Minimum Grid Cutwidth of Complete Bipartite Graphs

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Abstract

We shall determine the minimum grid cutwidth of the complete bipartite graph, $K_{m,n}$, for m and n odd. We will also consider embedding the complete bipartite graph, $K_{m,n}$, in grids with more than two rows.

1 Introduction

A graph $G = (V, E)$ consists of a set of vertices, V, and edges, E, connecting pairs of vertices. A complete bipartite graph, $K_{m,n}$, consists of two disjoint sets of vertices A and B such that every vertex in A is connected by an edge to every vertex in B where $|A| = a$ and $|B| = b$ and no vertices in the same set are joined by an edge. Figure 1 is an example of a complete bipartite graph, $K_{5,3}$.

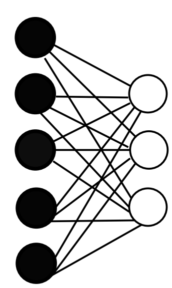

Figure 1: $K_{5,3}$

2 Background

In 1998, Mario Rocha[1] determined the grid cutwidth for any complete bipartite graph, $K_{m,n}$, for m and n even and $m \leq n$. In his paper, Rocha proved that $K_{m,n}$ can be embedded in a 2 x $[\frac{n+m}{2}]$ grid, such that:

$$gcw(K_{m,n}) = \frac{mn}{4}.$$

Rocha also showed that any complete bipartite graph with m even, n odd and n≠3 can be embedded in a 2 x $[\frac{n+m+1}{2}]$ grid, such that:

$$gcw(K_{m,n}) = \begin{cases} \frac{mn}{4} & \text{if } m \text{ is a multiple of 4} \\ \frac{mn+2}{4} & \text{if } m \text{ is not a multiple of 4.} \end{cases}$$

In 2002, Matt Johnson[2] proved that for any complete bipartite graph $K_{m,n}$:

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & mn \text{ even} \\ \frac{mn+1}{2} & mn \text{ odd.} \end{cases}$$

Johnson's vertex formula, which Rocha uses to embed complete bipartite graphs in grids of two rows, states that:

$$\lfloor \frac{xm}{m+n} + 1/2 \rfloor,$$

where given the position x on a linear host, the formula tells you how many vertices of a set A should be placed to the left of x.

According to the Bowles-Chavez-Hartung (BCH)[3] theorem, for any complete bipartite graph $K_{m,n}$ with $m \leq n$, the cut of each region in a linear embedding is minimized by placing $\frac{2x+m-n}{4}$ black vertices from A to the left of x.

Additionally, in 2002 Bradley Marchand[4] showed that the grid cutwidth of a complete graph embedded in a m x n grid where $m = n$ and m and n even is:

$$gcw(K_{mxn}) = \begin{cases} \frac{mn^2}{4} & m \text{ a multiple of 4} \\ \frac{mn^2}{4} + 1 & m \text{ not a multiple of 4.} \end{cases}$$

3 Grid Cutwidth of any Complete Bipartite Graph: The Odd Cases

Theorem 1: For m and n odd, $m \leq n$, and $m + n > 6$ any complete bipartite graph, $K_{m,n}$, can be embedded in a 2 x $[\frac{n+m}{2}]$ grid, such that:

$$gcw(K_{m,n}) = \begin{cases} \frac{mn-1}{4} & m = n \\ \frac{mn+1}{4} & m+n \text{ is a multiple of 4} \\ \frac{mn-1}{4} & m+n \text{ is not a multiple of 4.} \end{cases}$$

Proof

By discovering the minimum grid cutwidth of $K_{m,n}$ for the odd case, the general problem of finding the grid cutwidth of any complete bipartite graph embedded in two rows will be solved. Each proof will consist in showing that the lower bound for the grid cutwidth will correspond to the upper bound. Unlike Rocha, we will embed any complete bipartite graph around the centerline in an alternating and symmetric fashion using the BCH[3] formula, with the bottom row always having one more black vertex than the top.

