An Upper Bound on Stick Numbers for Links

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Abstract

We employ a supercoiling tangle construction to economically construct any pieceweise linear integral tangle containing seven or more crossings. This method, which can be extended to construct links of large crossing numbers with minimal stick numbers, gives a new upper bound for the stick number of a minimal crossing algebraic link with crossing number greater than or equal to seven times the number of tangles and non-algebraic links with crossing number greater than four times the number of tangles.

1 Introduction

We will introduce some background information on integral tangles, algebraic tangles, arborescent links, basic polyhedrons, and non-algebraic links. We will also introduce the technique of constructing algebraic knots from the PL standard form of trees created by McCabe [2], and explain how non-algebraic can be constructed from their graphs.

A series of adjacent vertical or horizontal crossings between two strands with four arcs emerging in the four compass directions, NW, NE, SW, and SE, is called an *integral tangle*. Examples of integral tangles are depicted in part (a) of Diagram 1.1. Notice that each tangle has four strands leaving it. If we connect two arcs from one tangle to two arcs from another tangle, then the two tangles are said to be connected algebraically and create an *algebraic tangle*. The most basic type of algebraic tangle is an integral tangle. We can construct more complex algebraic tangles by connecting two integral or algebraic tangles together by two of their exiting strands. So an algebraic tangle is any combination of integral tangles connected by two strands with four strands exiting it. Examples of two algebraic tangles are depicted in part (b) Diagram 1.1.

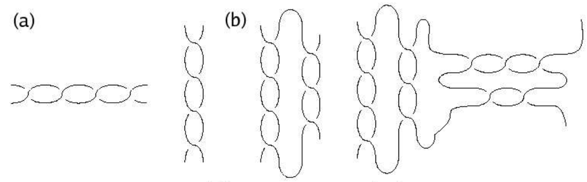


Diagram 1.1

Integral Tangles and Algebraic Tangles

According to Conway [3], any link projection can be viewed as a collection of algebraic tangles, and a link which possesses a projection consisting of only one algebraic tangle is called an *algebraic link*, as shown in Diagram 1.2.

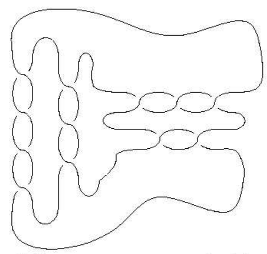


Diagram 1.2

An Algebraic Link

Arborescent links are simply links that can be constructed by connecting up the strands and tangles as specified by a tree. McCabe [2] states that all algebraic links are arborescent, and she goes on to give a standard form for the trees corresponding to algebraic links. In her standard form, McCabe defines *stumps* to be vertices of a tree that have only one edge connecting them to the rest of the tree. *Twigs* are defined to be vertices with two adjacent edges, one connecting it to the rest of the tree and one extending down to a stump. An example of a twig is circled in (c) of Diagram 1.3.1. According to McCabe, if there is a vertex in a PL standard form tree that adjoins only stumps and twigs with the possible exception of one edge that extends upward, and if there are at least two stumps and/or twigs extending downward from that vertex, then that vertex together with the adjoining stumps and twigs below it is a defined to be a *fan*. Some examples of fans are represented in Diagram 1.3.1 and their corresponding tangle projections are represented in 1.3.2.

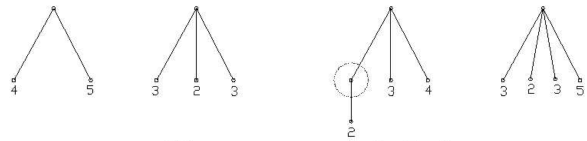


Diagram 1.3.1

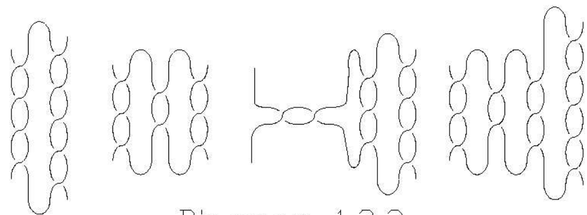


Diagram 1.3.2

Link Construction from a Tree in PL Standard Form

We will now run through two examples of construction from the trees (b) and (d) in Diagram 1.4.1. Let's start with tree (b). We will say that every stump has degree 4 for simplicity, and we will start our construction from the lower right fan because it is the lowest part of the tree. Since the four stumps of the fan lie in the third row, we will construct an algebraic tangle with four horizontal integral tangles stacked on top of each other, each containing four crossings. Now if we look at the vertex adjacent to the fan we see that it has a twig connected to it, and that twig has a stump located lying in the third row just to the left of the fan. This means that we need to connect another horizontal tangle with four twists to the left side of the algebraic tangle we just made. Now we can complete the link by connecting the strands leaving the algebraic tangle, NW to NE and SW to SE. The resulting link is shown in Diagram 1.4.2

Now we will construct the link for tree (d). Again we will say that each vertex is of degree 4, and we will start by constructing the lower right fan because it is in the lowest row. Its stumps are in the fourth row of vertices so they represent vertical tangles set side by side. The vertex above this fan has one stump in the third row extending off of it and to the left of the fan, so we will connect a horizontal integral tangle to the left side of the fan. Now the vertex above that has another fan extending down from it with stumps in the third row. So we will attach a fan with two horizontal tangles to the left side of the algebraic tangle we just constructed. Now to finish up the link, again we connect the strands exiting the algebraic tangle, NW to NE and SW to SE. The resuling link is shown below in Diagram 1.4.2.

