The Linear and Cyclic Wirelength of Complete Bipartite Graphs

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Abstract

This paper finds the linear wirelength of complete bipartite and tripartite graphs. A model for the graph with minimum wirelength is created by examining the wirelength directly in the bipartite case as well as by minimizing the cut at each region to find the wirelength for both bipartite and tripartite graphs.

1 Introduction

A complete bipartite graph $K_{m,n}$ consists of two sets of vertices A and B in which each of the m vertices in A is joined by an edge to each of the n vertices in B, but no vertex is joined to another in the same set. In this paper, the vertices from set A will be white and the vertices from set B will be black. Figure 1 is a representation of the complete bipartite graph $K_{2,3}$.

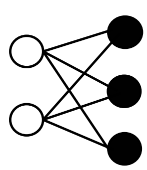

Figure 1: $K_{2,3}$

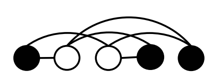

Figure 2: linear embedding

This paper minimizes the linear embedding of complete bipartite and tripartite graphs, in which each vertex of the graph is embedded onto a line. In a linear embedding, the edges are connected by the same rule. Figure 2 is a representation of a linear embedding of $K_{2,3}$.

The linear wirelength of an arrangement of vertices is the sum of the lengths of all the edges. The linear wirelength of the above arrangement is $1 + 2 + 1 + 2 + 2 + 3 = 11$. However, the linear wirelength of a graph, denoted lwl(G) is the minimum wirelength over all numberings. So the $lwl(K_{2,3})$ is ≤ 11. The first proof in this paper will show the diagram of the minimum wirelength for any complete bipartite graph.

A Complete Tripartite graph $K_{r,s,t}$ consists of three sets of vertices $|A| = r$, $|B| = s$ and $|C| = t$ in which each of the vertices in one set is joined by an edge to each of the vertices in the other two, but no vertex is joined to another in the same set. Figure 3 is one representation of $K_{1,2,4}$.

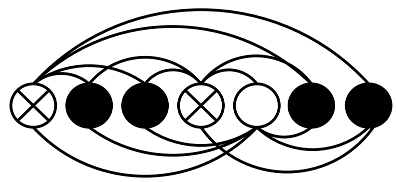

Figure 3: One arrangement of $K_{1,2,4}$.

This paper also uses linear cutwidth as a method for calculating wirelength. The cut of a region between two vertices is the number of edges that cross the region from the left or right. For example, the cut of Figure 4 between vertices 2 and 3 is 6.

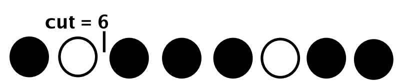

Figure 4: One arrangement of $K_{2,6}$.

The maximum cut of an arrangement of a graph is the largest cut that occurs in that arrangement. The linear cutwidth of a graph is the minimum of all maximum cuts over all possible arrangements.

2 Background

Applications of cutwidth and wirelength include networking, circut layout, and code design. Many papers on the cutwidth and a few papers on the wirelength of graphs have been written. F. Rios [12] and D. Clarke [7] found the linear and cyclic cutwidth of any complete graph K_n. J. Chavez and R. Trapp [5] found the cyclic cutwidth of trees and S. Bezrukov and U.-P. Schroeder [3] wrote a corresponding paper on the cyclic wirelength of trees. M. Johnson [11] found the linear cutwidth of $K_{m,n}$ and M. L. Holben [9] found the cyclic cutwidth of $K_{m,n}$.
Johnson used a specific graph numbering to find the minimum cutwidth of any complete bipartite graph, which was $\frac{mn}{2}$ for mn even and $\frac{mn+1}{2}$ for mn odd. Johnson's numbering consisted of an algorithm in which he placed $\frac{xm}{m+n}$ vertices from the smaller set to the left of each position x where x is the number of

vertices to the left of the cut. Holben used Johnson's linear cutwidth as a lower-bound for finding cyclic cutwidth. Johnson also claimed that his arrangement minimized the cut at each region, which would then also give the wirelength since the sum of the cuts equals the wirelength. However, while Johnson's linear cutwidth was correct, there was a flaw in his proof that his arrangement minimized the cut of each region. Within this paper, we find an arrangement which minimizes the cut of each region (resulting in the same linear cutwidth as Johnson), allowing us to find the wirelength.

3 Structure for the Minimum Wirelength of a Complete Bipartite Graph

Theorem 1 *The minimum linear wirelength for a complete bipartite graph* $K_{m,n}$ *where* $n \geq m$ *occurs when the graph is arranged such that the* m *vertices from set* A *and* $m-1$ *vertices from set* B *alternate between* $\lceil \frac{n-m+1}{2} \rceil$ *vertices from set* B *on the left and* $\lfloor \frac{n-m+1}{2} \rfloor$ *vertices from set* B *on the right.*

Figure 5 is an example of the arrangement described in the theorem.

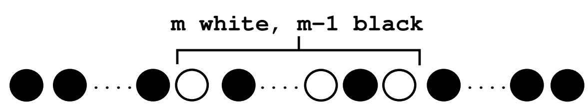

Figure 5: optimal arrangement

Proof

Let $K_{m,n}$ be a complete bipartite graph with $n \geq m$ represented by n black vertices and m white vertices. First, we want to place the two outer white vertices so that the wirelength contributed by these two will be minimized. Without loss of generality, consider the left side of the graph. Suppose that there are j black vertices on the left of the first white vertex as in the diagram below.

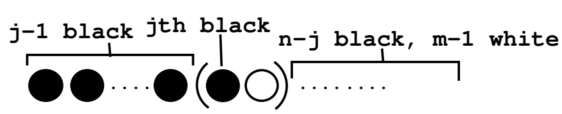

Figure 6: switching vertices

Since we are minimizing the contribution by this white to the wirelength, we want to choose the biggest j such that switching the jth black vertex with the first white vertex will result in a positive change in wirelength. Observe that switching these two will bring the white vertex closer to $j-1$ black vertices and further from $n - j$ black vertices, and bring the black vertex closer to $m - 1$ white vertices. So the change to the wirelength is $-(j-1)+(n-j)-(m-1) = n - m - 2j + 2$. We want this to be the least integer greater than zero, so $n - m - 2j + 2 > 0 \Rightarrow 2j < n - m + 2 \Rightarrow j < \frac{n-m}{2} + 1$.

