The Linear Cutwidth of Complete Bipartite and Tripartite Graphs

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Abstract

This paper looks at complete bipartite graphs, $K_{m,n}$, and complete tripartite graphs, $K_{r,s,t}$. The main focus is cutwidths. By looking at the linear embedding of these graphs, each cut can be minimized, resulting in the linear cutwidth.

1 Introduction

1.1 Complete Bipartite and Tripartite Graphs

A graph consists of vertices and edges, where one edge connects two vertices. When looking at a diagram of a graph, dots represent the vertices and lines represent the edges. There are many different types of graphs. In this paper, we focus on complete bipartite graphs and complete tripartite graphs. We will start by defining a complete bipartite graph, denoted $K_{m,n}$. A bipartite graph consists of two sets of vertices A and B, where $|A| = m$ and $|B| = n$. The vertices of one set, A, are connected only to the vertices of the other set, B. To make it a complete bipartite graph, each vertex of A is connected to all the vertices in B. As shown in Figure 1, the vertices of A are on the left and the vertices of B are on the right. Each vertex of A is connected to all of the vertices of B.

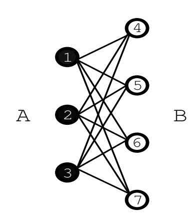

Figure 1: $K_{3,4}$

This outline can be extended to a complete tripartite graph, denoted $K_{r,s,t}$. In a tripartite graph, there are three sets of vertices A, B, and C, where $|A| = r$, $|B| = s$, and $|C| = t$. The vertices of one set, A, are connected only vertices of the other two sets, B and C. To make it a complete tripartite graph, all the vertices from one set are connected to all the vertices of the other two sets. Figure 2 is an example of complete tripartite graph.

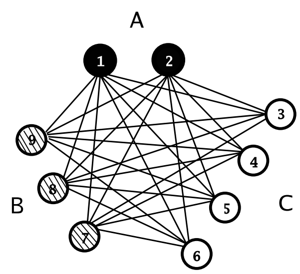

Figure 2: $K_{2,3,4}$

1.2 Linear Embedding

The vertices of a complete n-partite graph can be arranged in a many different ways. The arrangement we are going to focus on is a linear embedding. This is a simple arrangement of all of the vertices on a single line, as shown in Figure 3. The vertices that were connected in the original graph will also be connected in this arrangement.

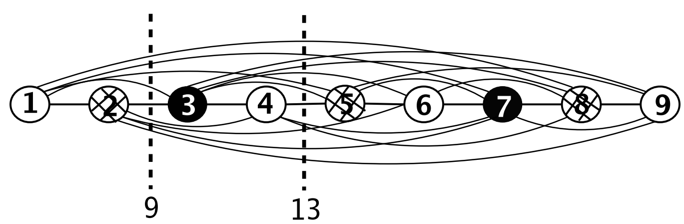

Figure 3: $K_{2,3,4}$

The regions of a linearly embedded graph, denoted $(a, x-a)$ for $K_{m,n}$ and $(a, b, x-a-b)$ for $K_{r,s,t}$, is the area between two adjacent vertices, where x represents the number of vertices to the left of the cut, and there are a vertices from A, $(x-a)$ or b vertices from B, and $(x-a-b) = c$ vertices from C. The cut of a region for a complete bipartite linear embedding, denoted $cut(a, x-a)$, is the number of edges that cross through the area between two vertices, which can be found using:

$$cut(a, x-a) = a(n-(x-a)) + (x-a)(m-a).$$

For example, in Figure 4, the cut between vertices 1 and 2 is 2. The cut of a region for a complete tripartite linear embedding, denoted $cut(a, b, x-a-b)$, is the number of edges that cross through the area between two vertices, which can be found using:

$$\begin{aligned} cut(a, b, x-a-b) &= a[(s-b)+(t-(x-a-b))] + b[(r-a)+(t-(x-a-b))] \\ &\quad +(x-a-b)[(r-a)+(s-b)]. \end{aligned}$$

For example, in Figure 3, the cut between vertices 2 and 3 is 9.

The maximum cut of a linear embedding is the the region with the most edges crossing through it. As you can see in Figure 3 the maximum cut is 13, and in Figure 4, the maximum cut is 3. The linear cutwidth of a graph, is the smallest maximum cut for all the different arrangements of the vertices within a linear embedded graph, which is what we are looking at throughout this paper.

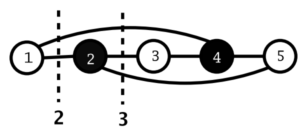

Figure 4: $K_{2,3}$

Many numbers will need to be rounded to their nearest whole numbers. The function we will be using is $[x] = \lfloor x + .5 \rfloor$.

