The Supercrossing Index Of Torus Knots

Ryan Lain Petitfils

Abstract

ABSTRACT. The crossing number of a given knot is the least possible number of crossings in any configuration of the knot. The supercrossing number, of a configuration on the other hand, represents the greatest number of crossings possible. The minimal stick representations of Torus knots are used to find bounds for the supercrossing index.

1 INTRODUCTION

A torus knot is any knot that can be put onto a torus such that no two strands cross on the surface of the torus. Torus knots can be represented by the notation $T_{p,q}$, where p is the number of times that the knot wraps around the meridian, and q is the number of times that the strands cross the longitude, also known as the lengthwise surface of the knot. The ordering actually does not matter and in fact a $T_{p,q}$ knot is equivalent to a $T_{q,p}$ knot. When p and q are relatively prime, the knot has one component. The gcd of p and q determines the number of components that the torus knot has. The stick number, $s[K]$, of a torus knot K, is the minimum number of sticks needed to create it.

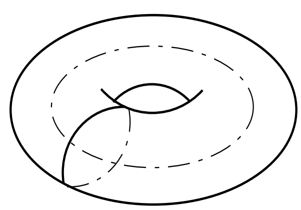

Figure 1: A Torus on which stick knots can be placed.

2 SUPERCROSSING INDEX

A knot invariant is a function on the set of knots which describes a property of a knot, such that the same value is assigned to equivalent knots. The supercrossing index, a known knot invariant of a given knot, was first studied by Adams [2]. The supercrossing index is the minimum supercrossing number taken over all possible configurations of a given knot. By Lemma 2.5, he proved that for the supercrossing index of a knot K,

$$scr[K] \leq \begin{cases} s(s-3)/2 & \text{if } s \text{ is odd;} \\ s(s-4)/2+1 & \text{if } s \text{ is even.} \end{cases}$$

Where s is the stick number of the given knot.

These results come directly from the fact that there can be at most two all-crossing segments in any given knot projection. An all-crossing segment is one which crosses every other segment except for its adjacent segments. For an knot constructed with an odd number of sticks, each of the s segments can cross at most s-3 other segments. For an even number of sticks, all other segments can cross at most s-4 other segments, and since the all-crossing segments must cross each other, an extra crossing is always added. His results describe the trefoil knot as having a supercrossing index of either 6 or 7. This is because it is not known if every configuration of a trefoil knot can be made to have 7 crossings. Adams also has been able to bound the supercrossing index in terms of the crossing number.

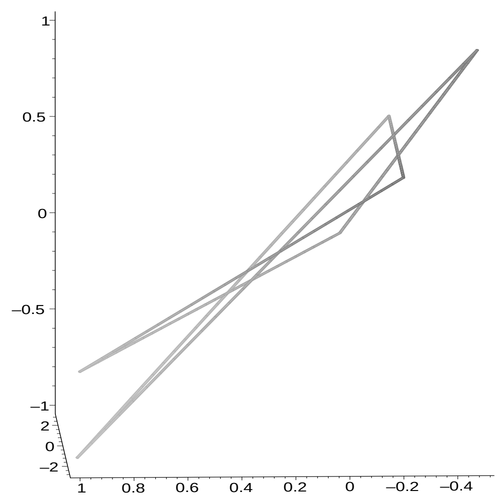

Figure 2: A 7 crossing representation of a 6 stick trefoil knot.

3 SUPERCROSSING OF TORUS KNOTS

Torus knots are easiest drawn by a process of a braiding knots. For example to find a representation of $T_{2,5}$, draw two top points and two bottom points then starting with the top points, draw braids that cross each other five times, until they reach the bottom points. Next, whichever braid ends up at the left most bottom point connects to the left most top point and same with the right points. From these basic projections I will look at three types of Torus knots and their stick numbers:

1. Torus knots of the form $T_{q,q-1}$,
2. Torus knots of the form $T_{q,3q}$
3. Torus knots of the form $T_{2,q}$

3.1 Example 1, $T_{q,q-1}$

One can bound supercrossing numbers for any type of knot as long as one knows the stick number and/or crossing number of that knot.

For Torus knots of the form $T_{q,q-1}$, the stick number and crossing number are known, they are $2q$ and $q^2 - 2q$ respectively.