3.1 For $\mathbf{m = n}$

We can obtain a lower bound for the hcut at the centerline, where the hcut is greatest, by using the cutwidth counting technique. Connecting the black vertices on the left side of the grid with the white vertices on the right, and vice versa, we get the hcut by dividing by two, the number of rows.

$$\begin{aligned} hcut &= \frac{[(\frac{m-1}{2})(\frac{n+1}{2}) + (\frac{m+1}{2})(\frac{n-1}{2})]}{2} \\ &= \frac{mn-1}{4}. \end{aligned}$$

We now must show that the lower bound for the grid cutwidth cannot be less than $\frac{mn-1}{4}$. Suppose we interchange the black vertices on the left with the white vertices on the right. Let i represent the number of these switches about the vertical line. Using the cutwidth counting technique, we can determine the hcut about the vertical line, noting that with each switch there will be more white and less black vertices on the left side.

$$\begin{aligned} hcut &= \frac{[(\frac{m-1}{2} - i)(\frac{n+1}{2} - i) + (\frac{m+1}{2} + i)(\frac{n-1}{2} + i)]}{2} \\ &= \frac{mn-1}{4} + i^2. \end{aligned}$$

When $i = 0$, the smallest hcut of $\frac{mn-1}{4}$ occurs.

3.1.1 Case 1: $\mathbf{m = 4k + 1}$

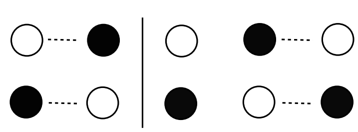

Figure 2: $m = 4k + 1$

We need to show that the upper bound for the grid cutwidth of our embedding process is equal to $\frac{mn-1}{4}$, the lower bound. In our embedding process only hv-edges are used to connect black vertices on one row to white vertices on the other. Calculating the hcut about the vertical line on the second row, the row with more vertices, will not, however, give us the lower bound.

$$\begin{aligned} hcut &= [(\frac{m-1}{4})(\frac{m+1}{2}) + (\frac{\frac{m+1}{2}+1}{2})(\frac{m-1}{2})] \\ &= \frac{m^2-1}{4} + \frac{m-1}{4}. \end{aligned}$$

When connecting the black vertices of the second row on each side of the grid to all the white vertices on the other side, using only hv-edges, we achieve the lower bound plus the term $\frac{m-1}{4}$. This term, which is also k, corresponds to the number of edges that need to run vh instead of hv in order to obtain the minimum grid cutwidth. We need to send this number of edges vh to the outside white vertices from the black vertices farthest from the vertical line. Running only hv-edges between rows, the vcut equals the number of black vertices on the bottom row, the row with more black vertices. Therefore, we can run k edges vh without the vcut exceeding the hcut.

To verify that the upper bound does not exceed $\frac{mn-1}{4}$, we must move the vertical line around. Let i represent the number of black and j the number of white vertices that get switched from one side to the other as a result of the shift, either to the left or right. By shifting the vertical line to the right, for example, we can calculate the hcut at different locations.

$$\begin{aligned} hcut &= [(\frac{m-1}{4}+i)(\frac{m+1}{2}-j) + (\frac{\frac{m+1}{2}+1}{2}-i)(\frac{m-1}{2}+j)] \\ &= \frac{m^2-1}{4} + \frac{m-1}{4} - 2ij + i + j. \end{aligned}$$

When $i = j = 0$, the largest possible hcut is obtained. Thus, after running k edges vh instead of hv, the upper bound for the grid cutwidth must be $\frac{mn-1}{4}$.

3.1.2 Case 2: $\mathbf{m = 4k - 1}$

Again we need to show that the upper bound corresponds to the lower bound.