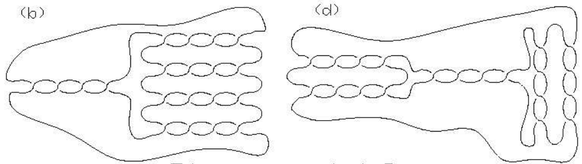


Diagram 1.4.2

Now that we have seen a couple examples of fan construction from the PL standard form of a tree, we should note a few of the PL standard forms attributes.

(1) Every PL standard form tree has at least one fan.

(2) If there is more than one fan in a tree, then each fan will appear at the bottom of the branch it is connected to.

(3) Starting at the bottom fan, the vertex above it will connect to one of the following (a) a stump, (b) a twig, (c) another fan, or (d) a more complex portion of the tree with at least one fan farther down, as shown in Diagram 1.4.1.

(4) The top vertex of the tree always has degree zero and represents a horizontal zero integral tangle, the second row of vertices represent vertical integral tangles, the third row of vertices represent horizontal integral tangles, and this alternating pattern continues all the way down the tree. So the odd rows represent horizontal tangles and the even rows represent vertical tangles.

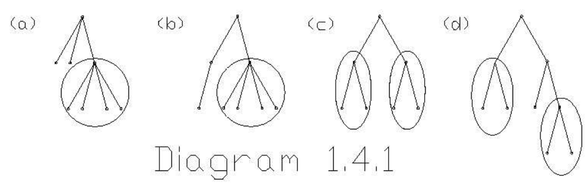


Diagram 1.4.1

Basic Polyhedron and Non-Algebraic Links

According to Conway [3], a *basic polyhedron* is an edge-connected 4-valent planar map containing no regions with two vertices. We can obtain knot diagrams from polyhedra by substituting algebraic tangles in for vertices. Note that since an algebraic link has only one algebraic tangle, its basic polyhedron will have only one vertex, and will look like a figure eight, as shown in Diagram 1.5.

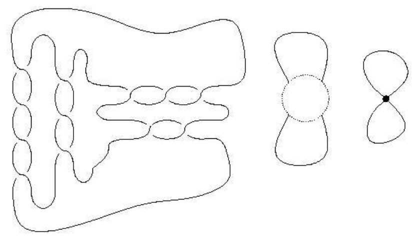


Diagram 1.5

Any non-algebraic link can be constructed from a basic polyhedron with every vertex connecting to four other distinct vertices. Two basic examples of graphs that can be used to construct non-algebraic links are depicted in Diagram 1.6 as well as a non-algebraic link constructed by replacing each vertex with an algebraic tangle. Note that since a non-algebraic link comes from a polyhedron with regions surrounded by four vertices, every algebraic tangle in a non-algebraic link will be connected to four distinct algebraic tangles.

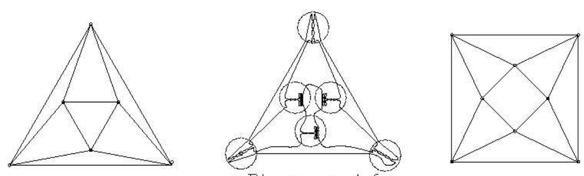


Diagram 1.6

2 The Supercoiling Construction Method of an Integral Tangle

Let's begin by examining how supercoiling works. If we look at the two integral tangles in Diagram 2.0, notice that there are two outside strands and inside of them is a dotted core. We will say that the number of times that the outer two strands cross each other is the *crossing number*, and that the number of times that the dotted core crosses over itself is the *writhe*. In the case of the integral tangle (a) in Diagram 2.0 the two outer strands cross eachother 4 times, yet the dotted core is planar and does not cross itself at all. Therefore the crossing number is 4 and the writhe is 0. Now if we look at integral tangle (b), it is evident that the writhe is 1, and it appears that the crossing number is only 2. In fact however, wherever the core crosses itself, the two strands cross each other twice, as shown in the circled regions. Therefore integral tangle (b) has four crossings. Now if we grab ahold of the ends of the core and the two strands in (b) and pull on their edges, straightening the core out so that it is planar, then the writhe becomes 0, but the crossing number is 4. Hence (a) and (b) are projections of the same integral tangle. This means that we can add two crossings to a tangle by either adding a full twist to the two outside strands or adding a crossing to the inner core. When we choose to add a crossing to the inner core we increase the writhe by 1. This process of adding crossings to an integral tangle by writhing is known as *supercoiling*, and is found naturally in DNA strands [6].

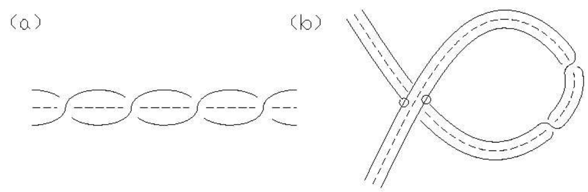


Diagram 2.0

Need to fix crossings!