Case 1:

When m is odd and n is even or m is even and n is odd, $\frac{n-m}{2} + 1$ is not an integer, so the greatest integer less than this is $\frac{n-m-1}{2} + 1$. So there are $\frac{n-m-1}{2} + 1$ black vertices on each side of the two outer white vertices. So there are a total of $2 \cdot \frac{n-m-1}{2} + 1 = n - m + 1$ black vertices on the outside and $n - (n - m + 1) = m - 1$ black vertices inside the two outer white vertices.

Case 2:

When m and n are both even or odd, $\frac{n-m}{2} + 1$ is an integer, so $\frac{n-m}{2}$ is the greatest integer less than $\frac{n-m}{2} + 1$. So there are $\frac{n-m}{2}$ black vertices on the left before the first white vertex. Similarly, there are $\frac{n-m}{2}$ black vertices on the right following the last white vertex. However, observe that switching the first white vertex on the left with adjacent $j + 1$th black vertex will not change the wirelength since this makes the white vertex further from j black vertices and closer to $n - j - 1$ black vertices and makes the black vertex further from $m - 1$ white vertices, resulting in a change to the wirelength of

$$\begin{aligned}
& j - (n - j - 1) + (m - 1) \\
= \;& m - n + 2j \\
= \;& m - n + 2 \cdot \frac{n-m}{2} \text{ (Since } j = \tfrac{n-m}{2}) \\
= \;& m - n + n - m \\
= \;& 0
\end{aligned}$$

Since the change in wirelength is zero, either arrangement of those two vertices is equivalent, and we will use the latter. This diagram has $\frac{n-m}{2}$ black vertices on one side of the outer white vertices and $\frac{n-m}{2} + 1$ black vertices on the other side, totaling $n - m + 1$ black vertices on the outside and $m - 1$ black vertices on the inside of the two outer white vertices.

Now that we know how many black vertices are on the ends, we can focus on minimizing the wirelength of the center of the graph. With both cases, given a $K_{m,n}$ graph, the piece inside and including the two outer white vertices will consist of m white vertices and $m - 1$ black vertices. So now we can focus on minimizing the complete bipartite graph $K_{m,m-1}$.

We want to show that the wirelength is minimized when this graph is alternating, so assume we are starting with a white vertex from the left (because of the above arrangement) and that the graph is alternating until the gth vertex from one set, which begins a string of h adjacent vertices from the same set.

Case 1: Suppose the vertex which begins the string of non-alternating vertices is the gth from the smaller set (see Figure 7). So switching the hth black vertex in this string with the adjacent white vertex will result in a change of wirelength totaling

$$g - (m - g - 1) + m - g - h - (g - 1) - (h - 1) \quad = \quad 2g - m + 1 + m - 2g - 2h + 2$$

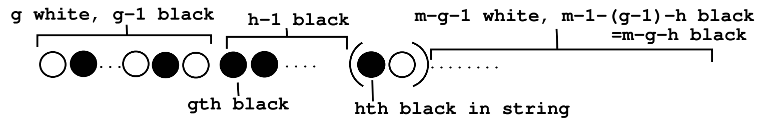

Figure 7: $K_{m,n}$ with h adjacent white vertices

$$= \quad -2h + 3$$

Thus, if $h \geq 2$, the equation is negative, so making the switch decreases the wirelength. Therefore whenever there are 2 or more black vertices in a row, it is better to switch them so that each white vertex is between two white vertices.

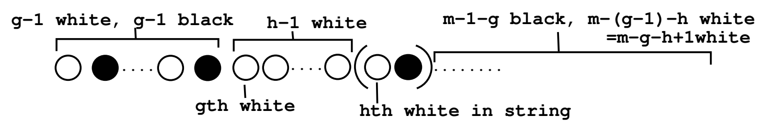

Figure 8: $K_{m,n}$ with h adjacent black vertices

Case 2: Supppose the vertex which begins the string of non-alternating vertices is the gth vertex from the larger set as in Figure 8. So switching the hth white vertex in this string with the adjacent black vertex will result in a change of wirelength totaling

$$\begin{aligned} & m-(g-1)-h-(g-1)-(h-1)+(g-1)-(m-1-g) \\ = \quad & m-2g-2h+3+2g-1-m+1 \\ = \quad & -2h+3 \end{aligned}$$

We observe that if $h \geq 2$, the equation is negative, so making the switch decreases the wirelength. Consequently, whenever there are 2 or more white vertices together, it is better to make switches so that each white vertex is adjacent to two black vertices. So if we decrease the diagram by switching until there are no strings of white or black vertices, the diagram will be alternating with white vertices on each end. Therefore, we know know that when n is odd and m is even or n is even and m is odd, the graph is minimized when there are $m-1$ black vertices alternating with m white vertices and there are $\frac{n-m-1}{2}+1$ black vertices on each side of the two outer white vertices. In the case where both n and m are even or odd, the diagram will be the same except that there will be $\frac{n-m}{2}$ black vertices on one side of the outer white vertices and $\frac{n-m}{2}+1$ black vertices on the other side. $\square$

4 Linear Cutwidth of Complete Bipartite Graphs

Theorem 1 gives the model for the minimum wirelength of a complete bipartite graph. It was noted, however, that this does not match Johnson's model for cutwidth, in which he claimed that the cut at every region was minimized. If this claim was true, the models would be identical, because when the cut is minimized at every region, the sum of the cuts equals the wirelength of a graph. For example, Figure 9 is a $K_{2,6}$ complete bipartite graph arranged by Johnson's arrangement, which places $[\frac{xm}{m+n}]$ vertices from the smaller set to the left of each position x where x denotes the number of vertices to the left of the cut. This arrangement gives a linear wirelength of 32.

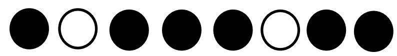

Figure 9: Johnson's Model of $K_{2,6}$; Wirelength is 32.

Figure 12 is $K_{2,6}$ with the numbering described in Theorem 1. The linear wirelength of this arrangement is 30, which is a smaller wirelength than the previous model.

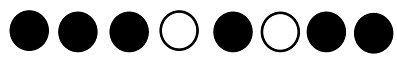

Figure 10: Model of $K_{2,6}$ from theorem 1; Wirelength is 30.