2 Background

Graph theory can be applied to many real world applications. Some of these include circuit layout, code design, and networking. Having the cutwidth of a graph is helpful when working with some of these applications, particularly with optimally arranging a network or circuit. A few people have worked with the cutwidth of graphs. The equations for the linear cutwidth and the cyclic cutwidth of the complete graph K_n were discovered by F. Rios[10]. D. Clarke[6] modified H. Schroder's[11] work on the cyclic cutwidth of two-dimensional mesh, $P_m \times P_n$, and got results for $m \geq n \geq 3$. These results were extended to three-dimensional mesh,$P_2 \times P_2 \times P_n$ by V. Sciortino[12]. Results were found for the cyclic cutwidth for trees by J. Chavez and R. Trapp [2]. Together, they also made a conjectured on the cyclic cutwidth for an n-dimensional cube, Q_n[4]. Although the conjecture has not been proven for Q_n, Q_3 and Q_4 have been proven by B. James[8], Q_5 has been proven by R. Aschenbrenner[1], and Q_6 has been proven by C. Castillo[3]. M. Holben[7] discovered equations for the cyclic cutwidth of a complete bipartite graph, $K_{m,n}$. Although many people have studied the cutwidth of many different types of graphs, M. Johnson[9] has developed the most recent ideas behind the linear cutwidth of complete bipartite graphs.

2.1 Johnson's Arrangement of Vertices

Johnson came up with a way to arrange the vertices of a graph $K_{m,n}$ along the linear embedding, such that the cutwidth for a particular graph would be minimized. He begins this process by letting x be the number of vertices to the left of a cut (in Figure 4 $x = 2$). To find the number of vertices from from A that are to the left of the cut, he used the equation $[\frac{xm}{m+n}]$, and rounded it to the nearest whole number. To find the number of vertices from B that are to the left of the cut, he used the equation $x - [\frac{xm}{m+n}]$. In other words, the vertices from B took the remaining spots to the left of the cut. If you do this for every cut, starting with the left most cut and continuing to its closest cut to it's right, you will end up with Johnson's arrangement.

2.2 Johnson's Linear Cutwidth of $K_{m,n}$

From Johnson's arrangement of vertices within a linearly embedded complete bipartite graph, he comes up with two simple equations on how to find the cutwidth of complete bipartite graphs. To find these equations, he uses the fact that his arrangement results in the maximum cut of all the regions being the center cut. Knowing this he can determine the number of vertices from A and B that will be to the left of the center cut. For example, if m and n were both even, then the number of vertices from A would be $\frac{m}{2}$ and the number of vertices from B would be $\frac{n}{2}$. He uses these numbers along with the equation $cut(a, b)$, where $a \in A$ and $b \in B$, to find the cut of the middle region. He looks at four different cases, where m and n are even or odd, and after simplifying his equation he concluded for any complete bipartite graph $K_{m,n}$ where $m \leq n$,

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & \text{for } mn \text{ even} \\ \frac{mn+1}{2} & \text{for } mn \text{ odd.} \end{cases}$$

Johnson was successful at finding a way to get the linear cutwidth of complete bipartite graphs, there are many other complete n-partite graphs to which the cutwidth is unknown. Thus it is important to concentrate on generalizing the cutwidth of complete n-partite graphs.

3 Linear Cutwidth of Complete Bipartite Graphs

Even though Johnson's arrangement of vertices gives the cutwidth, it does not minimize every region's cut. Together with J. Chavez and E. Hartung, we discovered a new way to arrange the vertices that does minimize the cut of each region.

Theorem 1 (Bowles-Chavez-Hartung (BCH1)): *Let $K_{m,n}$ be a complete bipartite graph with two sets of vertices A and B, where $|A| = m$ and $|B| = n$, and let $m \leq n$. Then the cut of each region of a linear embedding for $K_{m,n}$ is minimized by placing $\frac{2x+m-n}{4}$ vertices from A to the left of the cut.*

Proof Let x be the number of vertices to the left of a cut. Given x, suppose there are a vertices from A to the left of the cut. Thus, there are $x - a$ vertices from B to the left of the cut. From this we can conclude that on the right side of the cut there are $m - a$ vertices from A and $n - (x - a)$ vertices from B. We know that the cut of region $(a, x - a)$ is

$$\begin{aligned} cut(a, x-a) &= a(n-(x-a)) + (x-a)(m-a) \\ &= an - ax + a^2 + mx - ax - am + a^2 \\ &= 2a^2 + a(n - 2x - m) + mx. \end{aligned}$$

Let $f(a) = 2a^2 + a(n - 2x - m) + mx$. Notice that $f(a)$ is a continuous function of $a \in \mathbb{R}$, and that $f(a) = cut(a, x - a)$ for $0 \leq a \leq m$ and $a \in \mathbb{Z}$. Since $f(a)$ is a positive quadratic, we know that it is a parabola opening upward. Thus, the minimum of $f(a)$ is found by taking its derivative and setting it equal to zero. So,

$$\begin{aligned} f'(a) &= 4a + n - 2x - m = 0 \\ \Rightarrow a &= \frac{m + 2x - n}{4}. \end{aligned}$$

If a is not be an integer, it is rounded to the nearest whole number, denoted $[a]$. We then place $[a]$ vertices from A to the left of the cut. Thus for all of the regions of the linear embedding of $K_{m,n}$, its cut is minimized by placing $\left[\frac{m+2x-n}{4}\right]$ vertices from A to the left of each cut. ∎

This new arrangement results in the next corollary.