PUT IN PICTURE OF MCCABES TANGLE!!!!!!!!!!!!!!!!!!!!!!! McCabe [2] constructed tangles with stick number equal to the crossing number plus one. With the supercoiling method of constructing piecewise linear integral tangles we can match or improve on this upper bound for tangles where the crossing number of the tangle is greater than or equal to seven times the number of tangles. Below is the supercoiling construction method for integral tangles.

Theorem 1. *Let T be an integral tangle with crossing number $c(T) \geq 7$, then the upper bound on stick number is $s(T) \leq \left\lceil \frac{2c(T)}{3} + \frac{10}{3} \right\rceil$.*

Proof

We will start by drawing a horizontal dotted line and labeling its left endpoint 1 and its right endpoint 2. Then we will draw another dotted line extending down from vertex 2 at a -45 degree angle with the horizontal and label its endpoint 3. Now we can draw another dotted line from vertex 3 staight up at a 90 degree angle with the horizontal, labeling its vertex 4. If we think of the line $(1, 2)$ as lying on a plane, we can lower point 3 so that it lies just below that plane. After lowering point 3 just slightly, the line $(3, 4)$ ends up crossing under $(1, 2)$. Our result is the core of our tangle and is pictured below in Diagram 2.1 below.

Then at endpoint 1 of the line $(1, 2)$, we can push off a small distance directly below the vertex, draw a point, and label it A_1. Then we can push off the same distance directly above vertex 1 and draw another point labeled B_1. At endpoint 2 we will repeat this process labeling the point below the vertex B_2 and the point above A_2.. We will draw a line connecting the upper left point B_1 to the upper right point A_2, and another line connecting the lower left point A_1 with the lower right point B_2. The resulting graph will look like Diagram 2.2.

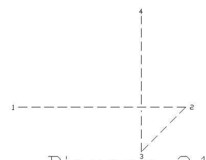


Diagram 2.1

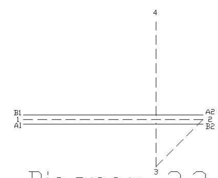


Diagram 2.2

Now at vertex 3 we will push off to the right and draw a point A_3 and to the left draw a point B_3. Now we draw the lines (B_2, A_3) and (A_2, B_3), and the resulting graph should look like Diagram 2.3 below.

We will now move to vertex 4, pushing off to the right to draw point B_4 and to the left to draw point A_4. Now we can draw the lines (A_3, B_4) and (B_3, A_4). The result is Diagram 2.4.

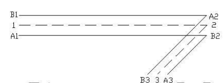


Diagram 2.3

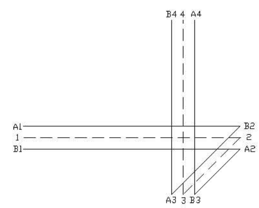


Diagram 2.4

Recall that we have drawn line $(3, 4)$ so that it lies just below line $(1, 2)$. If we think of lines (B_3, A_4) and (A_3, B_4) as running along at an arbitrarily small distance from $(3, 4)$, then (B_3, A_4) and (A_3, B_4) will also cross just under (B_1, A_2) and (A_2, B_1). So we can trim (B_3, A_4) and (A_3, B_4) to show that they cross under $(1, 2)$,(B_1, A_2), and (A_2, B_1). Diagram 2.5 is a close up view of how this crossing should look.

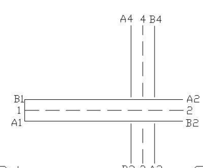


Diagram 2.5

Let's now consider the point labeled A_1 as being raised up an arbitrarily small distance above point 1, and B_1 as lowered below point 1. Continue this process, raising all points $(A_2, \cdots, A_4)$ above their nearest vertex on the dashed core, and lowering all points $(B_2, \cdots, B_4)$ below their nearest vertex on the dashed core. (Think of A as standing for above and B as standing for below.)

Now that all $(A_1, \cdots, A_4)$ are above $(1, \cdots, 4)$ which are above $(B_2, \cdots, B_4)$ we can determine whether a strand has over or under crossing by looking at its nearest vertex. If its vertex is labeled B then it will be an under-crossing. If it is labeled A then it will be an over-crossing. If it is on the dashed line (ie. it is labeled by a whole number) then it will be above the B but below the A. Hence we end up with Diagram 2.6.

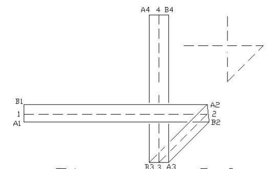


Diagram 2.6

If we were to replace the two rigid outside strands of Diagram 2.6 with flexible strands then we would end up with the usual representation of an integral tangle, two strands twisted around each other. This transformation is shown in Diagram 2.6.1. Note that the crossing number for this tangle is $c(T) = 4$.

