The Linear Cutwidth of Complete n-Partite Graphs

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August 26, 2005

Abstract

This paper looks at complete n-partite graphs, $K_{m_1,m_2,m_3,\ldots,m_n}$. The main focus is to find a linear embedding for the vertices that minimizes the linear cutwidth. The linear cutwidth equation in general is found for a complete n-partite graph.

1 Introduction

1.1 Complete Bipartite Graphs

A graph $G = (V, E)$ consists of a finite set, V, of vertices and a set, E, of edges joining different pairs of distinct vertices. In graphs that are presented in this paper, an edge is represented by a line and a vertex is represented by a dot. Graphs is this paper have a few restrictions. An edge cannot "loop" around so that the ends terminate at the same vertex. Also, two edges cannot connect the same pair of vertices in a graph. A bipartite graph consists of two sets of vertices, A and B, with edges that link one vertex in A to another vertex in B. However, edges cannot link two vertices from the same set. A complete bipartite graph is a graph where all the vertices from one set are connected to all of the vertices in the other set. Complete bipartite graphs are denoted as $K_{m,n}$ where $|\mathrm{A}| = m$ and $|\mathrm{B}| = n$. Figure 1 gives the complete bipartite graph $K_{3,4}$.

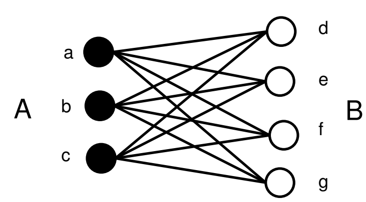

Figure 1: Bipartite Graph
$K_{3,4}$

1.2 Linear Embedding

There are many different ways to represent a graph. One of these ways is through linear embedding, which is focused on in this paper. When a graph is represented on a linear embedding, the vertices are put on a straight line with edges connecting the vertices that were adjacent in the original graph. Figure 2 gives a linear embedding of $K_{3,4}$.

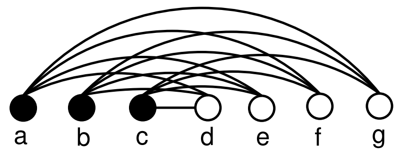

Figure 2: Bipartite Graph
$K_{3,4}$

A region of a linear embedding graph is the area between two adjacent vertices. The amount of vertices to the left of the region is indicated by $(a, x-a)$ for a bipartite graph, where x is the total number of vertices to the left of the particular cut and a is the number of vertices from set A that is to the left of that same cut. There are $(x-a)$ vertices from set B to the left of the cut. The cutwidth of a region is represented by $cut(a, x-a)$. This is defined as the number of edges that cross through a region between two adjacent vertices. The following equation is the cutwidth for a bipartite linear embedding:

$$cut(a, x-a) = a(n-(x-a)) + (x-a)(m-a).$$

When looking at Figure 2, the cutwidth when $x = 1$ is four (this is between vertices a and b). The maximum cut of a linear embedding is the cut with the most number of edges running through a region, which would be twelve in the above figure occuring between c and d. Another arrangement of the linear embedding of $K_{3,4}$ could give a different maximum. For example, the arrangement shown in Figure 3 will actually give a smaller maximum, in particular it will give eight. The linear cutwidth of a graph is defined as the smallest maximum cut for all the different arrangements of the vertices within a linear embedding. It is important to note that the main goal when considering linear cutwidth is to find an arrangement of the vertices that minimizes the cut. Therefore, eight would be the linear cutwidth of $K_{3,4}$ when considering the two different arrangements, but there could be an even better arrangement that would give a lower cutwidth.

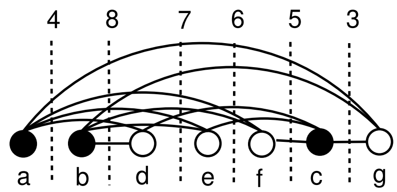

Figure 3: Bipartite Graph
$K_{3,4}$

2 Background

Graph theory can be applied to daily life and everyday the use for graphs keeps increasing and increasing. They can be used for road maps, subway systems, and telephone communication. Besides everyday life, graph theory can be used in the business world as well. Graph theory is used in network analysis, molecular bonding, circuit layout, and computer data structures. The cutwidth of a graph is important when working with some of these applications. For example, one would want to minimize the cutwidth when arranging a computer chip. There have been a lot of people who have looked at cutwidth for certain types of graphs. J.D. Chavez and R. Trapp [3] found that the cyclic cutwidth of trees equals the linear cutwidth. The exact values for the cyclic cutwidth of two-dimensional ordinary and cylindrical meshes, $P_m \times P_n$ and $P_m \times P_c$ in particular was found by H. Schröder, et al [6]. The equations for the linear cutwidth and the cyclic cutwidth of the complete graph K_n was found by F. Rios [5]. D. Thilikos, et al [7] found a polynomial time algorithm for the cutwidth of bounded degree graphs with small treewidth.