Corollary 1 *Let $K_{m,n}$ be a complete bipartite graph whose linear embedding is arranged by the BCH1 theorem. Then the maximum cut will occur when $x = \frac{m+n}{2}$ for $m + n$ even and when $x = \frac{m+n-1}{2}$ and $x = \frac{m+n+1}{2}$ for $m + n$ odd. Also, the cuts to the left of the middle cut will be strictly increasing and the cuts to the right of the middle cut will be strictly decreasing.*

Proof Let

$$\begin{aligned} cut(a, x-a) &= 2a^2 + a(n - 2x - m) + mx \\ &= 2a^2 - 4a^2 + mx (\text{since } a = \tfrac{2x+m-n}{4} \text{ gives the minimum cut}) \\ &= -2a^2 + xm \\ &= -2 \cdot \left(\frac{2x + m - n}{4}\right)^2 + xm \\ &= -\frac{1}{8} \cdot (2x + m - n)^2 + xm. \end{aligned}$$

Let $f(x) = -\frac{1}{8} \cdot (2x+m-n)^2 + xm$ be a continuous function of $x \in I\!R$ and notice that $-\frac{1}{8} \cdot (2x+m-n) + xm$ gives the cut of the region for $1 \leq x \leq m+n$. Since $f(x)$ is a negative quadratic, we know that it is a parabola opening downward. Thus, by using $f'(x) = 0$ we can show that the maximum cut occurs at the middle region where $x = \frac{n-m}{2}$ for $m+n$ even and where $x = \frac{n-m-1}{2}$ for $m+n$ odd.

$$
\begin{aligned}
f'(x) &= -\frac{1}{4} \cdot (2x+m-n) \cdot 2 + m \\
&= -\frac{1}{2} \cdot (2x+m-n) + m \\
&= -x - \frac{m}{2} + \frac{n}{2} + m \\
&= \frac{m+n}{2} - x = 0
\end{aligned}
$$

Observe that $\frac{m+n}{2} - x > 0$ when $1 \leq x < \frac{m+n}{2}$, thus $f(x)$ is increasing, and $\frac{m+n}{2} - x < 0$ when $\frac{m+n}{2} \leq x \leq m+n$, thus $f(x)$ is decreasing. So for $m+n$ even, the maximum cut occurs at $\frac{m+n}{2}$, but when $m+n$ is odd, $\frac{m+n}{2}$ is not an integer, but is equally spaced between the integers $\frac{m+n-1}{2}$ and $\frac{m+n+1}{2}$, so the maximum cut of the graph occurs at these two points. ∎

Once we have the arrangement of the vertices we can use this to find the cutwidth of the complete bipartite graph.

Theorem 2 (Bowles-Chavez-Hartung (BCH2)) *Let $K_{m,n}$ be a complete bipartite graph. Then*

$$
lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & \textit{for } mn \textit{ even} \\ \frac{mn+1}{2} & \textit{for } mn \textit{ odd} \end{cases}
$$

Proof We have shown that the linear embedding given by BCH Theorem 1 minimizes the cut of each region and that the center cut is the maximum of the linear embedding. Thus the $lcw(K_{m,n})$ occurs when $x = \frac{m+n}{2}$ for $m+n$ even and when $x = \frac{m+n-1}{2}$ and $x = \frac{m+n+1}{2}$ for $m+n$ odd. So we will look at each case to find the $lcw(K_{m,n})$.

Case 1: Let $m+n$ be even, then $x = \frac{m+n}{2}$. Place $\lceil \frac{2x+m-n}{4} \rceil$ vertices from A to the left of the center cut. Substituting $\frac{m+n}{2}$ for x gives $\lceil \frac{m+n+m-n}{4} \rceil = \lceil \frac{m}{2} \rceil$. So $a = \frac{m}{2}$ when m is even and $a = \frac{m+1}{2}$ when m is odd. Since $m+n$ is even, we know that when m is even, n is even, and when m is odd, n is odd. Therefore, the number of vertices from B on the left is

$$
\begin{cases} x - \frac{m}{2} = \frac{n}{2} & \text{for } n \text{ even} \\ x - \frac{m+1}{2} = \frac{n-1}{2} & \text{for } n \text{ odd} \end{cases}
$$

Taking the cut of $(\frac{m}{2}, \frac{n}{2})$ gives:

$$
\begin{aligned}
cut(\frac{m}{2}, \frac{n}{2}) &= \frac{m}{2} \cdot \frac{n}{2} + \frac{n}{2} \cdot \frac{m}{2} \\
&= \frac{mn}{2}.
\end{aligned}
$$

and taking the cut of $(\frac{m+1}{2}, \frac{n-1}{2})$ gives:

$$\begin{aligned} cut(\frac{m+1}{2}, \frac{n-1}{2}) &= \frac{m+1}{2} \cdot \frac{n+1}{2} + \frac{n-1}{2} \cdot \frac{m-1}{2} \\ &= \frac{mn+1}{2} \end{aligned}$$

Case 2: Let $m+n$ be odd. WLOG, let $x = \frac{m+n-1}{2}$. Place $[\frac{2x+m-n}{4}]$ vertices from A to the left of the center cut. Substituting $\frac{m+n-1}{2}$ for x gives $[\frac{m+n-1+m-n}{4}] = [\frac{2m-1}{4}] = [\frac{m}{2} - \frac{1}{4}]$. So $a = \frac{m}{2}$ when m is even and $a = \frac{m-1}{2}$ when m is odd. Since $m+n$ is odd, we know that when m is even, n is odd, and when m is odd, n is even. Therefore, the number of vertices from B on the left is

$$\begin{cases} x - \frac{m}{2} = \frac{n-1}{2} & \text{for } n \text{ odd} \\ x - \frac{m+1}{2} = \frac{n}{2} & \text{for } n \text{ even} \end{cases}$$