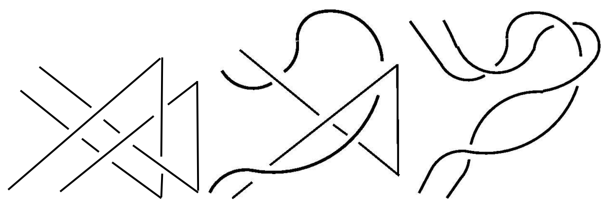


Diagram 2.6.1

We could add an additional 3 crossings for every two sticks by repeating the following process. Draw another dotted line segment from endpoint 4 down over the line segment $(1, 2)$ creating an endpoint 5. A_5 should lie to the left of 5 and B_5 should lie to the right. Then draw the solid lines (B_4, A_5) and (A_4, B_5). The result should be Diagram 2.7. With the addition of two more sticks we now have a tangle with three more crossings. Thus it has $s(T) = 8$ and $c(T) = 7$. Notice that we can continue this process of adding two sticks and three crossings indefinitely.

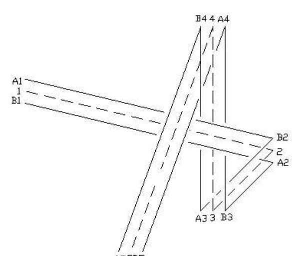


Diagram 2.7

So we have a method for constructing supercoiled integral tangles with $c(T) \equiv 1 \bmod 3$. Now if we take the tangle in Diagram 2.6 and erase lines (A_2, B_3) and (B_2, A_3) and draw lines (A_2, A_3) and (B_2, B_3), we end up with Diagram 2.8.1. Notice that (A_2, A_3) will cross over (B_2, B_3) three different times. The tangle is not alternating, but if use a type II Reidemeister move and push the point A_3 up so that it lies on the other side of (B_2, B_3), as shown in Diagram 2.8.2, we have constructed an alternating tangle with $s(T) = 6$ and $c(T) = 3$. Now if we were to follow our general process of adding two more sticks and three more crossings, as we did before, we will get a tangle with $s(T) = 8$ and $c(T) = 6$. This process can be repeated to obtain any tangle with crossing number $c(T) \equiv 0 \bmod 3$.

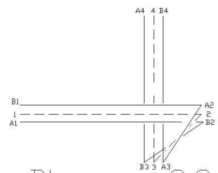


Diagram 2.8.1

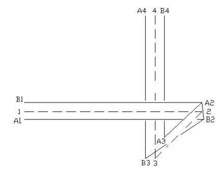


Diagram 2.8.2

For tangles with crossing numbers $c(T) \equiv 2 \bmod 3$ we use a slightly different process to modify the tangle in Diagram 2.6. Starting with that tangle we erase lines A_2, B_3 and B_2, A_3, and place an extra point E to the right of A_3, as shown in Diagram 2.9.1. Then we draw line (A_2, A_3), and lines (B_2, E) and (E, B_3). Point E should be raised above both points A_2 and A_3. So line (B_3, E) will cross under (A_3, B_4) and over (A_2, A_3). The result is the tangle in Diagram 2.9.2. Notice that this tangle has a stick number $s(T) = 7$ and crossing number $c(T) = 5$. If we added another two sticks and three crossings we would get $s(T) = 9$ and $c(T) = 8$. Indeed we can continue this process for all super-coiled integral tangles of crossing numbers $c(T) \equiv 2 \bmod 3$.

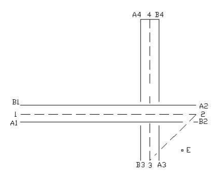


Diagram 2.9.1

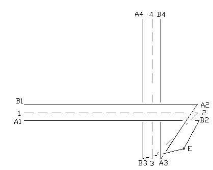


Diagram 2.9.2

Using this construction, the stick number for a tangle $s(T)$ for a tangle with crossing number $c(T)$ is $s(T) \leq \left\lceil \frac{2c(T)}{3} + \frac{10}{3} \right\rceil$. The $\frac{2}{3}$ comes from adding two sticks and three crossings and the $\frac{10}{3}$ comes from the fact that you do not get any crossings until the fourth stick, (Consult Diagram 2.3). □

It should be noted that, although McCabe's tangle construction is more efficient for tangles with crossing number $1 \leq c(T) \leq 6$, the supercoiling construction is as effective as McCabe's for $7 \leq c(T) \leq 9$, and is the better method for minimizing stick number of an integral tangle of any crossing number $c(T) \geq 10$.

3 Standard Form of Integral Tangle

Lemma 1. *We can construct any supercoiled integral tangle while keeping the endpoints of the strands exiting the tangle fixed in the same plane.*

Proof

We will first draw the core. Looking at Diagram 3.1, notice that points 1, 2, 3, and 4 are all fixed in the same plane and that the crossing between, $(1,4)$ and $(2,3)$ is simply a double point. If we lower point 4 so that it is just slightly below the plane of $(1,2,3)$ then $(1,4)$ will cross under $(2,3)$, and points 1 and 2 remain fixed. Diagram 3.2 shows the resulting core.