This paper will look at the minimized linear cutwidth of complete n-partite graphs. To begin, we will look at the findings of Bowles [2] and Johnson [4] on bipartite graphs and then proceed to show how this can be applied to complete n-partite graphs.

2.1 Bowles' Arrangement of Vertices

Bowles investigated what the best arrangement of the vertices of a complete bipartite graph would be so that the cuts of the graph are minimized in each region. She did this for both bipartite and tripartite graphs. Through these arrangements she found that the maximum cut occurs at $x = \frac{m+n}{2}$ for $m+n$ even and when $m+n$ is odd, the maximum cut occurs at both $x = \frac{m+n+1}{2}$ and $x = \frac{m+n-1}{2}$ for a bipartite graph. This is shown in the following Corollary.

Corollary 1 (Bowles) Let $K_{m,n}$ be a complete bipartite graph whose linear embedding is arranged according to Bowles' arrangement of vertices. Then the maximum cut will occur at $x = \frac{m+n}{2}$ for $m+n$ even and at $x = \frac{m+n-1}{2}$ and $x = \frac{m+n+1}{2}$ for $m+n$ odd. Also, the cuts to the left of the middle cut(s) will be strictly increasing and the cuts to the right of the middle cut(s) will be strictly decreasing.

Her arrangement of vertices is described in the next section and from this value of x, it is found that the maximum cut of all the regions will be the center cut if arranged in the specified way. Johnson found that the linear cutwidth for a complete bipartite graph can be summarized by the following equations after considering all the possibilties when m and n are even or odd.

Theorem 1 (Johnson) Let $K_{m,n}$ be a complete bipartite graph. Then

$$\mathrm{lcw}(K_{m,n}) = \begin{cases} \frac{mn}{2} & \text{for } mn \text{ even} \\ \frac{mn+1}{2} & \text{for } mn \text{ odd.} \end{cases}$$

Bowles was successful in finding equations for tripartite graphs as well. This is discussed in more detail towards the end of the paper. From analyzing these results, we can now generalize them to apply to complete n-partite graphs.

3 New Expression for Bowles' Arrangement

Bowles' arrangement does give the minimized cut for each region of a complete bipartite graph. However, you must calculate how many vertices from set A and from set B are to the left of each cut. So whenever you increase x a new calculation must be done. We have found a different calculation that tells how to place the vertices that will still give the same arrangement that Bowles achieves for a complete bipartite graph. This will involve a different expression and will include less calculations.

Some of the numbers that are calculated may need to be rounded to the next whole number. This is done by the following equation, where x is the number that needs to be rounded.

$$\lfloor x + 0.5 \rfloor$$

This will be represented as $[x]$ throughout the paper.

Lemma 1 (Weitzel-Chavez 1 (WC1)): Let $K_{m,n}$ be a complete bipartite graph with two sets of vertices, A and B, where $|\mathrm{A}| = m$ and $|\mathrm{B}| = n$ and $m \leq n$. Begin by placing a vertex from set A in spot $[\frac{n-m}{2}] + 1$. Continue placing the rest of the vertices from A in every other spot to the right. To finish the linear embedding, place the vertices from set B in the remaining empty spots. This will achieve the minimized cutwidth for each region for a complete bipartite graph when both $m + n$ is even or odd.

For example, using this new arrangement we will put the graph of $K_{3,4}$ onto a linear embedding. We will place our first vertex from A in spot two ($[\frac{4-3}{2}]+1 = 2$). The arrangement of the vertices from set A can be seen below in Figure 4.

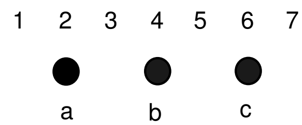

Figure 4: Beginning of Bipartite Graph
$K_{3,4}$

Now we will place the vertices from set B in the remaining spots. Figure 5 shows the complete arrangement of $K_{3,4}$.

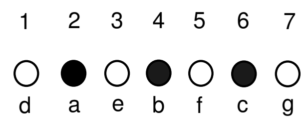

Figure 5: Bipartite Graph
$K_{3,4}$

It can be noticed that because $m + n$ is odd, the vertices from B are symmetrically placed around the vertices from A. When $m + n$ is even, there will be an extra vertex from B on the far rightside. It can also be seen that only one calculation was used in arranging the vertices.

Proof of Lemma 1
For a complete bipartite graph, Bowles found that the cut of each region of a linear embedding is minimized by placing $[\frac{2x+m-n}{4}]$ vertices from A to the left of each cut x. If you set this expression equal to 0.5, this will give the value for x that will result in placing the first vertex from set A in the linear embedding. It is set equal to 0.5 because this will give you the first possibility of rounding the number to one (definition of the rounding function) which means that a vertex from A is put to the left of the cut. Rearranging this equation and solving for x gives the cut where the first vertex from set A will appear (will give a number that is ≥ 0.5).