Taking the cut of $(\frac{m}{2}, \frac{n-1}{2})$ gives

$$\begin{aligned} cut(\frac{m}{2}, \frac{n-1}{2}) &= \frac{m}{2} \cdot \frac{n+1}{2} + \frac{n-1}{2} \cdot \frac{m}{2} \\ &= \frac{mn+m}{4} + \frac{mn-m}{4} \\ &= \frac{mn}{2}. \end{aligned}$$

and taking the cut of $(\frac{m-1}{2}, \frac{n}{2})$ gives

$$\begin{aligned} cut(\frac{m-1}{2}, \frac{n}{2}) &= \frac{m-1}{2} \cdot \frac{n}{2} + \frac{n}{2} \cdot \frac{m+1}{2} \\ &= \frac{mn-n}{4} + \frac{mn+n}{4} \\ &= \frac{mn}{2}. \end{aligned}$$

Therefore,

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & \text{for } mn \text{ even} \\ \frac{mn+1}{2} & \text{for } mn \text{ odd.} \end{cases} \quad \blacksquare$$

As you can see our optimal values match that of Johnson's, which was expected. However, from this new arrangement we can minimize every cut. This is helpful when looking at generalizing the linear cutwidth of complete n-partite graphs, which is explained later in the paper. Therefore, this model is broader in its applications.

4 Linear Cutwidth of Complete Tripartite Graphs

In this section, we minimize the cut of every region, and find the linear cutwidth of the linear embedding of a complete tripartite graph $K_{r,s,t}$. To do this, we will refer to the middle and outer regions of the graph. The middle region consists of r sets of three vertices, each set including one black, one white, and one gray vertex. The outside consists of the remaining vertices.

Theorem 3 *(*Bowles-Chavez-Hartung (BCH3)) *Let $K_{r,s,t}$ be a complete tripartite graph with three sets of vertices A, B, and C, where $|A| = r$, $|B| = s$, and $|C| = t$, and let $r \leq s \leq t$. To minimize each cut of the linear embedding for $K_{r,s,t}$, the middle and outer sections of the graph are minimized independently. The middle cuts are minimized by placing $\frac{2x+2r-s-t}{6}$ vertices from A, $\frac{2x+2s-r-t}{6}$ vertices from B, and $\frac{2x+2t-r-s}{6}$ vertices from C to the left of each cut. The outer sections are minimized according to the BCH arrangement for complete bipartite graphs.*

Proof This proof is separated into three sections: the minimizing of the middle region, the minimizing of the outer regions, and the placement of the middle region.

Minimizing the Middle Region
Let x be the number of vertices to the left of a cut. Given x, suppose there are a vertices from A, b vertices from B, and $c = x - a - b$ vertices from C to the left of the cut. From this we can conclude that on the right side of the cut there are $r - a$ vertices from A, $s - b$ vertices from B, and $t - (x - a - b)$ vertices from C. We know that the cut of region $(a, b, x - a - b)$ is

$$\begin{aligned}
cut(a, b, x - a - b) &= a[(s - b) + (t - (x - a - b)] + b[(r - a) + (t - (x - a - b))] \\
&\quad +(x - a - b)[(r - a) + (s - b)] \\
&= as - ab + at - ax + a^2 + ab + br - ba + bt - bx + ba + b^2 \\
&\quad +xr - xa + xs - xb - ar + a^2 - as + ab - br + ab - bs + b^2 \\
&= 2a^2 + 2b^2 + a(t - r - 2x) + b(t - s - 2x) + 2ab + x(r + s)
\end{aligned}$$

Let $f(a, b) = 2a^2 + 2b^2 + a(t-r-2x) + b(t-s-2x) + 2ab + x(r+s)$. Notice that $f(a, b)$ is a continuous function of $a, b \in \mathbb{R}$, and that $f(a, b) = cut(a, b, x-a-b)$ for $0 \leq a \leq r$ and $a \in \mathbb{Z}$ and for $0 \leq b \leq s$ and $b \in \mathbb{Z}$. Since $f(a, b)$ is a positive paraboloid, its opening is upward. Thus, we can find the minimum of $f(a, b)$ by taking its derivative and setting it equal to zero. So,

$$\begin{aligned}
\frac{\partial f(a, b)}{\partial a} &= 4a + (t - r - 2x) + 2b = 0 && (1) \\
\Rightarrow 2b &= -4a - t + r + 2x. && (2) \\
\frac{\partial f(a, b)}{\partial b} &= 4b + (t - s - 2x) + 2a = 0 && (3) \\
\Rightarrow 2a &= -4b - t + s + 2x. && (4)
\end{aligned}$$

Substituting the value of $2b$ from (2) into (3) and gives

$$0 = 2(-4a - t + r + 2x) + t - s - 2x + 2a$$

$$
\begin{aligned}
\Rightarrow 6a &= 2x + 2r - s - t \\
\Rightarrow a &= \frac{2x + 2r - s - t}{6}.
\end{aligned}
$$