Lemma 1 *For any knot K, of the form* $T_{q,q-1}$, $s[K]^2/4 - s[K] + 3 \leq scr[K]$

Proof: Since $s[K] = 2q$ and $crn[K] = q^2 - 2q$. By Lemma 2.6 from Adams' et. al. paper:

$$cr[K] + 3 \leq scr[K] \leq 2cr^2[K] - 3cr[K]$$

In this inequality, cr[K] represents the crossing index, the minimum crossing number taken over all projections of a knot. It follows that

$$crn[K] = (s[K]/2)^2 - 2(s[K]/2)$$

and for $T_{q,q-1}$ torus knots, since the stick number is always even, by substitution, $s[K]^2/4 - s[K] + 3 \leq scr[K]$.

3.2 Example 2, $T_{q,3q}$

It should be noted that every $T_{q,3q}$ knot is actually a link of q components, where each component is the unknot. Since links will at most have the same upper bound as knots for supercrossings, we can use Adam's same bound.

Lemma 2 *For any knot K, of the form* $T_{q,3q}, 3(s[K]^2/16 - s[K]/4 + 1) \leq scr[K]$

Proof: Since $s[K] = 4q$ and $crn[K] = 3q^2 - 3q$,
By Theorem 6.1 from Mills'[4] paper:

$$if q \geq 2, s(T_{q,3q}) = 4q$$

it follows that:

$$crn[K] = 3(s/4)^2 - (s/4) = 3(s^2/16 - s/4)$$

By substitution, $crn[K] + 3 = 3(s^2/16 - s/4 + 1) \leq scr[K]$.

There exists a certain type of knot, $T_{2,q}$ and although the stick number is unknown, I can still bound the supercrossing number for a specific case.

3.3 Example 3, $T_{2,q}$

For knots of the form $T_{2,q}$, G.T. Jin [3] has proposed an equation which will generate stick knots on a Torus, where the gcd of 2 and q must be 1. Although the stick number of $T_{2,q}$ knots is unknown,his equation generates Torus knots with exactly $2q$ sticks. In his construction we use i, a counter that determines the number of vertices. It starts at 0 and goes to $2q-1$. Also used is his equation is α, which is a variable that is always between $\frac{\pi(p)}{q}$ and $min(\pi, 2\frac{\pi(p)}{q})$, in this case I chose $\alpha = 3\pi/q$. For $T_{2,q}$ his equation becomes:

$$|X_i| = \begin{cases} (cos(2\pi(i)/q), sin(2\pi(i)/q), -1) & \text{if } i \text{ is even;} \\ (cos((2\pi(i-1))/q + \alpha), sin((2\pi(i-1))/q) + \alpha, 1) & \text{if } i \text{ is odd.} \end{cases}$$

The $T_{2,q}$ knots that are generated by Jin's construction will have a vertex every $\frac{\pi}{q}$ rotations. In any of Jin's $T_{2,q}$ constructions, an angle $\frac{\pi}{q}$ is made by rotating the radius that goes from the center to any vertex to the next consecutive vertex. Similarly, an angle $\frac{3\pi}{q}$ is made by rotating the radius that goes from the center to any vertex to a vertex that is three vertices away counterclockwise. By his construction, q short sticks and q long sticks are created. Every vertex is considered either a top vertex or a bottom vertex and at $\phi = 90$, top vertices are created along z=1 and bottom vertices are created along z=-1. The central angle made between the vertices of the short sticks and longs sticks is $\frac{\pi}{q}$ and $\frac{3\pi}{q}$ respectively. By using symmetry arguments, I can determine the supercrossing number for these knots.

Lemma 3 *The supercrossing number for $T_{2,q}$, q odd, q$\geq$ 5 using Jin's construction is less than 4q.*

Proof: This proof has 2 main cases. Case 1: Each short stick crosses at most 3 other sticks. Case 2: Each long stick crosses at most 5 other sticks.

For any $T_{2,q}$ knot, under Jin's construction, at $\phi = 0$, the short sticks are the ones which contain vertices that are connected by an edge going from top to bottom, counterclockwise. The long sticks are connected by edges that are bottom to top, counterclockwise.

3.3.1 The Viewing Rectangle

In order to describe the maximum number of crossings possible for $T_{2,q}$, one must constrain a region from which to view the potential crossings for each stick. This region is a viewing rectangle and it iks a region formed by 4 vertices from which one can look through and determine crossings. Two of the vertices of this rectangle include those given by any stick for which one would like to find crossings for. The other two vertices are formed by any two vertices on the circle which are separated such that they create a central angle of either $\frac{\pi}{q}$ or $\frac{3\pi}{q}$. This angle corresponds to whether a short stick or a long stick was chosen for the first two vertices.