$$x = \left[\frac{2 + n - m}{2}\right]$$

Above, it is seen that the WC1 arrangement gives you the spot where the first vertex from set A is placed. The expression for the spot is equal to $[\frac{n-m}{2}] + 1$ when $m + n$ is odd and when $m + n$ is even. Manipulating this equation will show that Bowles' equation for placing the first vertex from set A is exactly the same as the WC1 equation for complete bipartite graphs. We will let y equal the spot where the first vertex from set A will be placed.

$$y = \left[\frac{n - m}{2}\right] + 1$$

$$
\begin{aligned}
&= \left[\frac{2+n-m-2}{2}\right]+1 \\
&= \left[\frac{2}{2}+\frac{n}{2}-\frac{m}{2}-\frac{2}{2}\right]+1 \\
&= \left[\frac{2+n-m}{2}\right]-\left[\frac{2}{2}\right]+1 \\
&= \left[\frac{2+n-m}{2}\right]-1+1 \\
&= \left[\frac{2+n-m}{2}\right] \\
\Longrightarrow\ & x=y
\end{aligned}
$$

This shows that by using either one of the two expressions, the same arrangement is given which minimizes the linear cutwidth. By using the WC1 arrangement, it is a lot easier to find the spots for the vertices from set A rather than doing all the calculations when varying x.

For an example, let's compare Bowles' arrangement of $K_{3,4}$ with the one that was shown earlier. When $x = 1$ we will see that the first vertex of the linear embedding will be from set B because there are zero vertices from set A to the left of the first cut.

$$\left[\frac{2(1)+3-4}{4}\right]=0$$

After increasing the value for x, it is seen that the first vertex from set A is placed to the left of the cut when $x = 2$. Therefore, we have the vertices in the following order thus far.

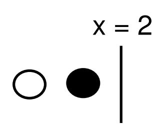

Figure 6: Beginning of Bipartite Graph
$K_{3,4}$

The next vertex from set A will be placed to the left of the cut when $x = 4$. Finally, when $x = 6$ the last vertex from set A will be placed to the left of the cut. After all of the vertices from A have been used, the remaining B vertices are placed to the right of the growing linear embedding. Once all the values for x have been plugged in (from $x = 1$ to $x = 6$) and the remaining vertices from B are in the appropriate spots, the linear embedding will look like Figure 7.

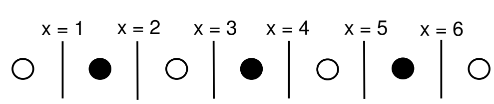

Figure 7: Bipartite Graph
$K_{3,4}$

This arrangement is identical to the arrangement shown earlier in Figure 5. Therefore the same linear cutwidth equation can be applied to the WC1 arrangement for complete bipartite graphs.

4 An Upper Bound for the Linear Cutwidth of Complete n-Partite Graphs

Theorem 2 (Weitzel-Chavez (WC2)): An upper bound for the linear cutwidth of a complete n-partite graph is:

$$lcw(K_{m_1,m_2,m_3,\ldots,m_n}) \leq \sum_{i=0}^{n-2} \left\lceil \frac{(m_n + \ldots + m_{n-i})m_{n-i-1}}{2} \right\rceil .$$

Proof
This proof assumes that $m_1 \leq m_2 \leq \ldots \leq m_n$ and will be performed by induction. For the base case, we will consider a complete bipartite graph, K_{m_1,m_2}. It was shown by Theorem 1 that the linear cutwidth for a complete bipartite graph is:

$$lcw(K_{m_1,m_2}) = \begin{cases} \frac{m_1m_2}{2} & \text{if } m_1m_2 \text{ is even} \\ \frac{m_1m_2+1}{2} & \text{if } m_1m_2 \text{ is odd.} \end{cases}$$

In this situation, $n = 2$ in WC2 because there are two components (m_1 and m_2). Therfore, plugging this value into the equation given by WC2 shows that it gives the same result as when one compares it to Johnson's equation for the linear cutwidth of complete bipartite graphs.

$$\begin{aligned} lcw(K_{m_1,m_2}) &\leq \sum_{i=0}^{0} \left\lceil \frac{m_2m_1}{2} \right\rceil \\ &= \left\lceil \frac{m_1m_2}{2} \right\rceil \\ &= \begin{cases} \frac{m_1m_2}{2} & \text{if } m_1m_2 \text{ is even} \\ \frac{m_1m_2+1}{2} & \text{if } m_1m_2 \text{ is odd} \end{cases} \end{aligned}$$

Now, we will assume that the following statement is true about the linear cutwidth of a complete $(n-1)$-partite graph:

$$lcw(K_{m_2,m_3,m_4,\ldots,m_n}) \leq \sum_{i=0}^{(n-1)-2} \left\lceil \frac{(m_n + \ldots + m_{n-i})m_{n-i-1}}{2} \right\rceil .$$

The above equation gives the minimized linear cutwidth for an $(n-1)$-partite graph when ignoring the smallest set of vertices, m_1. First, we will lay out the $(n-1)$-partite graph optimally. Then for the n-partite graph, we will consider the $(n-1)$-partite vertices $(m_2, m_3, \ldots, m_n)$ as one set (represented by m_z) and m_1 as another set of vertices. When combining the two sets, the problem simplifies to a complete bipartite case. The linear cutwidth will then be known because of Theorem 1. There will be fewer m_1 vertices than m_z vertices, so the following graph is a representation of the linear embedding of the two sets where the black dots represent m_1 and the white dots represent m_z. All of the m_z vertices will be connected to all of the m_1 vertices.