Substituting the value of $2a$ from (4) into (1) gives

$$
\begin{aligned}
0 &= 2(-4b - t + s + 2x) + t - r - 2x + 2b \\
\Rightarrow 6b &= 2x + 2s - r - t \\
\Rightarrow b &= \frac{2x + 2s - r - t}{6}.
\end{aligned}
$$

Knowing a and b we can get c:

$$
\begin{aligned}
c &= x - a - b \\
&= x - \left[\frac{2x + 2r - s - t}{6}\right] - \left[\frac{2x + 2s - r - t}{6}\right] \\
&= \frac{2x + 2t - r - s}{6}.
\end{aligned}
$$

If a, b, and c are not integers, it is rounded to the nearest whole number, denoted $[a]$, $[b]$, and $[c]$. Now given a, b, and c, let $1 \leq a \leq r$ since we are only concerned with the middle of the graph where all of the sets are active. If we minimize the cuts of the middle region, they will not be affected by the arrangement of the vertices on the outside as long as we know how many vertices from each set are on the left and right of the middle region. Given the middle region and the number of vertices to the left and right of it, we know that each vertex in the middle will be connected to the same number of edges regardless of the structure of the outside vertices. Therefore we can minimize the inside independently of the arrangement of the outside vertices.

If a chart of the values a, b, and c is constructed, each vertex may not have a unique position. Below in Figure 5, the x represents the number of vertices to the left of a cut. The variables a, b and c represent the number of vertices from each set A, B, C to the left of a cut. Every time a, b or c increases by one, a new vertex is added to that spot. For example, in Figure 5, at vertex 5 ($x = 5$), there are no vertices from A to the left. However, at vertex 6, there is one vertex from set A, so the sixth vertex will be black.

Since $1 \leq a \leq r$, there will be r groups of three in the middle region. Beginning with the first position where $a = 1$, each group of three vertices will have one black, one white and one gray vertex, but the arrangement of those within the group of three is not necessarily unique. So let the three positions of each group be denoted x_1, x_2, and x_3. There are three cases:

Case 1: Every black, white and gray has a unique position. For every x_i the value of only the vertex from a, b, or c will change and it will be consistent throughout the r groups. In this case, $x_i = a + b + c$ for all x_i while $1 \leq a \leq r$. For example, in $K_{1,4,7}$, the order of the middle must be black, white,gray.

3r

x_1 x_2 x_3

	x	...	3	4	5	**6**	**7**	**8**	9	10	11	...
●	a		0	0	0	**1**	**1**	**1**	2	2	2	
⊗	b		1	1	2	**2**	**2**	**3**	3	3	4	
○	c		2	3	3	**3**	**4**	**4**	4	5	5	

Figure 5: $K_{1,4,7}$

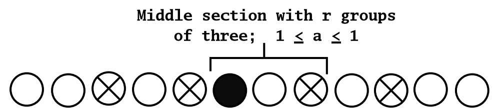

Figure 6: $K_{1,4,7}$

Case 2: Either black, white or gray will have a unique position consistently throughout the r groups. For the remaining two sets, the vertices from those sets will be interchangeable within each group of three. Again, remember that for each group of three, we are denoting the three positions x_1, x_2 and x_3. In this case, either at x_1, $a+b+c = x_1+1$ or at x_2, $a+b+c = x_2+1$. For example, in $K_{4,8,11}$, the black and the gray vertices can be interchanged between positions x_1 and x_2, as shown in Figure 8, but the white vertices are always in position x_3.

3r

x_1 x_2 x_3

	x	...	4	5	6	7	8	9	**10**	**11**	**12**	13	14	15	16	17	18	19	20	21	...
●	a		0	0	0	1	1	1	**2**	**2**	**2**	3	3	3	4	4	4	5	5	5	
⊗	b		2	2	2	3	3	3	**4**	**4**	**4**	5	5	5	6	6	6	7	7	7	
○	c		3	3	4	4	4	5	**5**	**5**	**6**	6	6	7	7	7	8	8	8	9	

Figure 7: $K_{4,8,11}$

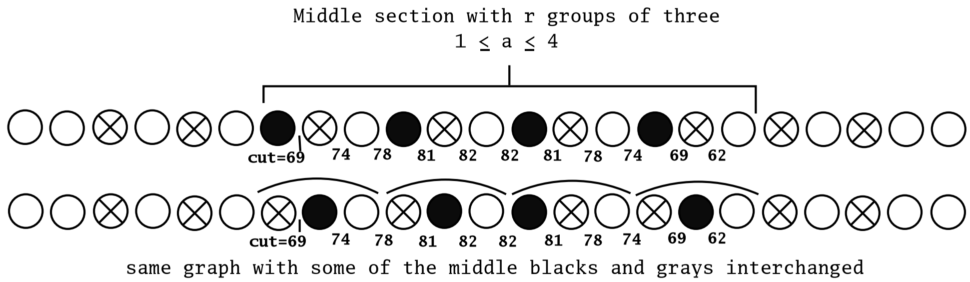

Figure 8: $K_{4,8,11}$

Case 3: The black, gray and white vertices in each group of three are all interchangeable. In this case, at x_1, $a+b+c = x_1+2$ and at x_2, $a+b+c = x_2+1$. $K_{3,9,13}$ is an example of this where the whites, grays and blacks can be in any position within groups of three.