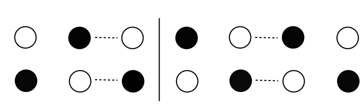

Figure 3: $m = 4k - 1$

$$\begin{aligned} hcut &= [(\frac{m+1}{4})(\frac{m+1}{2}) + (\frac{m+1}{4})(\frac{m-1}{2})] \\ &= \frac{m^2-1}{4} + \frac{m+1}{4}. \end{aligned}$$

$$
\begin{aligned}
hcut &= [(\frac{m+1}{4}+i)(\frac{m+1}{2}-j)+(\frac{m+1}{4}-i)(\frac{m-1}{2}+j)] \\
&= \frac{m^2-1}{4}+\frac{m+1}{4}-2ij+i.
\end{aligned}
$$

When $i=j=0$, the largest possible hcut is obtained. Instead of needing to send $\frac{m-1}{4}$ edges vh, here we need to send $\frac{m+1}{4}$ edges vh in order to achieve the minimum grid cutwidth.

3.2 For $\mathbf{m+n}$ not a multiple of 4

We will show that the lower bound is the same as the upper bound after sending k edges vh, much as we did in the previous section.

$$
\begin{aligned}
hcut &= \frac{[(\frac{m-1}{2})(\frac{n+1}{2})+(\frac{m+1}{2})(\frac{n-1}{2})]}{2} \\
&= \frac{mn-1}{4}.
\end{aligned}
$$

Again, we need to make sure that a smaller lower bound cannot be achieved.

$$
\begin{aligned}
hcut &= \frac{[(\frac{m-1}{2}-i)(\frac{n+1}{2}-i)+(\frac{m+1}{2}+i)(\frac{n-1}{2}+i)]}{2} \\
&= \frac{mn-1}{4}+i^2.
\end{aligned}
$$

When $i=0$, the smallest hcut of $\frac{mn-1}{4}$ occurs.

3.2.1 Case 1: $\mathbf{n=4k-1}$

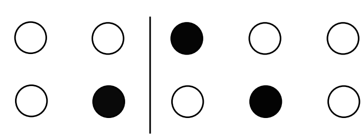

Figure 4: Ex. $K_{3,7}$

$$
\begin{aligned}
hcut &= [(\frac{m+1}{4})(\frac{n+1}{2})+(\frac{m+1}{4})(\frac{n-1}{2})] \\
&= \frac{mn-1}{4}+\frac{n+1}{4}.
\end{aligned}
$$

$$
\begin{aligned}
hcut &= [(\frac{m+1}{4}+i)(\frac{n+1}{2}-j)+(\frac{m+1}{4}-i)(\frac{n-1}{2}+j)] \\
&= \frac{mn-1}{4}+\frac{n+1}{4}-2ij+i.
\end{aligned}
$$

When $i=j=0$, the largest possible hcut is obtained.

3.2.2 Case 2: $\mathbf{n = 4k + 1}$

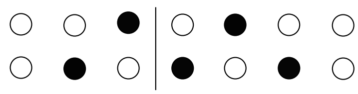

Figure 5: Ex. $K_{5,9}$

$$
\begin{aligned}
hcut &= [(\frac{m-1}{4})(\frac{n+1}{2}) + (\frac{m+3}{4})(\frac{n-1}{2})] \\
&= \frac{mn-1}{4} + \frac{n+3}{4}.
\end{aligned}
$$

$$
\begin{aligned}
hcut &= [(\frac{m-1}{4} + i)(\frac{n+1}{2} - j) + (\frac{m+3}{4} - i)(\frac{n-1}{2} + j)] \\
&= \frac{mn-1}{4} + \frac{n+3}{4} - 2ij + j + i.
\end{aligned}
$$

When $i = j = 0$, the largest possible hcut is obtained. Thus, the lower bound can be achieved.