If we look at the cuts of the same two arrangements of $K_{2,6}$, we see that the maximum cut of each graph is 6, but that there are some regions where the cuts are not the same. When $x = 2$, the cut of the Johnson arrangement is 6, but the cut at $x = 2$ of the arrangement given in Theorem 1 is 4.

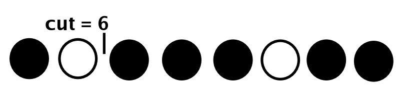

Figure 11: Johnson Arrangement of $K_{2,6}$; Cut at $x = 2$ is 6.

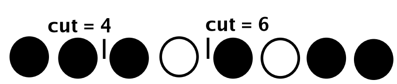

Figure 12: Model of $K_{2,6}$ from theorem 1; Cut at $x = 2$ is 4.

Johnson's arrangement does solve the cutwidth problem (finding the minimum of all maximum cuts of any linear arrangement of a complete bipartite graph), but does not solve the wirelength problem or minimize the cut of each region. Together with S. Bowles and J. Chavez, we show that the arrangement from Theorem 1 minimizes the cut of each region. From this, we can calculate the wirelength of a complete bipartite graph by adding the cuts of each region for any $K_{m,n}$ size graph.

Bowles-Chavez-Hartung (BCH) Theorem 1:
Let $K_{m,n}$ be a complete bipartite graph with two sets of vertices A and B, where $|A| = m$ and $|B| = n$, and let $m \leq n$. Then the cut of each region of a linear embedding for $K_{m,n}$ is minimized by placing $\frac{2x+m-n}{4}$ vertices from A to the left of the cut.

Proof: Let x be the number of vertices to the left of a cut. Given x, suppose there are a vertices from A to the left of the cut. Thus, there are $x - a$ vertices from B to the left of the cut. From this we can conclude that on the right side of the cut there are $m - a$ vertices from A and $n - (x - a)$ vertices from B. We know that the cut of region $(a, x - a)$ is

$$\begin{aligned} cut(a, x - a) &= a(n - (x - a)) + (x - a)(m - a) \\ &= an - ax + a^2 + mx - ax - am + a^2 \\ &= 2a^2 + a(n - 2x - m) + mx \end{aligned}$$

Let $f(a) = 2a^2 + a(n - 2x - m) + mx$. Notice that $f(a)$ is a continuous function of $a \in \mathbf{R}$, and that $f(a) = cut(a, x - a)$ for $0 \leq a \leq m$ and $a \in \mathbf{Z}$. Also, we can observe that $f(a)$ is a positive quadratic, so the graph is a parabola opening upwards. Thus, the derivative at zero gives the minimum of $f(a)$.

$$\begin{aligned} f'(a) &= 4a + n - 2x - m = 0 \\ \Rightarrow a &= \frac{m + 2x - n}{4} \end{aligned}$$

a may not be an integer, and if it is not, it is rounded to the nearest whole number, denoted $[x] = \lfloor x + .5 \rfloor$ and put $[a]$ vertices from A to the left of the cut. Thus for all of the regions of the linear embedding of $K_{m,n}$, its cut is minimized by placing $\left[\frac{m+2x-n}{4}\right]$ vertices from A to the left of each cut.

Corollary:
Let $K_{m,n}$ be a complete bipartite graph whose linear embedding is arranged by the BCH theorem. Then the maximum cut will occur when $x = \frac{n+m}{2}$ for $m + n$ even and when $x = \frac{n+m-1}{2}$ and $x = \frac{n+m+1}{2}$ for $m + n$ odd. Also, the cuts to the left of the middle cut will be strictly increasing and the cuts to the right of the middle cut will be strictly decreasing.

Proof: Let

$$\begin{aligned} cut(a, x - a) &= 2a^2 + a(n - 2x - m) + xm \\ &= 2a^2 - 4a^2 + xm \text{ (since } a = \tfrac{2x+m-n}{4} \text{ gives the minimum cut)} \end{aligned}$$

$$
\begin{aligned}
&= -2a^2 + xm \\
&= -2(\frac{2x+m-n}{4})^2 + xm \\
&= -\frac{1}{8}\cdot(2x+m-n)^2 + xm
\end{aligned}
$$

Let $f(x) = -\frac{1}{8}\cdot(2x+m-n)^2 + xm$ be a continuous function of $x \in \mathbf{R}$ and notice that $-\frac{1}{8}\cdot(2x+m-n)+xm$ gives the cut of the region for $1 \le x \le m+n$. Since $f(x)$ is a negative quadratic equation, we know that it is a parabola opening downward. Thus, the maximum value occurs when the derivative is equal to zero. By taking the derivative, we can show that the maximum cut occurs at the middle region where $x = \frac{n+m}{2}$ for $m+n$ even and where $x = \frac{n+m-1}{2}$ for $m+n$ odd.

$$
\begin{aligned}
f'(x) &= -\frac{1}{4}\cdot(2x+m-n)\cdot 2 + m \\
&= -\frac{1}{2}\cdot(2x+m-n) + m \\
&= -x - \frac{m}{2} + \frac{n}{2} + m \\
&= \frac{m+n}{2} - x
\end{aligned}
$$

Observe that $\frac{m+n}{2} - x > 0$ when $1 \le x < \frac{m+n}{2}$, thus $f(x)$ is increasing, and $\frac{m+n}{2} - x < 0$ when $\frac{m+n}{2} \le x \le m+n$, thus $f(x)$ is decreasing. So for $m+n$ even, the maximum cut occurs at $\frac{m+n}{2}$, but when $m+n$ is odd, $\frac{m+n}{2}$ is not an integer, but is equally spaced between the integers $\frac{m+n-1}{2}$ and $\frac{m+n+1}{2}$, so the maximum cut of the graph occurs at these two points.

Bowles-Chavez-Hartung (BCH) Theorem 2: *Let $K_{m,n}$ be a complete bipartite graph. Then*

$$
lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & \textit{for } mn \textit{ even} \\ \frac{mn+1}{2} & \textit{for } mn \textit{ odd} \end{cases}
$$

Proof:

We have shown that the linear embedding given by BCH Theorem 1 minimizes the cut of each region and that the center cut is the maximum of the linear embedding. Thus the $lcw(K_{m,n})$ occurs when $x = \frac{m+n}{2}$ for $m+n$ even and when $x = \frac{m+n-1}{2}$ and $x = \frac{m+n+1}{2}$ for $m+n$ odd. So we will look at each case to find the $lcw(K_{m,n})$.

