same amount of sticks as possible. In the case of $T_{2,8}$ this occurs when each component is made of 5 sticks. The reason for the $n-2$ factor next to it accounts for the fact that its that component that will be repeated to continue making the knot. The stick number added to that adds in the contribution of the original two component link. This conjecture would be more or less easy to prove, all that needs to really be shown is that if a two component torus link can be made with n sticks then it can be made with one component made with $\lceil \frac{n}{2} \rceil$ sticks and one of$\lfloor \frac{n}{2} \rfloor$ sticks. With that proven, a similar but more general version of the proof found in section **??** should be able to prove this conjecture. The next idea that seems to fall out of the results above is that $s(T_{2,2k}) < s(T_{2,2k+2})$. To prove this it has to be shown that $s(T_{2,2k})$ can be made with at least one less stick than $s(T_{2,2k+2})$. If this proves true a few more stick numbers would quickly be confirmed and the monotonic tendencies of the stick number would exist. This would lead to my last and final conjecture. I believe that

$$s(T_{2,2k}) = \begin{cases} \frac{4}{3}k + \frac{14}{3} & \text{when } 3|k+1 \\ \frac{4}{3}k + \frac{13}{3} & \text{when } 3|k+2 \\ \frac{4}{3}k + 4 & \text{when } 3|k \end{cases} = \lfloor \frac{4}{3}k + \frac{14}{3} \rfloor$$

from the numbers that I have collected and from the apparent stick numbers that come after it. The results in general might be useful in developing a general formula for the stick number of $T_{p,q}$, which would be quite good.

Acknowledgements

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