3.3 When $\mathbf{m + n}$ is a multiple of 4

Again, we will show that the lower bound is the same as the upper bound after sending k edges vh.

$$
\begin{aligned}
hcut &= \frac{[(\frac{m+1}{2})(\frac{n+1}{2}) + (\frac{m-1}{2})(\frac{n-1}{2})]}{2} \\
&= \frac{mn+1}{4}.
\end{aligned}
$$

We must again show that a smaller lower bound cannot be achieved.

$$
\begin{aligned}
hcut &= \frac{[(\frac{m+1}{2} - i)(\frac{n+1}{2} - i) + (\frac{m-1}{2} + i)(\frac{n-1}{2} + i)]}{2} \\
&= \frac{mn+1}{4} + i^2 - i.
\end{aligned}
$$

When $i = 0$, the smallest hcut of $\frac{mn+1}{4}$ occurs.

3.3.1 Case 1: $\mathbf{n = 4k + 1}$

$$
\begin{aligned}
hcut &= [(\frac{m+1}{4})(\frac{n+1}{2}) + (\frac{m+1}{4})(\frac{n-1}{2})] \\
&= \frac{mn+1}{4} + \frac{n-1}{4}.
\end{aligned}
$$

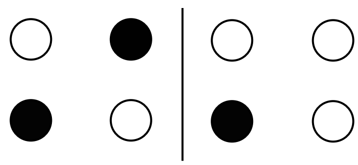

Figure 6: Ex. $K_{3,5}$

$$\begin{aligned} hcut &= [(\frac{m+1}{4}+i)(\frac{n+1}{2}-j)+(\frac{m+1}{4}-i)(\frac{n-1}{2}+j)] \\ &= \frac{mn+1}{4}+\frac{n-1}{4}-2ij+i. \end{aligned}$$

When $i=j=0$, the largest possible hcut is obtained.

3.3.2 Case 2: $\mathbf{n = 4k - 1}$

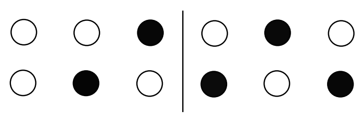

Figure 7: Ex. $K_{5,7}$

$$\begin{aligned} hcut &= [(\frac{m-1}{4})(\frac{n-1}{2})+(\frac{m+3}{4})(\frac{n+1}{2})] \\ &= \frac{mn+1}{4}+\frac{n+1}{4}. \end{aligned}$$

$$\begin{aligned} hcut &= [(\frac{m-1}{4}+i)(\frac{n-1}{2}-j)+(\frac{m+3}{4}-i)(\frac{n+1}{2}+j)] \\ &= \frac{mn+1}{4}+\frac{n+1}{4}-2ij-i+j. \end{aligned}$$

When $i=j=0$, the largest possible hcut is obtained. Thus, the lower bound can, once again, be achieved. So, in each case, the lower bound corresponds to the upper bound. Of note, we could have separated the proof into two cases: m congruent to m mod4 ($\frac{mn-1}{4}$) and m not congruent to m mod4 ($\frac{mn+1}{4}$).

4 Embedding Complete Bipartite Graphs in 3 Rows

Lemma 1: For any nonnegative m and$n$ and $m+n>3$, any complete bipartite graph of the form $K_{3m,3n}$, $K_{3m+1,3n-1}$ or $K_{3m+2,3n-2}$ can be embedded in a grid of 3 rows such that:

$$gcw(K_{3m,3n}) = \begin{cases} \frac{9mn}{6} & m+n \text{ even} \\ \frac{9mn-2}{6} & m+n \text{ odd.} \end{cases}$$

$$gcw(K_{3m+1,3n-1}) = \begin{cases} \frac{9mn+3n-3m-1}{6} & m+n \text{ even} \\ \frac{9mn+3n-3m-3}{6} & m+n \text{ odd.} \end{cases}$$

$$gcw(K_{3m+2,3n-2}) = \begin{cases} \frac{9mn+6n-6m-4}{6} & m+n \text{ even} \\ \frac{9mn+6n-6m-6}{6} & m+n \text{ odd.} \end{cases}$$

Instead of using Johnson's formula to place the black vertices of set A in a grid of three rows, and using it in reverse for the middle row, we will split the blacks nearly evenly between the three rows, with m black vertices in the first and third rows. We will use the BCH[3] formula to arrange the vertices from A in each row. Further, once a lower bound is found, we must verify that it is obtainable.