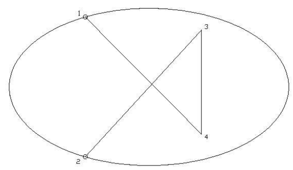


Diagram 3.1

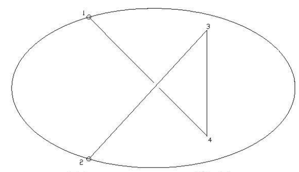


Diagram 3.2

Looking at Diagram 3.3, notice that to construct this core we simply replaced line $(3,4)$ with the two line segments $(3,5)$ and $(4,5)$. Therefore points 1, 2, and 3 still lie in the same plane. We can lower point 5 just slightly below point 4 and we will again have an alternating core with points 1 and 2 still fixed in the same plane as shown in Diagram 3.4.

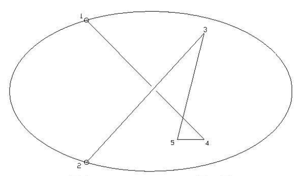


Diagram 3.3

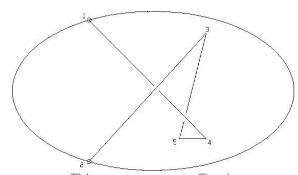


Diagram 3.4

The core in Diagram 3.5 can be constructed by simply replacing line segment $(4,5)$ with two segments $(4,6)$ and $(5,6)$, and raising point 6 slightly above point 4.

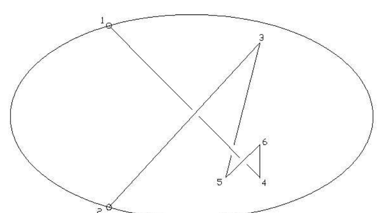


Diagram 3.5

We can repeat this process in order to construct a core for any crossing tangle we wish, while still keeping the strands exiting the tangle, $(1,4)$ and $(2,3)$, fixed throughout the process. Now that we know we can keep the points 1 and 2 fixed in the same plane while constructing an alternating core, we will add the two piecewise linear strands. We will keep the points A_1, B_1, A_2, and B_2 fixed in the same plane as 1 and 2, only raising the points $A_3, A_4, \cdots, A_6$ a small distance above the core and lowering points $B_3, B_4, \cdots, B_6$ below the core. As shown in Diagram 3.6. Note that the strands (A_1, B_4), (A_4, B_1), (A_2, B_3), and (A_3, B_2) all intersect the plane $(1, 2, 3)$ in the arbitrarily small neighborhoods of points 1 and 2. Note that while the tangle in Diagram 3.6 has ten crossings, we can extend this same process for any number of crossings and still keep the strands exiting the tangle fixed in the same place. □

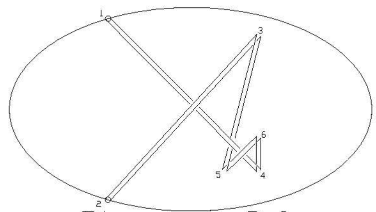


Diagram 3.6

One last note about Diagram 3.6 is that the neighborhoods of points 1 and 2 do not have to lie on the circle we can adjust the core so that the points 1 and 2 as far away from the tangle as we wish. We could extend these neighborhoods so far away from the tangle, that the tangle itself appears to be a vertex. This will become important in the construction of fans.

Now let's examine the characteristics of vertical and horizontal piecewise linear supercoiled integral tangles. Diagram 3.7 shows supercoiled vertical and horizontal tangles that are constructed using sticks. With supercoiled tangles we can see that any vertical tangle will have four strands exiting out of each of the four quadrants NW, NE, SW, and SE. So our tangles are in agreement with Conway's definition of an integral tangle[3].

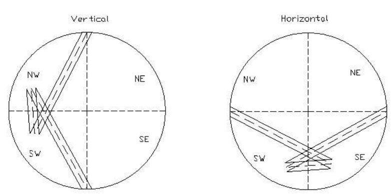


Diagram 3.7

We should note that there is also a great deal of flexibility as to where those strands exit out of their respective quadrants. Looking at Diagram 3.8 we can see that we can rotate the inside strands of a tangle from lying parallel to the outside strands to any angle inside the outer strands as long as they remain crossing. If we spread the outer strands to 179 degrees apart from one another, then the two inside strands can essentially rotate anywhere in that 179 degree span provided they remain crossing each other.

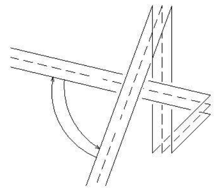


Diagram 3.8

Lemma 2. *All supercoiled piecewise linear integral tangles can be constructed so that their two outside strands exit at an angle arbitrarily close to 180 degrees from each other.*

Proof

If we look at the tangle in Diagram 3.9 and flatten its edges together like an accordion so that the vertices $3, 4, 5, 6, 7, 8$ are arbitrarily close, then we could create a tangle with the two outside edges extending out just shy of 180 degrees, as depicted in Diagram 3.10.