Case 1:

Let $m+n$ be even. So $x = \frac{m+n}{2}$. Place $\lceil \frac{2x+m-n}{4} \rceil$ vertices from A to the left of the center cut. Substituting $\frac{m+n}{2}$ for x gives $\lceil \frac{m+n+m-n}{4} \rceil = \lceil \frac{m}{2} \rceil$. Thus $a = \frac{m}{2}$ when m is even and $a = \frac{m+1}{2}$ when m is odd. Since $m+n$ is even, we know that when m is even, n is even, and when m is odd, n is odd. Therefore, the number of vertices from B on the left is

$$
\begin{cases} x - \frac{m}{2} = \frac{n}{2} & \text{for } n \text{ even} \\ x - \frac{m+1}{2} = \frac{n-1}{2} & \text{for } n \text{ odd} \end{cases}
$$

Taking the cut of $(\frac{m}{2}, \frac{n}{2})$ gives

$$\begin{aligned} cut(\frac{m}{2}, \frac{n}{2}) &= \frac{m}{2} \cdot \frac{n}{2} + \frac{n}{2} \cdot \frac{m}{2} \\ &= \frac{mn}{2} \end{aligned}$$

and taking the cut of $(\frac{m+1}{2}, \frac{n-1}{2})$ gives

$$\begin{aligned} cut(\frac{m+1}{2}, \frac{n-1}{2}) &= \frac{m+1}{2} \cdot \frac{n+1}{2} + \frac{n-1}{2} \cdot \frac{m-1}{2} \\ &= \frac{mn+1}{2} \end{aligned}$$

Case 2:
Let $m+n$ be odd. Wlog, let $x = \frac{m+n-1}{2}$. Place $\lfloor \frac{2x+m-n}{4} \rfloor$ vertices from A to the left of the center cut. Substituting $\frac{m+n-1}{2}$ for x gives $\lfloor \frac{m+n-1+m-n}{4} \rfloor = \lfloor \frac{2m-1}{4} \rfloor = \lfloor \frac{m}{2} - \frac{1}{4} \rfloor$. So $a = \frac{m}{2}$ when m is even and $a = \frac{m-1}{2}$ when m is odd. Since $m+n$ is odd, we know that when m is even, n is odd, and when m is odd, n is even. Therefore, the number of vertices from B on the left is

$$\begin{cases} x - \frac{m}{2} = \frac{n-1}{2} & \text{for } n \text{ odd} \\ x - \frac{m+1}{2} = \frac{n}{2} & \text{for } n \text{ even} \end{cases}$$

Taking the cut of $(\frac{m}{2}, \frac{n-1}{2})$ gives

$$\begin{aligned} cut(\frac{m}{2}, \frac{n-1}{2}) &= \frac{m}{2} \cdot \frac{n+1}{2} + \frac{n-1}{2} \cdot \frac{m}{2} \\ &= \frac{mn+m}{4} + \frac{mn-m}{4} \\ &= \frac{mn}{2} \end{aligned}$$

and taking the cut of $(\frac{m-1}{2}, \frac{n}{2})$ gives

$$\begin{aligned} cut(\frac{m-1}{2}, \frac{n}{2}) &= \frac{m-1}{2} \cdot \frac{n}{2} + \frac{n}{2} \cdot \frac{m+1}{2} \\ &= \frac{mn-n}{4} + \frac{mn+n}{4} \\ &= \frac{mn}{2} \end{aligned}$$

Therefore,

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & \text{for } mn \text{ even} \\ \frac{mn+1}{2} & \text{for } mn \text{ odd.} \end{cases}$$

5 Calculating the Wirelength of a Complete Bipartite Graph

Thus far, Theorem 1 gives the model for the minimum wirelength of the complete bipartite graph and the BCH theorem gives the model for the minimum cutwidth at each region of a complete bipartite graph. It is not a coincidence that these minimizations result in the same model; Adding up all the cuts of a graph gives the wirelength, so the graph with the minimum cutwidth at each region will also give the minimum wirelength. Because it is simpler to add the cuts of a graph, this is the method we will use to find wirelength.

Theorem:

$$lcw(K_{m,n}) = \begin{cases} \frac{-m^3}{12} + \frac{m^2 n}{2} + \frac{mn^2}{4} + \frac{m}{12} & \text{for } m+n \text{ odd} \\ -\frac{m^3}{12} + \frac{m^2 n}{2} + \frac{mn^2}{4} + \frac{m}{3} & \text{for } m+n \text{ even.} \end{cases}$$

Proof: Let i be the position with i vertices to its left. We will first look at the case when $m+n$ is odd. In this case, there is an even number of adjacent vertices from set B on each side of the graph, so the graph is symmetric. Since the graph is symmetric, we will calculate the cuts of the left side and multiply this by 2. Observe that there are $\frac{n-m+1}{2}$ adjacent black vertices on the left side of a $K_{m,n}$ bipartite graph and that each of these will have a cut of im since there will be i black vertices connecting to m white vertices.

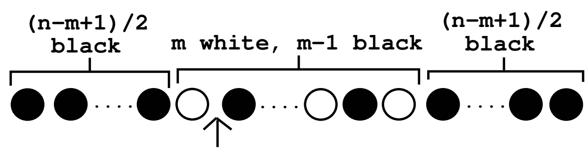

Figure 13: $K_{m,n}$ with m+n odd

So the sum of these cuts will be $\sum_{i=1}^{\frac{n-m+1}{2}} im$.
Beginning with the first position with a white vertex to the left, the cut will be different, and for the puposes of creating a summation, we will start with $i=1$ again. Here, there will be $\frac{n-m+1}{2}+i$ vertices to the left of each position i. However, when calculating the number of white or black vertices on each side of i, the number will be different depending on whether the vertex left of i is white (then i is odd) or black (then i is even), so we will split this into cases.
Case 1: i is odd. For each of these, there will be $\frac{n-m+1}{2}+\frac{i-1}{2}$ black vertices and $\frac{i+1}{2}$ white vertices on the left and $\frac{n+m-1}{2}-\frac{i-1}{2}$ black vertices and $m-\frac{i+1}{2}$ white vertices to the right. So the cut at i