Case 1: Let E be a short stick constructed by Jin's equation and viewed from $\phi = 90$. Create a viewing rectangle using the two vertices of E as one side. Next, constrain the viewing rectangle such that E and the arc on the opposite side of E both make a central angle of $\frac{\pi}{q}$. The vertices on the opposite side of E do not have to be vertices of the knot itself. When this region of equivalent arcs is created, 3 situations arise that describe the greatest possible contribution to the supercrossing number of E. In all three cases E can always cross the long stick, call it F, contained in the viewing rectangle, which is actually parallel to E at $\phi = 0$.

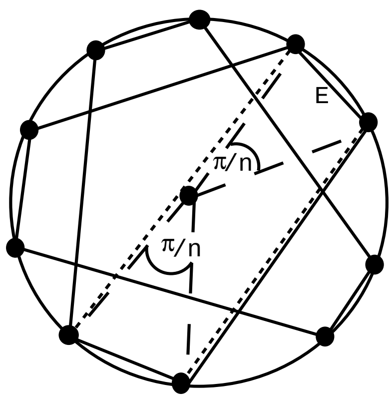

Figure 3: situation 1, $\phi = 0$

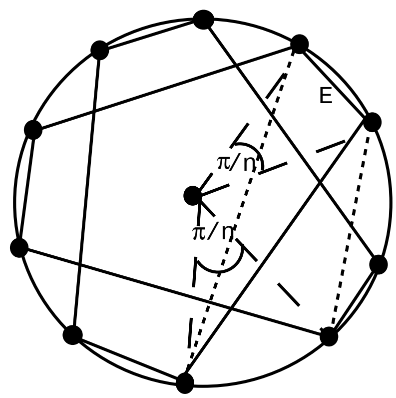

Figure 4: situation 2, $\phi = 0$

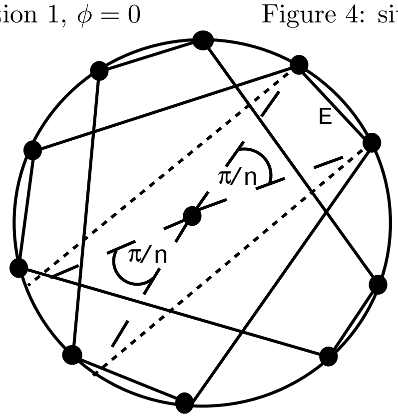

Figure 5: situation 3, $\phi = 0$

The first situation describes the viewing rectangle that contains the vertices of E and any bottom and top respective counterparts on any other side. In this situation, the edges of the viewing rectangle will connect the bottom vertex of E to any other bottom vertex and the top vertex of E to any other top vertex in the circle. In this situation, the viewing rectangle contains E, the long stick F, and two more sticks. In such a region containing E, F and two other sticks,

E can cross at most 3 others.

The second situation is similar to the first except that the viewing rectangle contains the vertices of E and instead, contains the top and bottom respective counterparts on any other side. In this region, E can potentially cross F as well as two other sticks. The short stick E can cross at most 3 others in this region as well.

The third situation describes a viewing rectangle that contains the vertices of E and vertices on any other side that are to the left or right of each existing vertex on the circle. With the exception of F, the short stick E can potentially cross 3 more sticks, two long and one short. However, since one of the long sticks must have the same top or bottom vertex as the short stick, then that vertex must be on the opposite side of E than the short stick's. However if that vertex is on the opposite side of E, then either the short stick or the long stick does not cross E. In this case, E can at most cross 3 others. The drawing below describes an example of this third situation (not including F).

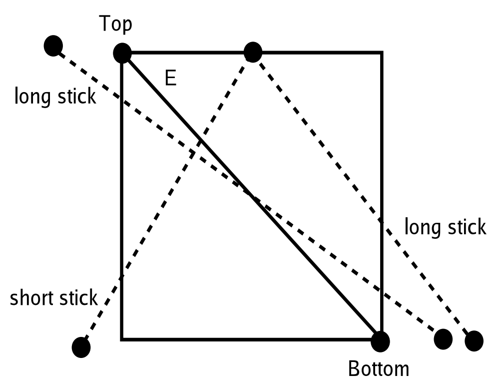

Figure 6: Situation 3, $\phi = 90$, E can cross only 2 sticks in addition to F.