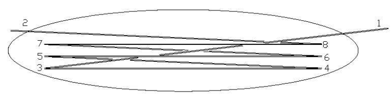


Diagram 3.9

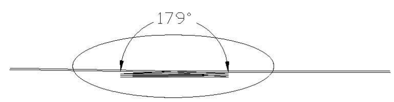


Diagram 3.10

Note that we could add as many vertices to this tangle as we like, and since the strands of a knot are of an infitesimal thickness, it follows that, no matter how many crossings a tangle has, its outside strands can still be made to exit at an angle slightly less than 180 degrees apart from one another. □

Furthermore, if we decide to construct all tangles so that their two outside strands extend outward at an angle of 179 degrees, then we can rotate the two inside strands to any angle inside that 179 degrees as long as they remain crossing, as shown shown in Diagram 3.11.

We will consider the form of the integral tangle where the outside strands are 179 degrees apart and the inside strands are free to rotate inside that 179 degrees to be our standard building blocks of piecewise linear algebraic links, and thus, we call this form our *PL standard form of integral tangles.*

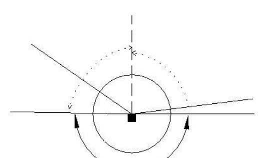


Diagram 3.11

PL Standard Form of a Supercoiled Integral Tangle

4 Construction of Fans

We now begin constructing fans using our supercoiled tangles made of sticks.

Theorem 2. *We can construct a vertical or horizontal fan with* $T_1, T_2, \ldots, T_n$ *supercoiled tangles with crossing numbers* $c(T)$ *by attaching the cores of those* n *tangles at two set vertices* 1 *and* 2*, and connecting the supercoiled tangles so that the fan's stick number is* $s(F) \leq \sum_{i=1}^{n} \left\lceil \frac{2c(T)}{3} + \frac{10}{3} \right\rceil$

Proof

Recall that the neighborhoods of points 1 and 2 can lie anywhere on the strands that we choose. These neighborhoods simply denote where the four exiting strands intersect the plane $(1, 2, 3)$. Now if we draw two cores with strands departing at slightly different angles, we can connect the tangles by adjoining the strands at the neighborhoods of points 1 and 2, as shown in Diagram 4.1.

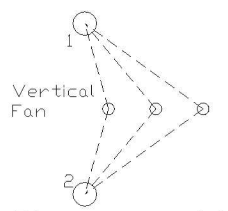


Diagram 4.1

THERE IS STILL THE FAN WITH ONE TWIG, BUT THIS CAN BE CONNECTED WITHOUT ADDING ANY ADDITIONAL STICKS AS SHOWN IN DIAGRAM 4.3.1.

Diagram 4.2 is a horizontal fan, but a vertical fan can be constructed by rotating a horizontal fan 90 degrees. Note that we can continue adding tangles to our fan by attaching more integral tangle cores to points 1 and 2, thus constructing a fan with $T_1, T_2, \ldots, T_n$ integral tangles as is depicted in Diagram 4.3. Since we can construct any fan of T_n tangles by using only the sticks in the integral tangles our stick number for the fan is simply the sum of the stick numbers of our n tangles $s(F) \leq \sum_{i=1}^{n} \left\lceil \frac{2c(T_i)}{3} + \frac{10}{3} \right\rceil$. $\square$

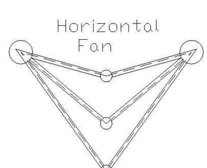


Diagram 4.2

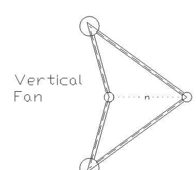
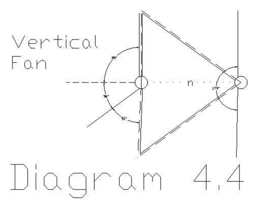
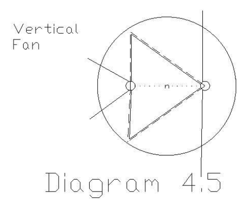


Diagram 4.3

Lemma 3. *Since we construct our fans from PL standard form integral tangles, the strands will exit the fan in the same manner as the strands exiting our integral tangles.*

Proof
Looking at Diagram 4.4 we can see that, like an integral tangle, we can rotate the exiting strands of a fan so that the outside strands exit the fan at an angle of 179 degrees from each other, and that the inside strands can rotate within that 179 degree span as long as they cross. □ Looking at Diagram 4.5, note that if we cover up the circle so all we see are the strands exiting it, we would not be able to tell a fan from our standard form of an integral tangle. Therefore, a fan of this form will be our *PL standard form of a fan*.





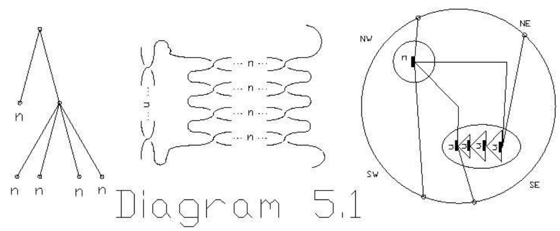
5 Constructing Algebraic Links by Connecting Tangles and Fans

Now that we know how to construct supercoiled integral tangles out of sticks and fans out of supercoiled integral tangles, all that remains is to connect them up.