$$\begin{aligned}
&= (\frac{n-m+1}{2}+\frac{i-1}{2})(m-\frac{i+1}{2})+(\frac{i+1}{2})(\frac{n+m-1}{2}-\frac{i-1}{2}) \\
&= m\cdot(\frac{n-m+1}{2}+\frac{i-1}{2})+\frac{i+1}{2}(\frac{-n+m-1}{2}-\frac{i-1}{2})+(\frac{i+1}{2})(\frac{n+m-1}{2}+\frac{-i+1}{2}) \\
&= m\cdot(\frac{n-m+1}{2}+\frac{i-1}{2})+\frac{i+1}{2}(\frac{2m-2}{2})+\frac{i+1}{2}(\frac{-i+1+1-i}{2}) \\
&= m\cdot(\frac{n-m+1}{2}+\frac{i-1}{2}+\frac{i+1}{2})+\frac{i+1}{2}(-1+\frac{2-2i}{2})
\end{aligned}$$

$$
\begin{aligned}
&= m \cdot (\frac{n-m+1}{2} + \frac{2i-1+1}{2}) + \frac{i+1}{2}(-i) \\
&= m \cdot (\frac{n-m+1}{2} + i) - \frac{i(i+1)}{2}
\end{aligned}
$$

Case 2: i is even. For each of these, there will be $\frac{n-m+1}{2} + \frac{i}{2}$ black vertices and $\frac{i}{2}$ white vertices on the left and $\frac{n+m-1}{2} - \frac{i}{2}$ black vertices and $m - \frac{i}{2}$ white vertices to the right. So the cut at i

$$
\begin{aligned}
&= (\frac{n-m+1}{2} + \frac{i}{2})(m - \frac{i}{2}) + (\frac{i}{2})(\frac{n+m-1}{2} - \frac{i}{2}) \\
&= m \cdot (\frac{n-m+1}{2} + \frac{i}{2}) + \frac{i}{2}(\frac{-n+m-1}{2} - \frac{i}{2}) + (\frac{i}{2})(\frac{n+m-1}{2} + \frac{-i}{2}) \\
&= m \cdot (\frac{n-m+1}{2} + \frac{i}{2}) + \frac{i}{2}(\frac{2m-2}{2}) + \frac{i}{2}(\frac{-i-i}{2}) \\
&= m \cdot (\frac{n-m+1}{2} + \frac{i}{2} - \frac{i}{2} - \frac{i}{2}(i) \\
&= m \cdot (\frac{n-m+1}{2} + i) - \frac{i(i+1)}{2}
\end{aligned}
$$

So for each case, the cut at i is equal to $m \cdot (\frac{n-m+1}{2} + i) - i\frac{i(i+1)}{2}$. This is equivalent to the sum $\sum_{i=1}^{m-1} \frac{i(i+1)}{2} - \sum_{i=1}^{m-1} \frac{i(i+1)}{2}$, which is the wirelength of the left half of the graph. Since the graph is symmetric, the total wirelength of the graph is $2 \cdot (\sum_{i=1}^{m-1} \frac{i(i+1)}{2} - \sum_{i=1}^{m-1} \frac{i(i+1)}{2})$. This can be further simplified to an algebraic formula as such:

$$
\begin{aligned}
& 2 \cdot (\sum_{i=1}^{m-1} \frac{i(i+1)}{2} - \sum_{i=1}^{m-1} \frac{i(i+1)}{2}) \\
&= 2m \cdot \sum_{i=1}^{m-1} i - \frac{2}{2} \cdot \sum_{i=1}^{m-1} i - \frac{2}{2} \cdot \sum_{i=1}^{m-1} i^2 \\
&= 2m \cdot (\frac{(\frac{m+n-1}{2}) \cdot (\frac{m+n+1}{2})}{2}) - \frac{(m-1)(m)(2(m-1)+1)}{6} - \frac{(m-1)(m)}{2} \\
&= m(\frac{m^2+mn+m+mn+n^2+n-m-n-1}{4}) + \frac{(m^2-m)(2m-1)}{6} + \frac{-m^2+m}{2} \\
&= \frac{m^3+2m^2n+mn^2-m}{4} + \frac{-2m^3+m^2+2m^2-m}{6} + \frac{-m^2+m}{2} \\
&= \frac{3m^3+6m^2n+3mn^2-3m-4m^3+6m^2-2m-6m^2+6m}{12} \\
&= \frac{-m^3}{12} + \frac{m^2n}{2} + \frac{mn^2}{4} + \frac{m}{12}
\end{aligned}
$$

We will now look at the case when $m+n$ is even. In this case, there are $\frac{n-m}{2} + 1$ adjacent vertices on one side of the graph and $\frac{n-m}{2} + 1$ on the other.

Without loss of generality, suppose that there are $\frac{n-m}{2}+1$ adjacent vertices on the left side of the graph. On the outer sections with consecutive blacks, the positions i (i being the number of vertices to the outside of the position) will have cuts of im. So the cuts of the outsides will add up to $\sum_{i=1}^{\frac{n-m}{2}+1} im$ on the left side and $\sum_{i=1}^{\frac{n-m}{2}} im$ on the right.

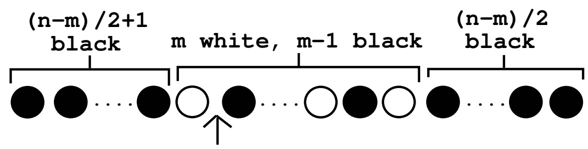

Figure 14: $K_{m,n}$ with m+n even

Beginning on the left with the first position where there is one white to the left, there will be a different cut for each position in the middle section with alternating vertices. Here, we will start with $i=1$ again, so let i denote the space with $\frac{n-m}{2}+1$ vertices to the left. Again, calculating the number of white and black vertices on each side of i will depend on whether i is odd (following a white vertex) or even (following a black vertex), so we will split into cases.
Case 1:
i is odd. For each of these, there will be $\frac{n-m}{2}+1+\frac{i-1}{2}$ black vertices and $\frac{i+1}{2}$ white vertices on the left and $\frac{n+m}{2}-1-\frac{i-1}{2}$ black vertices and $m-\frac{i+1}{2}$ white vertices to the right. So the cut at i