4.1 $\mathbf{K_{3m,3n}}$

We can find the lower bound for the minimum grid cutwidth of $K_{3m,3n}$ by embedding i, the optimal number of black vertices, on the left side of the grid and the remainder, $3m-i$, on the right. Then, connecting i black vertices on the left to the remaining white vertices on the right and adding the contribution of the $3m-i$ black vertices on the right connecting to the remaining white vertices on the left, we will find the lower bound for our embedding process. Once we obtain the minimizing function, we can take it's derivative and set it equal to zero to find the minimum value for i on the parabola. To determine the optimum grid cutwidth, we can substitute i back into the function to find the minimum total cut. Finally, we must divide the total cut by three to obtain the minimum cut on each row.

4.1.1 For $\mathbf{m+n}$ even

$$\begin{aligned} f_{min} &= i[(3n - 3\frac{m+n}{2} + i] + (3m-i)[3\frac{m+n}{2} - i] \\ &= 2i^2 + i[-6m] + 9m\frac{m+n}{2}. \\ f' &= 4i - 6m = 0. \\ i &= \frac{6m}{4}. \end{aligned}$$

$$\begin{aligned} f_{min}(\frac{6m}{4}) &= 2(\frac{6m}{4})^2 + (\frac{6m}{4})[-6m] + 9m\frac{m+n}{2} \\ &= \frac{9mn}{2}. \end{aligned}$$

4.1.2 For $\mathbf{m+n}$ odd

$$\begin{aligned}
f_{min} &= i[3n - \frac{3(m+n)+3}{2} + i] + (3m-i)[\frac{3(m+n)+3}{2} - i] \\
&= 2i^2 + i[-6m-3] + \frac{9m^2+9mn+9m}{2}. \\
f' &= 4i - 6m - 3 = 0. \\
i &= \frac{6m+3}{4}.
\end{aligned}$$

$$\begin{aligned}
f_{min}(\frac{6m+4}{4}) &= 2(\frac{6m+4}{4})^2 + (\frac{6m+4}{4})[-6m-3] + \frac{9m^2+9mn+9m}{2} \\
&= \frac{9mn-2}{2}.
\end{aligned}$$

4.2 $\mathbf{K_{3m+1,3n-1}}$

We can obtain a lower bound for the minimum grid cutwidth of $K_{3m+1,3n-1}$ just as we did for $K_{3m,3n}$.

4.2.1 For $\mathbf{m+n}$ even

$$\begin{aligned}
f_{min} &= i[3n - 1 - 3\frac{m+n}{2} + i] + (3m+1-i)[3\frac{m+n}{2} - i] \\
&= 2i^2 + i[-6m-2] + 3\frac{(3m+1)(m+n)}{2}. \\
f' &= 4i - 6m - 2 = 0. \\
i &= \frac{6m+2}{4}.
\end{aligned}$$

$$\begin{aligned}
f_{min}(\frac{6m+2}{4}) &= 2(\frac{6m+2}{4})^2 + (\frac{6m+2}{4})[-6m-2] + 3\frac{(3m+1)(m+n)}{2} \\
&= \frac{9mn+3n-3m-1}{2}.
\end{aligned}$$

4.2.2 For $\mathbf{m+n}$ odd

$$\begin{aligned}
f_{min} &= i[3n - 1 - \frac{3(m+n)+3}{2} - i] + (3m+1-i)[\frac{3(m+n)+3}{2} - i] \\
&= 2i^2 + i[-6m-3] + \frac{9m^2+9mn+12m+3n+3}{2}. \\
f' &= 4i - 6m - 3 = 0. \\
i &= \frac{6m+3}{4}.
\end{aligned}$$

$$
\begin{aligned}
f_{min}(\frac{6m+4}{4}) &= 2(\frac{6m+4}{4})^2 + (\frac{6m+4}{4})[-6m-3] + (3m+1)[\frac{3(m+n)+1}{2}] \\
&= \frac{9mn+3n-3m-3}{2}.
\end{aligned}
$$