Figure 8: Bipartite Graph

When considering the graph above, the linear cutwidth is $\lceil \frac{m_1 m_z}{2} \rceil$ because that is the known definition for the linear cutwidth of a complete bipartite case (Theorem 1). The linear cutwidth of $K_{m_2,m_3,\ldots,m_n}$ (or K_{m_z}) is known because that is assumed true in the $(n-1)$-partite case. Therefore, the upper bound for the linear cutwidth of a complete bipartite graph is equivalent to the following when considering the individual sets:

$$\left(\sum_{i=0}^{(n-1)-2} \left\lceil \frac{(m_n + \ldots + m_{n-i})m_{n-i-1}}{2} \right\rceil \right) + \left\lceil \frac{(m_n + m_{n-1} + \ldots + m_2)m_1}{2} \right\rceil .$$

This is the upper bound for the linear cutwidth of a complete n-partite graph as given in Theorem 2.

5 A Lower Bound for the Linear Cutwidth of n-Partite Graphs

Theorem 3 (Weitzel-Chavez (WC3)): A lower bound for the linear cutwidth of a complete n-partite graph is:

$$lcw(K_{m_1,m_2,m_3,\ldots,m_n}) \geq \sum_{i=0}^{n-2} \left\lceil \frac{(m_n + \ldots + m_{n-i})m_{n-i-1}}{2} \right\rceil .$$

Proof

First, we will consider the bipartite case $K_{m,n}$. For the lower bound, we will just be concerned with minimizing the middle region where the linear cutwidth will occur. To begin, we will find the middle cut of the graph where the maximum cutwidth will be. This is known to be the middle cut because of Corollary 1. If $m+n$ is even this will occur at $\frac{m+n}{2}$. If $m+n$ is odd, this will occur at both $\frac{m+n+1}{2}$ and $\frac{m+n-1}{2}$. This also shows that half of the total amount of vertices will be to the left of the cut and the other half will be to the right. This does not necessarily mean that half of the vertices from m will be to the left of the center cut and half of the vertices from n will be to the left of the center cut, etc. We aren't concerned with minimizing all the regions of the graph, just where the linear cutwidth occurs.

Consider that there will be a vertices from set m to the left of the middle cut(s) Therefore, there must be $m-a$ vertices to the right of the middle cut(s) from set m. This results in there being $\frac{m+n}{2}-a$ vertices from set n to the left of the cut(s) because there is a total of $\frac{m+n}{2}$ vertices in general to the left of the cut(s) To the right of the cut(s), there must be $\frac{m+n}{2}-m+a$ or $\frac{n-m}{2}+a$ vertices from set n. This case occurs if $m+n$ is even. The following table summarizes the amount of vertices from each set that will be to the left and to the right of the middle cut(s) when considering if $m+n$ is even or odd.

Table 1

Set	$m+n$ Even Left of cut	$m+n$ Even Right of cut	$m+n$ Odd Left of cuts	$m+n$ Odd Right of cuts
m	a	$m-a$	a	$m-a$
n	$\frac{m+n}{2}-a$	$\frac{n-m}{2}+a$	$\lfloor\frac{m+n}{2}\rfloor-a$	$\lceil\frac{n-m}{2}\rceil+a$

Now, if we can find the cutwidth of this middle cut, we will have the linear cutwidth. The cutwidth equation (represented by f) is shown below and simplified when $m+n$ is even.

$$\begin{aligned}
f(a) &= a\left(\frac{n-m}{2}+a\right)+(m-a)\left(\frac{m+n}{2}-a\right) \\
&= 2a^2+a\left(\frac{n-m}{2}-\left(\frac{m+n}{2}\right)-m\right)+m\left(\frac{m+n}{2}\right) \\
&= 2a^2+a(-2m)+m\left(\frac{m+n}{2}\right)
\end{aligned}$$

Taking the derivative of the above equation and solving for a, will give the number of vertices from set m that will be to the left of the cut. This is minimized because after taking the derivative, the equation is set equal to zero when a is solved for.

$$f' = 4a-2m$$

$$\Longrightarrow a = \frac{m}{2}$$

Taking this value for a (assuming m is even) and plugging it back into the function f will show what the minimized linear cutwidth will be for a complete bipartite graph when $m+n$ is even and in particular when m is even.

$$
\begin{aligned}
f\left(\frac{m}{2}\right) &= 2\left(\frac{m}{2}\right)^2 + \left(\frac{m}{2}\right)(-2m) + m\left(\frac{m+n}{2}\right) \\
&= \frac{m^2}{2} - m^2 + \frac{m^2}{2} + \frac{mn}{2} \\
&= \frac{mn}{2} \qquad (1)
\end{aligned}
$$