Thus, for every possible arrangement of any short stick, E, there are at most 3 crossings. By Jin's arrangment, there are q short sticks and therefore the contribution to supercrossing number for the short sticks is less than 3q.

Case 2: Let G be a long stick constructed by Jin's equation and viewed from $\phi = 90$. By a similar argument to Case 1, if one creates a viewing rectangle using the two vertices of G as one side, one can constrain a viewing rectangle, such that G and the arc on any other side both make a central angle of $\frac{3\pi}{q}$. The vertices on the other side do not have to be vertices of the knot itself. When this region of equivalent arcs is created, 3 situations arise that describe the greatest possible contribution to the supercrossing number of G. In all three cases G can always cross the short stick, H, that is contained between the 4 vertices of G.

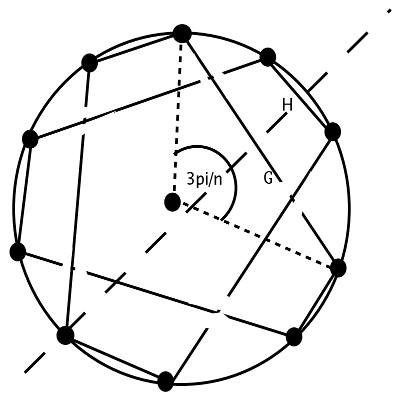

Figure 7: G must cross H from this viewing angle.

These cases follow similarly to Case 1 and, in general, G can potentially cross 5 sticks other than H. However, in any viewing rectangle of G that contains 5 sticks other than G, at least two edges will always be connected to a common vertex. Since for every bottom vertex, except one, there are two top vertices on opposite sides of G and for every top vertex, except one, there are two bottom vertices on opposite sides of G, then one edge must cross G and the other edge must not. Therefore, G can at most cross 4 of those sticks.

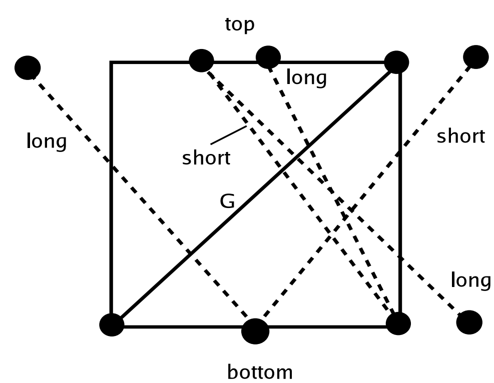

Figure 8: G can at most cross only 4 sticks in addition to H.

For every possible arrangement of any long stick, G, I get at most 5 crossings. By Jin's arrangment, there are q long sticks and therefore the supercrossing number for the short sticks is less than 5q.

By combining these two supercrossing upperbounds and dividing by two to make up for crossings counted twice, the supercrossing number for a $T_{2,q}$ knot when q is odd, and is constructed with 2q sticks is less than $4q$. □

4 The Equivalent Supercrossing Number

Consider that every $T_{2,q}$ knot in this section refers to those constructed by Jin's formula. In his construction, one can create a line, which I call the line of sight, which describes any diameter, such that by looking down it, one can realize the supercrossing number. To do so however, one must construct a new type of viewing rectangles. These rectangles are created by first constructing tangent lines to the circle that are parallel to the line of sight, then drawing two more parallel lines to those which intersect the vertices of any chosen stick. The tangent lines are necessary because they help determine which vertices to focus on in order to construct the additional parallel lines. By looking through the viewing rectangle created by the additional parallel lines, one can observe the exact number of crossings for a certain stick at fixed angles of ϕ and θ. At certain angles however, there exist degenerate cases for specific lines of sight.

4.1 The Degenerate Cases:

The supercrossing number for $T_{2,q}$ is recognized at $\phi = 90$. However, special restrictions on θ are necessary to make this argument. If a line of sight is drawn such that its parallel lines of tangency intersect points that lie on the knot's vertices, then when $\phi = 90$, the knot will appear to have an overlap of vertices and therefore crossings are not recognized as they should be. In Jin's construction, a degenerate case occurs whenever $\theta = 90 + \frac{i\pi}{q}$, where i is an integer. These degenerate cases occur for both long sticks and short sticks.