3r

$\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3$

	x	...	7	8	9	10	11	12	**13**	**14**	**15**	16	17	18	19	20	21	...
●	a		0	0	0	1	1	1	**2**	**2**	**2**	3	3	3	4	4	4	
⊗	b		3	3	3	4	4	4	**5**	**5**	**5**	6	6	6	7	7	7	
○	c		5	5	5	6	6	6	**7**	**7**	**7**	8	8	8	9	9	9	

Figure 9: $K_{3,9,13}$

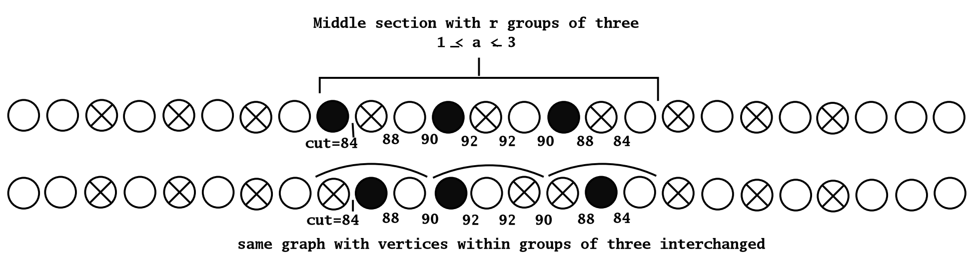

Figure 10: $K_{3,9,13}$

So when it is not clear whether a, b or c changes first, the cuts of the regions are equivalent regardless of which arrangement is chosen as long as there is a vertex from each set within each group of three. Thus the linear embedding of a complete tripartite graph $K_{r,s,t}$ is minimized within the region where $1 \leq a \leq r$ by placing $\frac{2x+2r-s-t}{6}$ vertices from A, $\frac{2x+2s-r-t}{6}$ vertices from B, and $\frac{2+2t-r-s}{6}$ vertices from C to the left of each cut.

Minimizing the Outer Regions

We will now consider the outer vertices. The chart of a complete tripartite graph gives the number of vertices from set B and C to the right and left of the middle region. The number of vertices from B and C to the left of the middle region is given by b and c at the last position where $a = 0$. For example, in Figure 5, when $x = 5$, there are 2 gray vertices and 3 white vertices. The number of vertices from B to the right of the middle region is given by $s - b$ where b is the first position where $a = r + 1$. The number of vertices from C to the right is given by $t - c$ where c is the first position where $a = r + 1$. For example, in Figure 5, when $x = 9$, $b = 3$ and $c = 4$, so there is 1 vertex from B and there are 3 vertices from C to the right of the middle. Also, we have r gray, r white and r black vertices in the middle region. As previously stated, the arrangement of the middle does not affect the cuts of the outside. Without loss of generality, we look at the vertices to the left of the middle region. Let i be the number of vertices to the left. Observe that each white on the outside is connected to r gray and r black vertices from the middle region and that each gray on the outside is connected to r white and r black vertices from the middle region. This results in the middle vertices contributing $i \cdot 2r$ to the cut of each region. The same is true of the vertices to the right of the middle region where i is the number of vertices to the right of each cut. This applies to any arrangement of the outside vertices. But the outside vertices make up the complete bipartite graph $K_{s-r,t-r}$, thus to minimize the cuts along outside vertices, we can use the BCH1 arrangement.

Placing the Middle Region Within the Bipartite Graph

We have shown that the middle region can be minimized independently of the outer regions and the outer regions can be minimized independently of the middle region. Since we know the number of vertices to the left of the middle region, we can place the minimized middle region after this number of vertices of the minimized bipartite graph. Notice that the left outer region consists of $\lceil \frac{s-r}{2} \rceil$ vertices from B and $\lceil \frac{t-r}{2} \rceil$ vertices from C and the right outer region consists of $\lfloor \frac{s-r}{2} \rfloor$ vertices from B and $\lfloor \frac{t-r}{2} \rfloor$ vertices from C. This results in the minimal cut for each region of the complete tripartite graph $K_{r,s,t}$. $\blacksquare$

This new arrangement results in the next corollary.

Corollary 2 *Let $K_{r,s,t}$ be a complete tripartite graph whose linear embedding is arranged by BCH3. Then the maximum cut will occur when $x = \frac{r+s+t}{2}$ for $r+s+t$ even and when $x = \frac{r+s+t-1}{2}$ and $x = \frac{r+s+t+1}{2}$ for $r+s+t$ odd.*

Proof Let

$$\begin{aligned}
cut(a,b,x-a-b) &= 2a^2+2b^2+a(t-r-2x)+b(t-s-2x)+2ab+x(r+s) \\
&= 2\left(\frac{2x+2r-s-t}{6}\right)^2+2\left(\frac{2x+2s-r-t}{6}\right)^2 \\
&\quad +\left(\frac{2x+2r-s-t}{6}\right)(t-r-2x)+\left(\frac{2x+2s-r-t}{6}\right)(t-s-2x) \\
&\quad +2\left(\frac{2x+2r-s-t}{6}\right)\left(\frac{2x+2s-r-t}{6}\right)+x(r+s) \\
&\quad (\text{since } a=\tfrac{2x+2r-s-t}{6} \text{ and } b=\tfrac{2x+2s-r-t}{6}) \\
&= \frac{2xt}{3}-\frac{r^2}{6}+\frac{2xs}{3}+\frac{2xr}{3}-\frac{2x^2}{3}+\frac{rs}{6}+\frac{rt}{6}-\frac{s^2}{6}+\frac{st}{6}-\frac{t^2}{6}.
\end{aligned}$$