The minimum of the biggest linear hcut among the three rows of $K_{3m+1,3n-1}$, for m odd and n even for example, is given by:

$$
\begin{aligned}
hcut &= (\frac{m-1}{2})(\frac{3n}{2}) + (\frac{m+1}{2})(\frac{3n}{2}-1) \\
&= \frac{6mn-2m-2}{4}.
\end{aligned}
$$

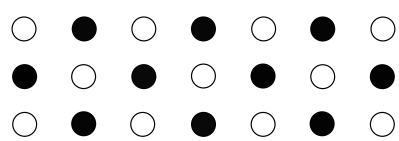

Figure 8: Ex. $K_{10,11}$

We can make sure this lower bound is achievable by comparing the greatest maximum vcut with only hv-edges in the embedding to the smallest maximum hcut of the three rows. In other words, we must verify that the smallest maximum hcut is at least as small as the biggest vcut, which is $4m+1$ in columns of all white vertices and $3m+1$ in columns containing a single black vertex. Running all the neccesary edges vh instead of hv to obtain the lower bound, from the blacks farthest from the centerline to the outside whites, will increase the largest vcut by the same as the smallest hcut.

$$
\begin{aligned}
3m+1 &\leq \frac{6mn-2m-2}{4} \\
n &\geq \frac{14}{6} + \frac{1}{m}
\end{aligned}
$$

Therefore, for m odd and n even, the vcut will not exceed the hcut when $n \geq 3$ and the lower bound corresponds to the grid cutwidth that we can obtain. We can show that the lower bound is achievable for m even and n odd in the same manner. Likewise, we can show that the lower bounds for all $K_{3m,3n}$, $K_{3m+1,3n-1}$ and $K_{3m+2,3m-2}$, for each case, can be gotten.

4.3 $\mathbf{K_{3m+2,3n-2}}$

We can obtain a lower bound for the minimum grid cutwidth of $K_{3m+2,3n-2}$ just as we did for $K_{3m+1,3n-1}$.

4.3.1 For $\mathbf{m+n}$ even

$$\begin{aligned}
f_{min} &= i[3n-2-3\frac{m+n}{2}+i]+(3m+2-i)[3\frac{m+n}{2}-i] \\
&= 2i^2+i[-6m-4]+3\frac{(3m+2)(m+n)}{2}. \\
f' &= 4i-6m-4=0. \\
i &= \frac{6m+4}{4}.
\end{aligned}$$

$$\begin{aligned}
f_{min}(\frac{6m+2}{4}) &= 2(\frac{6m+2}{4})^2+(\frac{6m+2}{4})[-6m-4]+3\frac{(3m+2)(m+n)}{2} \\
&= \frac{9mn+6n-6m-4}{2}.
\end{aligned}$$

4.3.2 For $\mathbf{m+n}$ odd

$$\begin{aligned}
f_{min} &= i[3n-2-\frac{3(m+n)+3}{2}+i]+(3m+2-i)[\frac{3(m+n)+3}{2}-i] \\
&= 2i^2+i[-6m-7]+\frac{9m^2+9mn+15m+6n+6}{2}. \\
f' &= 4i-6m-7=0. \\
i &= \frac{6m+7}{4}.
\end{aligned}$$

$$\begin{aligned}
f_{min}(\frac{6m+8}{4}) &= 2(\frac{6m+8}{4})^2+(\frac{6m+8}{4})[-6m-7]+\frac{9m^2+9mn+15m+6n+6}{2} \\
&= \frac{9mn+6n-6m-6}{2}.
\end{aligned}$$