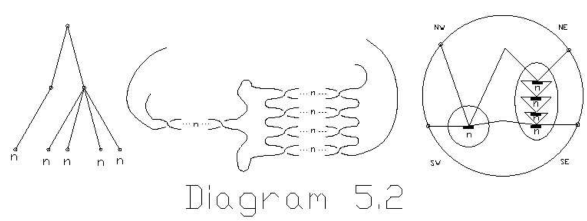
Theorem 3. *The stick number of any algebraic link with $T_1, T_2, \ldots, T_n$ integral tangles is* $s(K) \leq \sum_{i=1}^{n} \left\lceil \frac{2(c(T_i))}{3} + \frac{10}{3} \right\rceil + 2$.

Proof
From property 4 of a PL standard form tree in the introduction: Starting at the bottom fan and constructing it in our PL standard form for fans, we will have a stick number $s(F) \leq \sum_{i=1}^{n} \left\lceil \frac{2(c(T))}{3} + \frac{10}{3} \right\rceil$. Now there are four possible objects the fan will have to connect up to.

Case 1: When a fan is connected to a stump. The stump corresponds to a tangle of a different orientation (ie. if the fan is vertical, then it will connect to a horizontal tangle). This can be done without adding any sticks, as shown in Diagram 5.1. So the algebraic tangle formed in case 1, containing a fan and an integral tangle, will have a stick number $s(F) + s(T)$, which is the sum of the stick numbers of the integral tangles in the fan, plus the stick number of the one additional integral tangle. Therefore the stick number of the new algebraic tangle will simply be the sum of the stick numbers of all the integral tangles contained in that algebraic tangle. If we say that there are n integral tangles contained in this new algebraic tangle, then $s(C_1) \leq \sum_{i=1}^{n} \left\lceil \frac{2c(T)}{3} + \frac{10}{3} \right\rceil$. Note that since it is constructed from PL standard integral tangles, the algebraic tangle formed in case 1 has strands that exit in the same fashion as our PL standard integral tangle.



Case 2: When a fan is connected to a twig. The twig corresponds to a tangle with the same orientation (ie. if the fan is vertical, the tangle will be vertical as well). This can be done without adding any additional sticks, as shown in Diagram 5.2. So the stick number of the new algebraic tangle constructed in case 2 is just the sum of the stick numbers for all of the integral tangles in this new algebraic tangle. Again if we say that this tangle has n integral tangles, then $s(C_2) \leq \sum_{i=1}^{n} \left\lceil \frac{2c(T)}{3} + \frac{10}{3} \right\rceil$. Like the tangle formed in case 1, the algebraic tangle formed in case 2 also has strands that exit in the same fashion as our PL standard integral tangle.



Case 3: When a fan is connected to another fan. The fan that we are connecting to will be of the same orientation as the fan we started with, and this can be done without adding any new sticks, as shown in Diagram 5.3. Thus the algebraic tangle containing two fans will have a stick number of $s(F_1)+s(F_2)$ Here again if there are n integral tangles in the algebraic tangle constructed in case 3, then $s(C_3) \leq \sum_{i=1}^{n} \left\lceil \frac{2c(T)}{3} + \frac{10}{3} \right\rceil$, and the algebraic tangle has strands that exit in the PL standard form of an integral tangle.

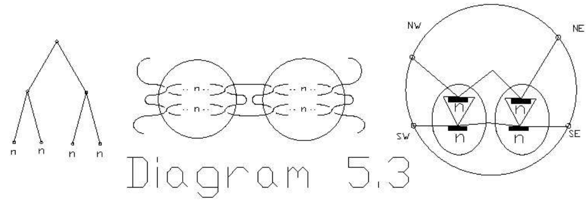


Diagram 5.3

Case 4: When a fan is connected to a more complex portion of a tree with a fan farther down. The complicated algebraic tangle will be a combination of the algebraic tangles in cases 1 through 3, and can be connected without adding anymore sticks, as shown in Diagram 5.4. The stick number of the more complicated portion of the tree $s(CP)$ will be the sum of the sticks used in all the integral tangles contained in that portion, so $s(CP) \leq s(T_1)+s(T_2)+\ldots+s(T_n)$. Thus the stick number of the algebraic tangle constructed in case 4 will be the sum of the stick numbers of the integral tangles in the fan plus the sum of the stick numbers of the integral tangles in the more complicated portion. Hence, if the algebraic tangle constructed in case 4 contains $T_1, T_2, \ldots, T_n$ integral tangles, then the stick number is the sum of the stick number of all the integral tangles contained in that algebraic tangle, $s(C_4) \leq \sum_{i=1}^{n} \left\lceil \frac{2c(T)}{3} + \frac{10}{3} \right\rceil$. Yet again, the algebraic tangle in case 4 has strands exiting in a similar fashion to the PL standard integral tangle.