$$\begin{aligned}
&= (\frac{n-m}{2}+1+\frac{i-1}{2})(m-\frac{i+1}{2})+(\frac{i+1}{2})(\frac{n+m}{2}-1-\frac{i-1}{2}) \\
&= m\cdot(\frac{n-m}{2}+1+\frac{i-1}{2})+\frac{i+1}{2}(\frac{-n+m-2}{2}-\frac{i-1}{2})+(\frac{i+1}{2})(\frac{n+m-2}{2}+\frac{-i+1}{2}) \\
&= m\cdot(\frac{n-m}{2}+1+\frac{i-1}{2})+\frac{i+1}{2}(\frac{2m-4}{2})+\frac{i+1}{2}(\frac{-i+1}{2})+(\frac{i+1}{2})(\frac{-i+1}{2}) \\
&= m\cdot(\frac{n-m}{2}+1+\frac{i-1}{2}+\frac{i+1}{2})+\frac{i+1}{2}(-2+\frac{i+1}{2})(-i+1)) \\
&= m\cdot(\frac{n-m}{2}+1+i)+\frac{i+1}{2}(-i-1) \\
&= m\cdot(\frac{n-m}{2}+1+i)-\frac{(i+1)(i+1)}{2}
\end{aligned}$$

Case 2:
i is even. For each of these, there will be $\frac{n-m+1}{2}+\frac{i}{2}$ black vertices and $\frac{i}{2}$ white vertices on the left and $\frac{n+m-1}{2}-\frac{i}{2}$ black vertices and $m-\frac{i}{2}$ white vertices to the right. So the cut at i

$$= (\frac{n-m}{2}+1+\frac{i}{2})(m-\frac{i}{2})+(\frac{i}{2})(\frac{n+m}{2}-1-\frac{i}{2})$$

$$
\begin{aligned}
&= m\cdot(\frac{n-m}{2}+1+\frac{i}{2})+\frac{i}{2}(\frac{-n+m-2}{2}-\frac{i}{2})+(\frac{i}{2})(\frac{n+m-2}{2}+\frac{-i}{2})\\
&= m\cdot(\frac{n-m}{2}+1+\frac{i}{2})+\frac{i}{2}(\frac{2m-4}{2})+\frac{i}{2}(\frac{-i}{2})+\frac{i}{2}(\frac{-i}{2})\\
&= m\cdot(\frac{n-m}{2}+1+\frac{i}{2}+\frac{i}{2})+(\frac{i}{2})(-2)+\frac{i}{2}(-i)\\
&= m\cdot(\frac{n-m}{2}+1+i)-\frac{i(i+2)}{2}
\end{aligned}
$$

So when i is odd, the cut at i is equal to $m\cdot(\frac{n-m}{2}+1+i)-\frac{(i+1)(i+1)}{2}$, and when i is even, the cut is equal to $m\cdot(\frac{n-m}{2}+1+i)-\frac{i(i+2)}{2}$. As when $m+n$ was odd, we see that this is m multiplied by the number of vertices to the left, but with each ith vertex after the first white, $\frac{(i+1)(i+1)}{2}$ is subtracted for each odd i and $\frac{i(i+2)}{2}$ is subtracted from each even i. So putting these together with the sums of the outside, we obtain

$$\sum_{i=1}^{\frac{n+3m-2}{2}} im+\sum_{i=1}^{\frac{n-m}{2}} im-\sum_{i=2}^{2m-2}\frac{i(i+2)}{2}\quad(\text{for } i \text{ even})-\textstyle\sum_{i=1}^{2m-1}\frac{(i+1)(i+1)}{2}\quad(\text{for } i \text{ odd}).$$

The first summation goes up to $\frac{n+3m-2}{2}$ to include all cuts except those on the outer right, where there are no longer alternating vertices. The last two summations go up to $2m-2$ and $2m-1$ because there are $2m-2$ cuts between the two outer white vertices. However, for these last two, we want a summation of consecutive integers (not just odds or evens), so when i is even, we will substitute $2j$ for i to get

$$\sum_{j=1}^{m-1}\frac{(2j)(2j+2)}{2} \quad=\quad \sum_{j=1}^{m-1}2j^2+2j$$

And for i odd, substitute $2j-1$ for i to get

$$\sum_{j=1}^{m-1}\frac{((2j-1)+1)((2j-1)+1}{2} \quad=\quad \sum_{j=1}^{m-1}2j^2$$

So the summation is

$$
\begin{aligned}
&\sum_{i=1}^{\frac{n+3m-2}{2}} im+\sum_{i=1}^{\frac{n-m}{2}} im-\sum_{j=1}^{m-1}2j^2+2j-\sum_{j=1}^{m-1}2j^2\\
&= m\cdot(\sum_{i=1}^{\frac{n+3m-2}{2}} i)+m\cdot(\sum_{i=1}^{\frac{n-m}{2}} i)-4(\cdot\sum_{j=1}^{m-1}j^2)-2\cdot(\sum_{j=1}^{m-1}j\\
&= m\cdot(\frac{(\frac{n+3m-2}{2})(\frac{n+3m}{2})}{2}+\frac{(\frac{n-m}{2}(\frac{n-m+2}{2})}{2}-4\cdot(\frac{(m-1)(m)(2(m-1)+1)}{6})-2\cdot(\frac{(m-1)(m)}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= m \cdot \left(\frac{2n^2+6mn+9m^2-2n-6m-2mn+2n-2m+m^2}{8}\right) - \frac{4}{6}\cdot(m^2-m)(2m-1) - m^2 + m \\
&= \frac{2mn^2+4m^2n+10m^3-8m^2}{8} - \frac{4}{6}\cdot(2m^3-m^2-2m^2+m) - m^2 + m \\
&= \frac{3mn^2+6m^2n+15m^3-12m^2-16m^3+24m^2-8m-12m^2+12m}{12} \\
&= -\frac{m^3}{12} + \frac{m^2n}{2} + \frac{mn^2}{4} + \frac{m}{3}
\end{aligned}
$$

6 Linear Cutwidth of Complete Tripartite Graphs

In this section, we minimize the cut of every region of the linear embedding of a complete tripartite graph $K_{r,s,t}$. To do this, we will refer to the middle and outer regions of the graph. The middle region consists of r sets of three vertices, each set including one black, one white, and one gray vertex. The outside consists of the remaining vertices.