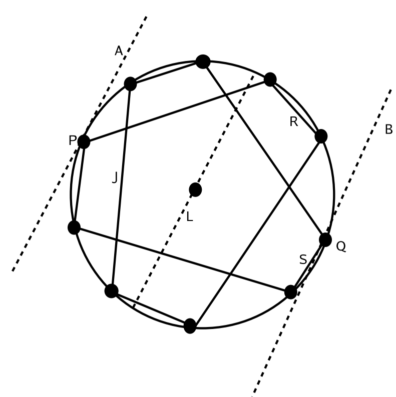

Figure 9: A Degenerate Case, when vertices overlap.

4.2 Short Sticks

Lemma 4 *In any $T_{2,q}$ knot by Jin's construction, when q is odd, $q \geq 5$ and $\phi = 90$, by any given non-degenerate line of sight,exactly 2 short sticks will each cross 2 other sticks and exactly 1 short stick will cross 0 other sticks. The rest of the short sticks will each cross exactly 3 other sticks.*

Proof:

Construct a circle O, with a $T_{2,q}$ knot, K, on it. Form a line of sight,L, and construct the two tangent lines A and B that are parallel to L. Under Jin's Construction, one of the points of tangency, call it P, that either A or B create will be part of an arc $\frac{\pi}{q}$ which is between two vertices that connect to form a short stick, whereas the other point of tangency, call it Q is part of an arc $\frac{\pi}{q}$ between two vertices that are not connected. Next create a viewing rectangle by constructing two lines parallel to L that intersect the vertices of the short stick that contains P in the arc between its vertices. In this viewing rectangle, that short stick will not cross any other sticks.

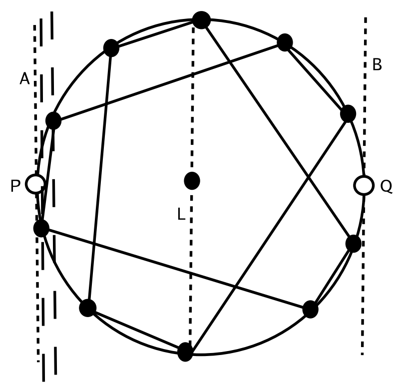

Figure 10: According the line of sight L, the short stick in the viewing rectangle cannot cross any others.

Exactly π degrees around the circle from P, Q is subtended by an arc that contains two vertices, each of which must connect to some other vertices to form two short sticks, call them R and S. Since these neccessary connecting vertices must be consecutive vertices around the circle, then one of the vertices must connect to the vertex just clockwise of it and the other vertex must connect to the one just counterclockwise of it. Construct four parallel lines to the line of sight, two that go through the vertices of R and two that go through the vertices of S. Although R and S cannot cross their adjacent edges, they must cross each other and each long stick that is parallel to the other small stick. In each of their viewing rectangles, R and S each cross exactly two other sticks, while the rest of the small sticks cross 3 each at $\phi = 90$.

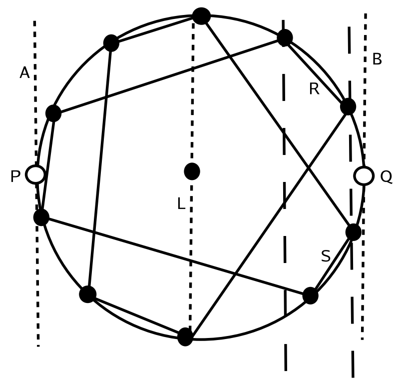

Figure 11: According to L, R and S each cross exactly 2 other sticks in their viewing rectangles.

4.3 Long Sticks

Lemma 5 *In any $T_{2,q}$ knot, by Jin's construction, when q is odd, $q \geq 5$ and $\phi = 90$, by any given non-degenerate line of sight,exactly 2 long sticks will each cross 3 other sticks and exactly 1 long stick will cross 2 other sticks. The rest of the long sticks will each cross exactly 5 other sticks.*

Proof:

Follow the construction by the previous lemma. Consider the long stick, call it J, that is parallel to the 0-crossing short stick. By constructing a viewing rectangle, which intersect the vertices of J, one can see 3 potential crossings. However, one of these sticks is always adjacent to J, so J cannot cross it. Therefore J will cross the two sticks adjacent to the 0-crossing short stick.

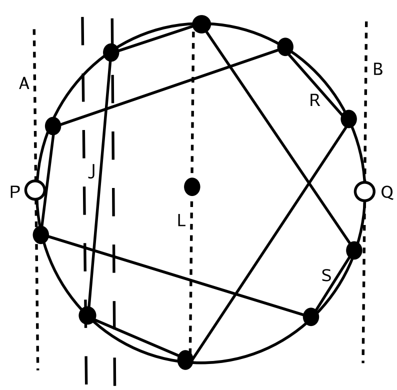

Figure 12: J will cross exactly 2 other sticks.