Let $f(x)=\frac{2xt}{3}-\frac{r^2}{6}+\frac{2xs}{3}+\frac{2xr}{3}-\frac{2x^2}{3}+\frac{rs}{6}+\frac{rt}{6}-\frac{s^2}{6}+\frac{st}{6}-\frac{t^2}{6}$ be a continuous function of $x\in I\!R$ and notice that $\frac{2xt}{3}-\frac{r^2}{6}+\frac{2xs}{3}+\frac{2xr}{3}-\frac{2x^2}{3}+\frac{rs}{6}+\frac{rt}{6}-\frac{s^2}{6}+\frac{st}{6}-\frac{t^2}{6}$ gives the cut of the region for $1\le x\le r+s+t$. By using $f'(x)$ we can show that the maximum cut occurs at the middle region where $x=\frac{r+s+t}{2}$ for $r+s+t$ even and where $x=\frac{r+s+t-1}{2}$ for $r+s+t$ odd.

$$\begin{aligned}
f'(x) &= \frac{-4x+2r+2s+2t}{3}=0 \\
x &= \frac{r+s+t}{2}.
\end{aligned}$$

So for $r+s+t$ even, the maximum cut occurs at $\frac{r+s+t}{2}$, but when $r+s+t$ is odd, $\frac{r+s+t}{2}$ is not an integer, but is equally spaced between the integers $\frac{r+s+t-1}{2}$ and $\frac{r+s+t+1}{2}$, so the maximum cut of the graph occurs at these two points. ∎

Once we have the arrangement of the vertices and where the maximum occurs within the arrangement, we can use this to find the cutwidth of the complete tripartite graph.

Theorem 4 (Bowles)*Let $K_{r,s,t}$ be a complete tripartite graph. Then*

$$lcw(K_{r,s,t})=\begin{cases}\frac{rs+rt+st}{2} & \text{for two or more } s,r,t \text{ even}\\ \frac{rs+rt+st+1}{2} & \text{otherwise}\end{cases}$$

Proof We have shown that the linear embedding given by the BCH3 arrangement minimizes the cut of each region and that the center cut is the maximum of the linear embedding. Thus the $lcw(K_{r,s,t})$ occurs when $x=\frac{r+s+t}{2}$ for $r+s+t$ even and when $x=\frac{r+s+t-1}{2}$ and $x=\frac{r+s+t+1}{2}$ for $r+s+t$ odd. So we will look at each case to find the $lcw(K_{r+s+t})$.

Case 1: Let $r+s+t$ be even, then $x=\frac{r+s+t}{2}$. Place $[\frac{2x+2r-s-t}{6}]$ vertices from A to the left of the center cut. Substituting $\frac{r+s+t}{2}$ for x gives $[\frac{r+s+t+2r-s-t}{6}]=[\frac{r}{2}]$. So $a=\frac{r}{2}$ when r is even and $a=\frac{r+1}{2}$ when r is odd.

Place $[\frac{2x+2s-r-t}{6}]$ vertices from B to the left of the center cut. Substituting $\frac{r+s+t}{2}$ for x gives $[\frac{r+s+t+2s-r-t}{6}] = [\frac{s}{2}]$. So $b = \frac{s}{2}$ when s is even, $b = \frac{s+1}{2}$ when s is odd and r is even, and $b = \frac{s-1}{2}$ when s is odd and r is odd. Since $r+s+t$ is even, we know that when r and s are even or r and s are odd, t is even, and when one out of r and s is odd, t is odd. Therefore, the number of vertices from C on the left of the cut is

$$c = \begin{cases} x - \frac{r}{2} - \frac{s}{2} = \frac{t}{2} & \text{for } r, s \text{ even} \\ x - \frac{r+1}{2} - \frac{s-1}{2} = \frac{t}{2} & \text{for } r, s \text{ odd} \\ x - \frac{r}{2} - \frac{s+1}{2} = \frac{t-1}{2} & \text{for } r \text{ even and } s \text{ odd} \\ x - \frac{r+1}{2} - \frac{s}{2} = \frac{t-1}{2} & \text{for } r \text{ odd and } s \text{ even} \end{cases}$$

Now recall that the

$$cut(a,b,c) = a[(s-b)+(t-c)] + b[(r-a)+(t-c)] + c[(r-a)+(s-b)].$$

Taking the cut of $(\frac{r}{2}, \frac{s}{2}, \frac{t}{2})$ gives:

$$\begin{aligned} cut(\frac{r}{2}, \frac{s}{2}, \frac{t}{2}) &= \frac{r}{2}[(s-\frac{s}{2})+(t-\frac{t}{2})] + \frac{s}{2}[(r-\frac{r}{2})+(t-\frac{t}{2})] + \frac{t}{2}[(r-\frac{r}{2})+(s-\frac{s}{2})] \\ &= \frac{rs+rt+st}{2}. \end{aligned}$$