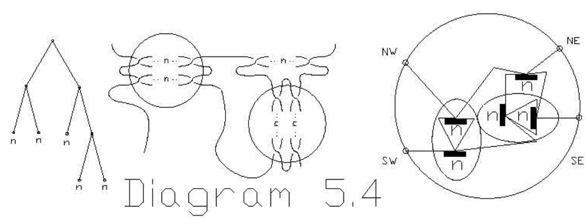


Diagram 5.4

To conclude, we can construct any algebraic tangle we wish without adding anymore sticks than those contained in the integral tangles. To connect up the four open strands of any algebraic tangle we will need at most 2 additional sticks. Ergo, $s(K) \leq \sum_{i=1}^{n} \left\lceil \frac{2(c(T))}{3} + \frac{10}{3} \right\rceil + 2$. □

Since all of the cases form integral tangles with strands exiting in the fashion of an integral tangle, we will define algebraic tangles of this form our *PL standard algebraic tangles.*

6 A New Upper Bound for Stick Number of Non-Algebraic Links

Negami proved that the stick number for any knot is $s(K) \leq 2c(K)$, which is the only known upper bound on the stick number of non-algebraic links. If the number of crossings in a non-algebraic link equals the number of tangles, then this bound is sharp. However if the crossing number is somewhat greater than the twist number, then a better bound can be determined.

Theorem 4. *If $T \leq c(K) \leq 4T$ then $s(K) \leq 2c(T)$, but if $c(K) \geq 4T$ then if there are $T_1, T-2, \ldots, T_n$ integral tangles contained a non-algebraic link, then* $s(K) \leq 2T + \sum_{i=1}^{n} \left\lceil \frac{2c(T_i)}{3} + \frac{10}{3} \right\rceil$.

Proof

Notice that the edge number of any graph representing a polyhedral link is two times the number of vertices, because each vertex has degree 4. If we let each edge correspond to a stick and each vertex correspond to either an algebraic or integral tangle, then the stick number of the graph is $s(K) = 2T$. Now let's take these vertices and blow them up into little circles, as shown in Diagram 6.1.

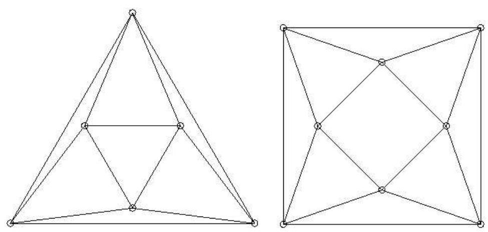


Diagram 6.1

Now if we can show that it is possible to take our supercoiled tangles and place them in each circle so that the strands exiting the tangle attach to the edges of the planar graph, then we can add the stick number of each tangle with the 2T sticks of the planar graph and we get a link with $s(K) \leq 2T + \sum_{i=1}^{n} \left\lceil \frac{2c(T_i)}{3} + \frac{10}{3} \right\rceil$.

Looking at any vertex of the graph in Diagram 6.1 it is easy to see that we can in fact connect up the 4 exiting strands of our tangle with the four edges of the graph. As long as we take the two outside edges of our tangle and connect them up with two adjacent edges of the graph. Then connecting the two inside strands to the two remaining edges of the graph is quite simple, as shown in Diagrams 6.2 and 6.3.

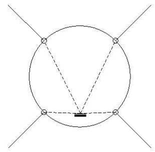


Diagram 6.2

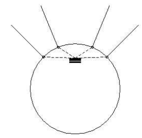


Diagram 6.3

If we look at Diagram 6.4 we can see that by connecting the two outside strands to two adjacent edges of the graph we leave the two inside strands open to connect to the other two edges of the graph, but the two inside strands are free to rotate within that 179 degree span. Ergo, no matter where the other two edges of the graph intersect the circle encompassing the tangle, we will be able to rotate the two inside strands so that they connect to these edges. Recall that the strands exiting any of our algebraic tangles behave just like the strands exiting our integral tangles; so the same construction process works for them as well. Thus, our stick number for any alternating non-algebraic knot is just the sum of the stick numbers of our integral tangles, plus the edges of the graph. If we have $T_1, T_2, \ldots, T_n$ integral tangles contained in our non-algebraic link, then the stick number will be $s(K) \leq 2T + \sum_{i=1}^{n} \lceil \frac{2c(T_i)}{3} + \frac{10}{3} \rceil$. $\square$

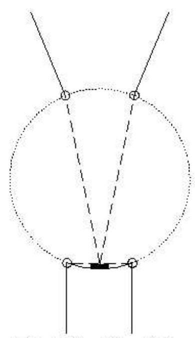


Diagram 6.4

7 Conclusions

Now that we know that the stick number for any knot with $c(K) \geq 7n$, where T_n is the number of integral tangles, is $s(K) \leq \sum_{i=1}^{n} \left\lceil \frac{2(c(T_i))}{3} + \frac{10}{3} \right\rceil + 2$ we could attempt to improve on this bound by saving additional sticks when connecting up separate algebraic tangles. If we are able to run one strand of an algebraic tangle straight into another algebraic tangle, then there is the possibility of a considerable reduction in stick number. Likewise, now that we know the stick number of any non-algebraic link is $s(K) \leq 2T + \sum_{i=1}^{n} \lceil \frac{2c(T_i)}{3} + \frac{10}{3} \rceil$, we can strive to improve this bound by allowing the edges of the graph to rotate out of the plane. If feasible, this rotation would allow us to run a strand from one algebraic tangle into another, saving us atleast one stick per tangle.

8 Acknowledgements

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