5 Embedding Complete Bipartite Graphs in m Rows

Theorem 2: For any nonnegative m, n and z, $m+n\geq 4$, and $z<a$, any complete bipartite graph of the form $K_{am+z,an-z}$ can be embedded in a grid of a rows such that:

$$gcw(K_{am+z,an-z})=\begin{cases}\frac{a^2mn+azn-azm-z^2}{2*a} & m+n \text{ even}\\ \frac{a^2mn+azn-azm-z^2-2}{2*a} & m+n \text{ odd.}\end{cases}$$

Proof
In the same fashion that we proved Lemma 1, we will prove the general case.

$$\begin{aligned}
f_{min} &= i[an - z - a\frac{m+n}{2} + i] + (am + z - i)[a\frac{m+n}{2} - i] \\
&= 2i^2 + i[-2am - 2z] + \frac{a^2m^2 + a^2mn + azn + azm}{2}. \\
f' &= 4i - 2am - 2z = 0. \\
i &= \frac{2am + 2z}{4}.
\end{aligned}$$

$$\begin{aligned}
f_{min}(\frac{2am+2z}{4}) &= 2(\frac{2am+2z}{4})^2 + (\frac{2am+2z}{4})[-2am - 2z] + \frac{a^2m^2 + a^2mn + azn + azm}{2} \\
&= \frac{a^2mn + azn - azm - z^2}{2}.
\end{aligned}$$

The proof for $m + n$ odd is done in the same manner, the result being to only subtract 2 from the numerator. Thus, as we expected, the minimum grid cutwidth of any complete bipartite graph that can be embedded evenly in a grid without grey vertices will have a lower bound of $\lceil \frac{lcw}{\#\text{ of rows}} \rceil$.
For $m + n$ odd, the minimum of the biggest linear hcut among the a rows of $K_{am+z,an-z}$, for m odd and n even for example (just like $K_{10,11}$ in Lemma 1), is given by:

$$\begin{aligned}
hcut &= (\frac{m-z}{2})(\frac{an}{2}) + (\frac{m+z}{2})(\frac{an}{2} - z) \\
&= \frac{2amn - 2zm - 2z^2}{4}.
\end{aligned}$$

$$\begin{aligned}
am + z &\leq \frac{2amn - 2zm - 2z^2}{4} \\
n &\geq \frac{2am + 2z + zm + z^2}{am}.
\end{aligned}$$

For $m + n$ odd and m odd and n even still, but $a(m + 1) < an$, the smallest maximum hcut will still be less than the largest maximum vcut for certain n.

$$\begin{aligned}
(a+1)m + z &\leq \frac{2amn - 2zm - 2z^2}{4} \\
n &\geq \frac{2am + 2z + zm + z^2 + 2z}{am}.
\end{aligned}$$

For $m + n$ odd, but m even and n odd, we can show that the lower bound can be obtained. So, we can achieve the lower bound in this particular case and very easily do the same for all remaining cases.

6 Conclusion

In this paper we completed the proof for the minimum grid cutwidth of any complete bipartite graph embedded in two rows and have examined the smallest achievable cutwidth of complete bipartite graphs when embedded in three rows. The next step would be to complete embedding any complete bipartite graph in grids of a rows and b columns, observing exceptions, including grey vertices, to what is found. Additionally, one could continue to look at embedding other types of graphs in grids a x b.

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References

[1] Mario Rocha, "The Embedding of Complete Bipartite Graphs Onto Gids with a Minimum Grid Cutwidth", *Preprint*, 2003.

[2] Matt Johnson, "The Linear and Cyclic Cutwidth of the Complete Bipartite Graph", *Preprint*, 2003.

[3] Stephanie Bowles, "The Linear Cutwidth of Complete Bipartite and Tripartite Graphs", *Preprint*, 2004.

[4] Bradley Marchand, "The Linear and Cyclic Cutwidth of the Complete Bipartite Graph", *Preprint*, 2003.