We will continue this process for all of the combinations for r, s, and t.
Taking the cut of $(\frac{r}{2}, \frac{s+1}{2}, \frac{t-1}{2})$ gives:

$$cut(\frac{r}{2}, \frac{s+1}{2}, \frac{t-1}{2}) = \frac{rs+rt+st+1}{2}.$$

Taking the cut of $(\frac{r+1}{2}, \frac{s}{2}, \frac{t-1}{2})$ gives:

$$cut(\frac{r+1}{2}, \frac{s}{2}, \frac{t-1}{2}) = \frac{rs+rt+st+1}{2}.$$

Taking the cut of $(\frac{r+1}{2}, \frac{s-1}{2}, \frac{t}{2})$ gives:

$$cut(\frac{r+1}{2}, \frac{s-1}{2}, \frac{t}{2}) = \frac{rs+rt+st+1}{2}.$$

Case 2: Let $r+s+t$ be odd, then $x = \frac{r+s+t-1}{2}$. Place $[\frac{2x+2r-s-t}{6}]$ vertices from A to the left of the center cut. Substituting $\frac{r+s+t-1}{2}$ for x gives $[\frac{r+s+t-1+2r-s-t}{6}] = [\frac{r}{2} - \frac{1}{6}]$. So $a = \frac{r}{2}$ when r is even and $a = \frac{r-1}{2}$ when r is odd. Place $[\frac{2x+2s-r-t}{6}]$ vertices from B to the left of the center cut. Substituting $\frac{r+s+t-1}{2}$ for x gives $[\frac{r+s+t-1+2s-r-t}{6}] = [\frac{s}{2} - \frac{1}{6}]$. So $b = \frac{s}{2}$ when s is even, $b = \frac{s+1}{2}$ when s is odd and r is odd, and $b = \frac{s-1}{2}$ when s is odd and r is even. Since $r+s+t$ is odd, we know that when r and s are odd or r and s are even, t is odd, and when one out of r and s is odd and the other is even, t is even. Therefore, the number of vertices from C on the left of the cut is

$$c = \begin{cases} x - \frac{r-1}{2} - \frac{s+1}{2} = \frac{t-1}{2} & \text{for } r, s \text{ odd} \\ x - \frac{r}{2} - \frac{s}{2} = \frac{t-1}{2} & \text{for } r, s \text{ even} \\ x - \frac{r}{2} - \frac{s-1}{2} = \frac{t}{2} & \text{for } r \text{ even and } s \text{ odd} \\ x - \frac{r-1}{2} - \frac{s}{2} = \frac{t}{2} & \text{for } r \text{ odd and } s \text{ even} \end{cases}$$

Similarly as used in case 1, we use the equation $cut(a, b, c)$.
Taking the cut of $(\frac{r-1}{2}, \frac{s+1}{2}, \frac{t-1}{2})$ gives:

$$cut(\frac{r-1}{2}, \frac{s+1}{2}, \frac{t-1}{2}) \quad = \quad \frac{rs+rt+st+1}{2}.$$

Taking the cut of $(\frac{r}{2}, \frac{s}{2}, \frac{t-1}{2})$ gives:

$$cut(\frac{r}{2}, \frac{s}{2}, \frac{t-1}{2}) \quad = \quad \frac{rs+rt+st}{2}.$$

Taking the cut of $(\frac{r-1}{2}, \frac{s}{2}, \frac{t}{2})$ gives:

$$cut(\frac{r-1}{2}, \frac{s}{2}, \frac{t}{2}) \quad = \quad \frac{rs+rt+st}{2}.$$

Taking the cut of $(\frac{r}{2}, \frac{s-1}{2}, \frac{t}{2})$ gives:

$$cut(\frac{r}{2}, \frac{s-1}{2}, \frac{t}{2}) \quad = \quad \frac{rs+rt+st}{2}.$$

Therefore,

$$lcw(K_{r,s,t}) = \begin{cases} \frac{rs+rt+st}{2} & \text{for two or more } s, r, t \text{ even} \\ \frac{rs+rt+st+1}{2} & \text{otherwise} \end{cases}$$

∎

Results

In conclusion, we were able to minimize every cut of the linear embedding for complete bipartite and tripartite graphs. We also derived equations for the $lcw(K_{m,n})$ and for the $lcw(K_{r,s,t})$. To continue in this topic we could generalize the linear cutwidth for complete n-partite graphs. This idea lead to our conjecture:

Conjecture:
Let $K_{a_1,a_2,..,a_n}$ be a complete n-partite graph. Then

$$lcw(K_{a_1,a_2,..,a_n}) = \begin{cases} \sum \frac{a_i a_j}{2} & a_i < a_j,\ i \neq j,\ i,j \in \{1,2,...,n\}\ \forall i,j, \\ & \text{with at most one } a_1, a_2, ..., a_n \text{ odd} \\ & \\ \sum \frac{a_i a_j}{2} + \frac{1}{2} & a_i < a_j,\ i \neq j,\ i,j \in \{1,2,...,n\}\ \forall i,j, \\ & \text{otherwise} \end{cases}$$

Two possible ideas to prove this conjecture is through induction, or through looking at $c(n, 2)$ bipartite graphs. When looking at the $c(n, 2)$ bipartite graphs we can find the linear cutwidths of each bipartite graph then add them together to get the linear cutwidth for the n-partite graph.

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