This agrees with Johnson's result because if m is even and $m+n$ is even, n must also be even. If m is odd, $a = \frac{m\pm1}{2}$. This value is then plugged back into the original function f and will result in giving the minimzed linear cutwidth as well.

$$
\begin{aligned}
f\left(\frac{m\pm1}{2}\right) &= 2\left(\frac{m\pm1}{2}\right)^2 + \left(\frac{m\pm1}{2}\right)(-2m) + m\left(\frac{m+n}{2}\right) \\
&= \frac{m^2 \pm 2m + 1}{2} - (m^2 \pm m) + \frac{m^2+mn}{2} \\
&= \frac{2m^2 \pm 2m + mn + 1}{2} - \frac{2m^2 \mp 2m}{2} \\
&= \frac{mn+1}{2} \qquad (2)
\end{aligned}
$$

The above result is the reason why the ceiling function is used in WC3. Since m is odd and $m+n$ is even, n must also be odd. Therefore mn is odd as well. If mn is odd, it must be rounded up by one so after dividing by two, a whole number will be the result. This will then give the minimized linear cutwidth for a complete bipartite graph.

When $m+n$ is odd the same procedure is done, but the new values for set n will be used (as seen in Table 1). These cases take a little more work when it comes to rounding values as can be seen below.

$$
\begin{aligned}
f(a) &= a\left(\left\lceil\frac{n-m}{2}\right\rceil + a\right) + (m-a)\left(\left\lfloor\frac{m+n}{2}\right\rfloor - a\right) \\
&= 2a^2 + a\left(\left\lceil\frac{n-m}{2}\right\rceil - m - \left\lfloor\frac{m+n}{2}\right\rfloor\right) + m\left(\left\lfloor\frac{m+n}{2}\right\rfloor\right) \\
&= 2a^2 + a\left(\frac{n-m+1}{2} - m - \left(\frac{m+n-1}{2}\right)\right) + m\left(\left\lfloor\frac{m+n}{2}\right\rfloor\right) \\
&= 2a^2 + a(1-2m) + m\left(\left\lfloor\frac{m+n}{2}\right\rfloor\right)
\end{aligned}
$$

Taking the derivative of the above equation will give a different result than what is given when $m+n$ is even. The derivative of the above equation is:

$$f' = 4a + (1-2m).$$

Setting this equal to zero (so we get the minimized linear cutwidth) and solving for a will give the value for the number of vertices from set m to the left of the first cut.

$$a = \frac{2m-1}{4}$$

First we will assume that m is even. If m is even in the above equation, $2m$ will always give you a multiple of four which is wanted since the divisor is four. Therefore, when one is subtracted from the product, the closest whole integer after performing the division is simply, $\frac{2m}{4}$ or $\frac{m}{2}$. Taking this value for a and plugging it into the above equation for f will give:

$$\begin{aligned} f\left(\frac{m}{2}\right) &= 2\left(\frac{m}{2}\right)^2 + \left(\frac{m}{2}\right)(1-2m) + m\left(\left\lfloor\frac{m+n}{2}\right\rfloor\right) \\ &= \frac{m^2}{2} - m^2 + \frac{m}{2} + m\left(\frac{m+n-1}{2}\right) \\ &= \frac{m^2}{2} - m^2 + \frac{m}{2} + \frac{m^2+mn-m}{2} \\ &= \frac{-m^2}{2} + \frac{m}{2} + \frac{m^2+mn-m}{2} \\ &= \frac{mn}{2}. \end{aligned} \tag{3}$$

When m is odd, $a = \frac{m-1}{2}$. This is because we want to get to a whole integer after the division has taken place, and the closest is received when we round down. For example, if $m=5$, $a = \frac{2\cdot5-1}{4} = \frac{9}{4}$. The closest integer to this fraction is two, which is the result when plugging $m=5$ into $a = \frac{5-1}{2} = 2$. Taking this new value for a and plugging it into the original function f will give:

$$\begin{aligned} f\left(\frac{m-1}{2}\right) &= 2\left(\frac{m-1}{2}\right)^2 + \left(\frac{m-1}{2}\right)(1-2m) + m\left(\left\lfloor\frac{m+n}{2}\right\rfloor\right) \\ &= \left(\frac{m^2-2m+1}{2}\right) - m^2 + m + \frac{m}{2} - \frac{1}{2} + m\left(\frac{m+n-1}{2}\right) \\ &= \frac{-m^2-m}{2} + m + \frac{m^2+mn-m}{2} \\ &= \frac{mn}{2}. \end{aligned} \tag{4}$$

When $m+n$ is odd, the same result is found when m is either odd or even for a complete bipartite graph.