Next, consider two long sticks, call them M and K, that are each parallel to the two short sticks with 2 crossings each, described in the previous lemma. Although M and K are parallel to one of the short sticks each, the other short stick that they are not parallel to, they are adjacent to. In viewing rectangles under any non-degenerate line of sight, at most 4 crossings are apparent for each long stick. However, M and K cannot cross either of their adjacent short sticks, therefore they must cross 3 other sticks, while the rest of the long sticks cross 5 each at $\phi = 90$.

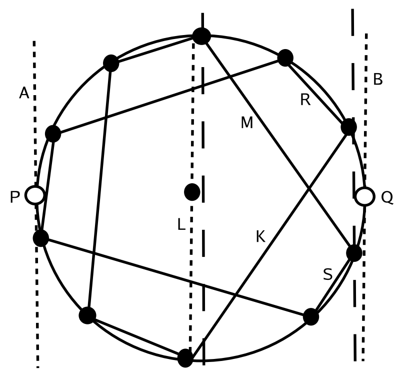

Figure 13: M and K will cross exactly 3 other sticks each.

The following diagram is a representation of the leftmost and rightmost sticks of any $T_{2,q}$ knot under Jin's construction, with q odd and $\phi = 90$. On one side, 1 0-crossing and 1 2-crossing sticks exist while on the other side, 2 2-crossing and 2 3-crossing sticks exist. Since these special cases exist for all of Jin's knots, their supercrossing numbers can be quantified and found by a formula.

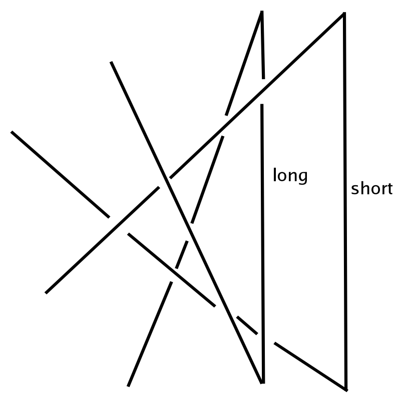

Figure 14: one side of a $T_{2,q}$ knot, $\phi = 90$

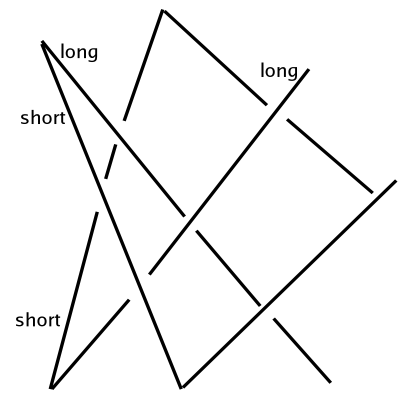

Figure 15: the other side of a $T_{2,q}$ knot, $\phi = 90$

4.4 Supercrossing Number and Index of $T_{2,q}$

Theorem 1 *Under Jin's Construction, $scr(T_{2,q}) = 4q - 6$, when q is odd and $q \geq 5$.*

Proof: This comes from the fact that, for $T_{2,q}$ from any non-degenerate line of sight, and at $\phi = 90$, q-3 long sticks cross exactly 5 other sticks and q-3 short sticks cross exactly 3 other sticks. Additionally by Lemma 4, 4 more crossings exist for the short sticks because of the 1-0 crossing and 2- 2 crossings short sticks. Dividing by two to dismiss crossings counted twice, the supercrossing number of short sticks is $(4(q-3)+4)/2$. Similarly, Lemma 5, 8 more crossings exist for the long sticks because of the 1-2 crossing and 2-3 crossing long sticks. Dividing by two to dismiss crossings counted twice, the supercrossing number of long sticks is $(5(q-3)+8)/2$. By adding these quantities together, the supercrossing number of Jin's $T_{2,q}$ knots is $4q - 6$. □

Corollary 1 *$scr[T_{2,q}] \leq 4q - 6$.*

Proof:

Since Torus knots of the form $T_{2,q}$, configured with 2q sticks can be constructed to have a supercrossing number of 4q-6, it follows that any Torus knot of the form $T_{2,q}$ has an index less than or equal to 4q-6.□

This improves the upperbound greatly, since not only is it linear but it comes directly from an exact stick number of $T_{2,q}$.

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