Bowles-Chavez-Hartung Theorem 3 (BCH3):
Let $K_{r,s,t}$ be a complete tripartite graph with three sets of vertices A, B, and C, where $|A| = r$, $|B| = s$, and $|C| = t$, and let $r \leq s \leq t$. To minimize each cut of the linear embedding for $K_{r,s,t}$, the middle and outer sections of the graph are minimized independently. The middle cuts are minimized by placing $\frac{2x+2r-s-t}{6}$ vertices from A, $\frac{2x+2s-r-t}{6}$ vertices from B, and $\frac{2x+2t-r-s}{6}$ vertices from C to the left of each cut. The outer sections are minimized according to the BCH arrangement for complete bipartite graphs.

Proof: This proof is separated into three sections: the minimizing of the middle region, the minimizing of the outer regions, and the placement of the middle region.

6.1 Minimizing the Middle Region

Let x be the number of vertices to the left of a cut. Given x, suppose there are a vertices from A, b vertices from B, and $c = x - a - b$ vertices from C to the left of the cut. From this we can conclude that on the right side of the cut there are $r - a$ vertices from A, $s - b$ vertices from B, and $t - (x - a - b)$ vertices from C. We know that the cut of region $(a, b, x - a - b)$ is

$$
\begin{aligned}
cut(a,b,x-a-b) &= a[(s-b)+(t-(x-a-b)]+b[(r-a)+(t-(x-a-b))] \\
&\quad +(x-a-b)[(r-a)+(s-b)] \\
&= as-ab+at-ax+a^2+ab+br-ba+bt-bx+ba+b^2 \\
&\quad +xr-xa+xs-xb-ar+a^2-as+ab-br+ab-bs+b^2 \\
&= 2a^2+2b^2+a(t-r-2x)+b(t-s-2x)+2ab+x(r+s)
\end{aligned}
$$

Let $f(a,b) = 2a^2+2b^2+a(t-r-2x)+b(t-s-2x)+2ab+x(r+s)$. Notice that $f(a,b)$ is a continuous function of $a, b \in I\!R$, and that $f(a,b) = cut(a,b,x-a-b)$ for $0 \le a \le r$ and $a \in Z\!\!\!Z$ and for $0 \le b \le s$ and $b \in Z\!\!\!Z$. The minimum of $f(a,b)$ is found by taking its derivative and setting it equal to zero. So,

$$\begin{aligned}
\frac{\partial f(a,b)}{\partial a} &= 4a + (t-r-2x) + 2b = 0 && (1)\\
\Rightarrow 2b &= -4a - t + r + 2x. && (2)\\
\frac{\partial f(a,b)}{\partial b} &= 4b + (t-s-2x) + 2a = 0 && (3)\\
\Rightarrow 2a &= -4b - t + s + 2x. && (4)
\end{aligned}$$

Substituting the value of $2b$ from (2) into (3) and gives

$$\begin{aligned}
0 &= 2(-4a - t + r + 2x) + t - s - 2x + 2a\\
\Rightarrow 6a &= 2x + 2r - s - t\\
\Rightarrow a &= \frac{2x + 2r - s - t}{6}.
\end{aligned}$$

Substituting the value of $2a$ from (4) into (1) gives

$$\begin{aligned}
0 &= 2(-4b - t + s + 2x) + t - r - 2x + 2b\\
\Rightarrow 6b &= 2x + 2s - r - t\\
\Rightarrow b &= \frac{2x + 2s - r - t}{6}.
\end{aligned}$$

Knowing a and b we can get c:

$$\begin{aligned}
c &= x - a - b\\
&= x - [\frac{2x + 2r - s - t}{6}] - [\frac{2x + 2s - r - t}{6}]\\
&= \frac{2x + 2t - r - s}{6}.
\end{aligned}$$

a, b, and c may not be integers, and if any one is not, it is rounded to the nearest whole number, denoted $[x] = \lfloor x + .5 \rfloor$. Now given a, b, and c, let $1 \le a \le r$ since we are only concerned with the middle of the graph where all of the sets are active. If we minimize the cuts of the middle region, they will not be affected by the arrangement of the vertices on the outside as long as we know how many vertices from each set are on the left and right of the middle region. Given the middle region and the number of vertices to the left and right of it, we know that each vertex in the middle will be connected to the same number of edges regardless of the structure of the outside vertices. Therefore we can minimize the inside independently of the arrangement of the outside vertices.

If a chart of the values a, b, and c is constructed, each vertex may not have a unique position. Below in Figure 15, the x represents the number of vertices to the left of a cut. The variables a, b and c represent the number of vertices from each set A, B, C to the left of a cut. Every time a, b or c increases by one, a new vertex is added to that spot. For example, in Figure 15, at vertex 5 ($x = 5$), there are no vertices from A to the left. However, at vertex 6, there is one vertex from set A, so the sixth vertex will be black.

Since $1 \leq a \leq r$, there will be r groups of three in the middle region. Beginning with the first position where $a = 1$, each group of three vertices will have one black, one white and one gray vertex, but the arrangement of those within the group of three is not necessarily unique. So let the three positions of each group be denoted x_1, x_2, and x_3. There are three cases:

Case 1: Every black, white and gray has a unique position. For every x_i the value of only the vertex from a, b, or c will change and it will be consistent throughout the r groups. In this case, $x_i = a + b + c$ for all x_i while $1 \leq a \leq r$. For example, in $K_{1,4,7}$, the order of the middle must be black, white, gray.

$$\overbrace{x_1 \ x_2 \ x_3}^{3r}$$

	x	...	3	4	5	**6**	**7**	**8**	9	10	11	...
●	a		0	0	0	**1**	**1**	**1**	2	2	2	
⊗	b		1	1	2	**2**	**2**	**3**	3	3	4	
○	c		2	3	3	**3**	**4**	**4**	4	5	5	

Figure 15: $K_{1,4,7}$

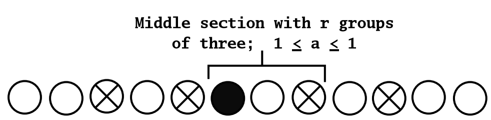

Figure 16: $K_{1,4,7}$

Case 2: Either black, white or gray will have a unique position consistantly throughout the r groups. For the remaining two sets, the vertices from those sets will be interchangable within each group of three. Again, remember that for each group of three, we are denoting the three positions x_1, x_2 and x_3. In this case, either at x_1, $a+b+c = x_1 + 1$ or at x_2, $a+b+c = x_2 + 1$. For example, in $K_{4,8,11}$, the black and the gray vertices can be interchanged between positions x_1 and x_2, as shown in Figure 18, but the white vertices are always in position x_3.