Recall from the upper bound that the expression on the right hand side of the inequality from WC3 can be rearranged as the following:

$$\left(\sum_{i=0}^{(n-1)-2}\left\lceil\frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{2}\right\rceil\right)+\left\lceil\frac{(m_n+m_{n-1}+\ldots+m_2)m_1}{2}\right\rceil.$$

Now to generalize the bipartite case to an n-partite case, we will assume that the lower bound for the linear cutwidth of an $(n-1)$-partite graph is:

$$lcw(Km_2,m_3,m_4,\ldots,m_n)\geq\sum_{i=0}^{(n-1)-2}\left\lceil\frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{2}\right\rceil.$$

We will let m represent $m_2+\ldots+m_n$ and we will let n represent m_1. To add in the last set of vertices, n, we will first look at when m is even (this means that $m_2+\ldots+m_n$ is even) and n is even. This will then give the same result as in case (1). Therefore, a factor of $\frac{mn}{2}$ must be added to the linear cutwidth of the $(n-1)$-partite case. This is equivalent to the following after the values for m and n have been plugged in:

$$\frac{m_1(m_2+\ldots+m_n)}{2}.$$

Now we must look at the case when n is odd and m is even. The extra term in this case will also be $\frac{mn}{2}$ as can be seen in case (3). Once again substituting the values in for m and n will give the result shown below:

$$\frac{m_1(m_2+\ldots+m_n)}{2}.$$

To summarize the case when m is even, the term above will be added to the linear cutwidth of the $(n-1)$-partite case to give the lower bound for the linear cutwidth of an n-partite graph:

$$\left(\sum_{i=0}^{(n-1)-2}\left\lceil\frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{2}\right\rceil\right)+\frac{(m_n+m_{n-1}+\ldots+m_2)m_1}{2}.$$

Next we must look at the two cases when m is odd (this means that $m_2+\ldots+m_n$ is odd). The first case will be that the last vertex, n, is even. Going back to the bipartite cases, we will see that case (4) is represented here. Therefore, the term to be added to the linear cutwidth is $\frac{mn}{2}$ or:

$$\frac{m_1(m_2+\ldots+m_n)}{2}.$$

Finally, the last case to look at is when m is odd and n is odd. This case is the reason why the ceiling function is used in the lower bound. This is also described in case (2). According to the bipartite case, the term that will be added is $\frac{mn+1}{2}$. Substituting in the ceiling function and the values for m and n will result in the following:

$$\left\lceil \frac{m_1(m_2+\ldots+m_n)}{2} \right\rceil.$$

When considering all the different cases described above, the lower bound for the linear cutwidth of an n-partite graph is now known. The ceiling function can be used in all of the cases because it will only apply when it is needed. Therefore, the lower bound for the linear cutwidth of an n-partite graph is:

$$\left(\sum_{i=0}^{(n-1)-2} \left\lceil \frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{2} \right\rceil \right) + \left\lceil \frac{(m_n+m_{n-1}+\ldots+m_2)m_1}{2} \right\rceil.$$

This can be rearranged to give the inequality in WC3. As can be seen the lower bound is the same as what we achieved in the upper bound. Therefore, the linear cutwidth of an n-partite graph is known and is minimized.

6 The Linear Cutwidth of a Complete Graph K_n

Now we will compare our known minimized linear cutwidth equation to the linear cutwidth equation for a complete graph K_n. It was shown by F. Rios that the linear cutwidth of K_n is equal to the following:

$$lcw(K_n) = \begin{cases} \frac{n^2}{4} & n \text{ even} \\ \frac{n^2-1}{4} & n \text{ odd.} \end{cases}$$

It can now be shown that the linear cutwidth of $K_{m_1,m_2,m_3,\ldots,m_{n-1},m_n}$, or K_n, where $m_1 = m_2 = m_3 = \ldots = m_{n-1} = m_n = 1$ will give the same result that Rios came up with. To begin we will state the linear cutwidth equation for an n-partite graph that includes the values of $m_1, m_2, m_3, \ldots, m_n$.

$$lcw(K_{m_1,m_2,m_3,\ldots,m_n}) = \sum_{i=0}^{n-2} \left\lceil \frac{(1+\ldots+1)1}{2} \right\rceil$$

In this case $n = n$ because we are dealing with a complete graph K_n. It can be seen that the sum above will give a result like the following when n is odd:

$$lcw(K_{m_1,m_2,m_3,\ldots,m_n}) = 1+1+2+2+3+3+\ldots+\frac{(n-1)\cdot 1}{2}+\frac{(n-1)\cdot 1}{2}.$$

It is also known that the sum of $1+2+\ldots+s$ is equal to the expression $\frac{s(s+1)}{2}$. Substituting $\frac{(n-1)\cdot 1}{2}$ for s, and noticing that each number is represented twice, we must multiply this expression by two to get the result that is wanted:

$$
\begin{aligned}
lcw(K_{m_1,m_2,m_3,\ldots,m_n}) &= 2\left\{\frac{\frac{n-1}{2}\cdot\left(\frac{n-1}{2}+1\right)}{2}\right\} \\
&= \frac{n-1}{2}\left(\frac{n-1}{2}+1\right) \\
&= \frac{n^2-2n+1}{4}+\frac{n-1}{2} \\
&= \frac{n^2-1}{4}.
\end{aligned}
$$

When n is even, we will use the same procedure. However, we must note that the sum of the linear cutwidth will now equal the following equation:

$$
lcw(K_{m_1,m_2,m_3,\ldots,m_n}) = 1+1+2+2+3+3+\ldots+\left\lceil\frac{(n-1)\cdot 1}{2}\right\rceil.
$$

Now s will be equal to $\frac{n\cdot 1}{2}$ because the ceiling function is taken into consideration so we can get an integer after the division has taken place. But notice that the last term is only represented once, not twice like it was when n was odd. Therefore, we will go through the same sort of equation and at the end we will subtract the last term since it is only included in the sum once, and at the beginning it is multiplied by two.

$$
\begin{aligned}
lcw(K_{m_1,m_2,m_3,\ldots,m_n}) &= 2\left\{\frac{\frac{n}{2}\cdot\left(\frac{n}{2}+1\right)}{2}\right\} \\
&= \frac{n}{2}\left(\frac{n}{2}+1\right) \\
&= \frac{n^2}{4}+\frac{n}{2} \\
&= \frac{n^2}{4}+\frac{n}{2}-\frac{n}{2} \\
&= \frac{n^2}{4}
\end{aligned}
$$

As can be seen these two results agree with Rios' equations for the linear cutwidth of complete graphs.

7 Comparing Different Arrangements

Now we will do an example showing that Bowles' arrangement minimizes the cutwidth of every region of a tripartite graph and compare it to the graph that is obtained when using the above Theorems. This will show that the equation for the linear cutwidth of complete n-partite graphs does not minimize every region, but does minimize the linear cutwidth.

Consider the complete tripartite graph $K_{2,5,7}$ (will also be noted as K_{m_1,m_2,m_3}). According to Bowles' the linear cutwidth will be:

$$lcw(K_{2,5,7}) = \frac{2 \cdot 5 + 2 \cdot 7 + 5 \cdot 7 + 1}{2} = 30$$

and the maximum cut will occur at:

$$x = \frac{2+5+7}{2} = 7.$$

To begin, the graph is just considered as a bipartite case. This occurs until $x = 6$ when the first vertex from m_1 shows up. Then it must be considered as a tripartite case until all of the vertices from m_1 have been placed in the linear embedding. It is noted that the amount of vertices from each set that is to the left of each cut can be found with equations from Bowles. From $x = 1$ through $x = 5$ we will consider the bipartite case K_{m_2,m_3}. These values will be used in the following equation which tells how many vertices from set m_2 will be to the left of each cut x.

$$\frac{2x + m_2 - m_3}{4}$$

From $x = 6$ through $x = 11$ we will consider the tripartite case K_{m_1,m_2,m_3}. The following equations will be used to show how many vertices from each set will be to the left of each cut. We will let a represent the number of vertices from m_1, b will represent the number of vertices from m_2, and finally c will represent the number of vertices from m_3.

$$a = \frac{2x + 2m_1 - m_2 - m_3}{6}$$

$$b = \frac{2x + 2m_2 - m_1 - m_3}{6}$$

$$c = \frac{2x + 2m_3 - m_1 - m_2}{6}$$

From $x = 12$ through $x = 13$ we will once again go back and consider this as a bipartite case. This is because we have run out of vertices from set m_1. There is one problem however. The value of x will not be twelve and thirteen because we are ignoring the vertices from set m_1 (because it is a bipartite case) and these values for x take into consideration all three sets of vertices. Therfore, x will really have values of ten and eleven when plugging these into the bipartite equation given above. This will give the correct amount of vertices from set m_2 that should be to the left of each cut. The following table gives a summary of the values that are received when using the equations described above. The empty spaces represent that that particular set has run out of vertices and does not need to be considered anymore.

Table 2

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14
m_1	0	0	0	0	0	1	1	1	2	2	2			
m_2	0	1	1	2	2	2	3	3	3	4	4	5	5	
m_3	1	1	2	2	3	3	4	4	4	5	5	5	6	7

By using the above table, the linear embedding of $K_{2,5,7}$ can be found. This linear embedding will have a minimized linear cutwidth and all of the regions will also have a minimized cutwidth. The following figures show the linear embedding using Bowles' arrangement (there are two possible ways to arrange the vertices in the middle). The m_1 vertices will be represented by white dots, the m_2 vertices will be represented by gray dots, and the m_3 vertices will be represented by black dots. The numbers above the cuts represent the cutwidth of each region and the numbers below the cuts represent what cut it is in accordance with the table above. The linear cutwidth is noted by placing the number in bold.