Case 3: The black, gray and white vertices in each group of three are all

3r

$\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3$

	x	...	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	...
●	a		0	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	
⊗	b		2	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	
○	c		3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	

Figure 17: $K_{4,8,11}$

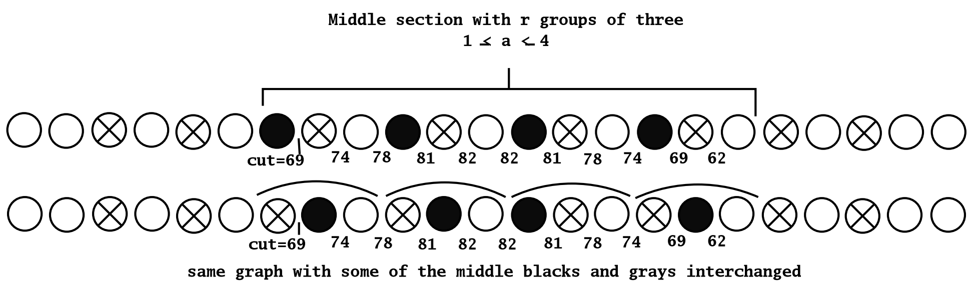

Figure 18: $K_{4,8,11}$

interchangable. In this case, at x_1, $a+b+c = x_1+2$ and at x_2, $a+b+c = x_2+1$. $K_{3,9,13}$ is an example of this where the whites, grays and blacks can be in any position within groups of three.

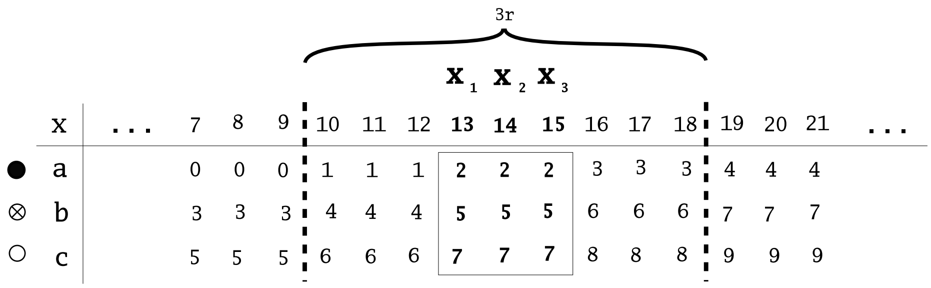

Figure 19: $K_{3,9,13}$

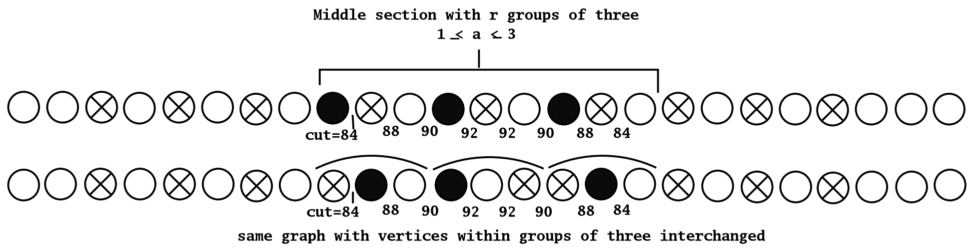

Figure 20: $K_{3,9,13}$

So when it is not clear whether a, b or c changes first, the cuts of the regions are equivalent regardless of which arrangement is chosen as long as there is a vertex from each set within each group of three. Thus the linear embedding of a complete tripartite graph $K_{r,s,t}$ is minimized within the region where $1 \leq a \leq r$ by placing $\frac{2x+2r-s-t}{6}$ vertices from A, $\frac{2x+2s-r-t}{6}$ vertices from B, and $\frac{2+2t-r-s}{6}$ vertices from C to the left of each cut.

6.2 Minimizing the Outer Regions

We will now consider the outer vertices. The chart of a complete tripartite graph gives the number of vertices from set B and C to the right and left of the middle region. The number of vertices from B and C to the left of the middle region is given by b and c at the last position where $a = 0$. For example, in Figure 15, when $x = 5$, there are 2 gray vertices and 3 white vertices. The number of vertices from B to the right of the middle region is given by $s - b$ where b is the first position where $a = r + 1$. The number of vertices from C to the right is given by $t - c$ where c is the first position where $a = r + 1$. For example, in Figure 15, when $x = 9$, $b = 3$ and $c = 4$, so there is 1 vertex from B and there are 3 vertices from C to the right of the middle. Also, we have r

gray, r white and r black vertices in the middle region. As previously stated, the arrangement of the middle does not affect the cuts of the outside. Without loss of generality, we look at the vertices to the left of the middle region. Let i be the number of vertices to the left. Observe that each white on the outside is connected to r gray and r black vertices from the middle region and that each gray on the outside is connected to r white and r black vertices from the middle region. This results in the middle vertices contributing $i \cdot 2r$ to the cut of each region. The same is true of the vertices to the right of the middle region where i is the number of vertices to the right of each cut. This applies to any arrangement of the outside vertices. But the outside vertices make up the complete bipartite graph $K_{s-r,t-r}$, thus to minimize the cuts along outside vertices, we can use the BCH arrangement.

6.3 Placing the Middle Region Within the Bipartite Graph

We have shown that the middle region can be minimized independently of the outer regions and the outer regions can be minimized independently of the middle region. Since we know the number of vertices to the left of the middle region, we can place the minimized middle region after this number of vertices of the minimized bipartite graph. Notice that the left outer region consists of $\lceil \frac{s-r}{2} \rceil$ vertices from B and $\lceil \frac{t-r}{2} \rceil$ vertices from C and the right outer region consists of $\lfloor \frac{s-r}{2} \rfloor$ vertices from B and $\lfloor \frac{t-r}{2} \rfloor$ vertices from C. This results in the minimal cut for each region of the complete tripartite graph $K_{r,s,t}$. $\square$

7 Future Research

Wirelength formula for tripartite: Because of time constraints, the wirelength of a complete tripartite graph has yet to be calculated. However, this is a short-term goal of mine, so a more profitable subject of research would be:
Can this concept be applied to n-partite graphs? The possibility is promising because it appears that each "inner" section of j sets of vertices can be minimized independently of each "outer" section of $j+1$ sets vertices containing it.

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