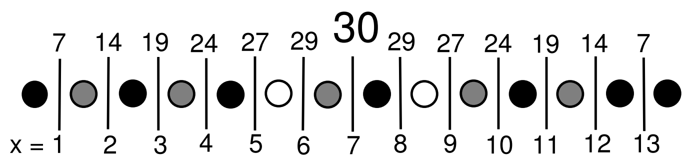

Figure 9: Tripartite Graph- Bowles' Arrangement 1
$K_{2,5,7}$

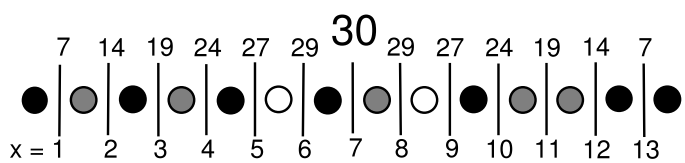

Figure 10: Tripartite Graph- Bowles' Arrangement 2
$K_{2,5,7}$

Now we will arrange the vertices with the help of the Theorems presented in this paper. To begin, we will look at the $(n-1)$-partite graph and then build off of that. Notice that we will build $K_{5,7}$ (K_{m_2,m_3}) since we are going to ignore the first set of vertices, m_1. The following figure gives the linear embedding of the $(n-1)$-partite graph where the gray dots represent m_2 and the black dots represent m_3.

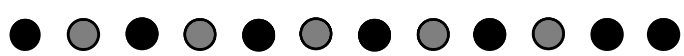

Figure 11: Bipartite Graph
$K_{5,7}$

To add the m_1 vertices, we will consider all of the vertices in the previous graph as one set (represented by X dots) and the m_1 vertices as another set (represented by white dots). This graph will then be a bipartite graph, in particular it will be $K_{2,12}$ ($K_{2,5+7}$). Therefore, we arrange the vertices the way it is outlined by WC1 and we receive the following linear embedding:

Figure 12: Bipartite Graph
$K_{2,12}$

Finally, to get the n-partite graph, we will impose the previous two linear embeddings on top of each other. This will give a linear embedding that has a minimized linear cutwidth. Once again, the bottom numbers represent which cut it is and the top numbers represent the cutwidth of each region. The linear cutwidth is noted in bold. Remember that the X dots in Figure 12 can either be a vertex from m_2 or a vertex from m_3, so one must go back to the bipartite graph of K_{m_2,m_3} to see which one it is. For example, the first dot in Figure 12 is a X dot, therefore we know we must look at $K_{5,7}$ to see which vertex comes first and it happens to be a vertex from m_3 which is represented by a black dot. So the first vertex for the n-partite graph is black and comes from m_3. The second dot in Figure 12 is a X dot as well, so we go back up to Figure 11 and see that the next dot is a gray dot, so we know the second vertex is a gray dot and comes from m_2 (we can't look at the first dot again because that vertex has already been layed down). We do this until we come to a white dot, which we know is simply a vertex from m_1. This procedure is done until all of the vertices have been used and the following linear embedding is obtained.

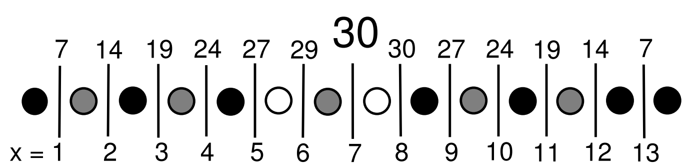

Figure 13: Tripartite Graph- WC2 Arrangement
$K_{2,5,7}$

But it can be seen that this arrangement does not give minimized cutwidths for every region. It does however give the minimized linear cutwidth which is what we claim in WC2 and WC3. It is also shown that the linear cutwidth still occurs at $x = 7$. Using WC2 and WC3 it can be seen that the calculated minimized linear cutwidth will agree with Bowles' results and the result shown in Figure 13 ($n = 3$ since it is a tripartite case- m_1, m_2, m_3).

$$lcw(K_{m_1,m_2,m_3}) = \sum_{i=0}^{3-2} \left\lceil \frac{(m_3 + \ldots + m_{3-i})m_{3-i-1}}{2} \right\rceil$$

$$= \left\lceil \frac{7 \cdot 5}{2} \right\rceil + \left\lceil \frac{(7+5) \cdot 2}{2} \right\rceil = 18 + 12 = 30$$

Therefore, it is shown that the minimized linear cutwidth is the result by using both Bowles' equation and by using WC2 and WC3.

8 Results

In conclusion, we were able to find an upper and lower bound for the linear cutwidth of an n-partite graph. These bounds happen to be equal to each other, therefore we know we have found the minimized linear cutwidth of an n-partite graph. We have also shown that this could be another proof for the linear cutwidth of complete graphs, K_n. For future research, we could find an arrangement of the vertices that would minimize the cuts in every region of an n-partite graph. This would therfore minimize the wirelength as well.

Acknowledgements

I would like to thank Dr.Chavez for being my mentor and for his guidance throughout the summer. I would also like to thank Dr.Trapp and all of the other REU participants for their encouragement and support. This project was completed during the 2005 REU in Mathematics at California State University at San Bernardino, jointly sponsored by CSUSB and NSF-REU Grant DMS-0453605.

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