On The Cyclic Cutwidth of Complete Tripartite and n-Partite Graphs

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Abstract

The cyclic cutwidth of the complete tripartite graph $K_{r,s,t}$ is explored. Previous work has been done on the cyclic cutwidth of complete bipartite graphs as well as the linear embedding of tripartite graphs. These results will be used to build on the cyclic cutwidth of the complete tripartite graph. The cyclic cutwidth of complete tripartite and n-partite graphs is found for some cases. An upper bound and lower bound for other cases is also explored.

1 Introduction

A graph G consists of a set of vertices and a set of edges that join pairs of vertices.

A complete tripartite graph $K_{r,s,t}$ consists of three disjoint sets of vertices A, B, and C, with $|A| = r$, $|B| = s$, and $|C| = t$, such that every vertex is joined to every other vertex except those that are in their own set. Unless otherwise stated, vertices from set A are denoted by white circles, vertices from set B are denoted by black circles and vertices from set C are denoted by gray circles for ease of interpretation. Figure 1 is one possible representation of $K_{1,2,3}$.

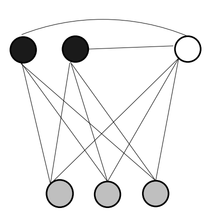

Figure 1

A linear embedding of a graph G is a representation of G with all of the vertices and edges of G embedded onto a line. The linear cutwidth of complete tripartite graphs has been explored. These findings help in the exploration of the cyclic cutwidth of complete tripartite graphs.

A cyclic embedding of a graph G is a representation of G with all of the vertices and edges of G embedded onto a circle. Any edges that connect vertices in the non-cyclic representation of G will also connect the same vertices in the cyclic embedding of G. To simplify the representation of a cyclic embedding on paper, we draw the edges inside the circle on which they are embedded. However, no edges on this paper-representation can pass through the exact center of the circle. That way we will know exactly which regions the edge is contributing to. Figure 2 is an example of a cyclic embedding of $K_{1,2,3}$.

A region in a cyclic embedding is the wedge-shaped area created by the center of the circle, two consecutive vertices, and the edge of the circle as in Figure 2 where one wedge shaped region is shaded.

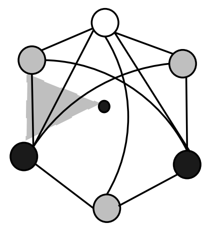

Figure 2

For a cyclic embedding the cut of a region is the number of edges that cross the region. The cut of the shaded region in Figure 2 is 3. The maxiumum cut of a particular embedding of a graph is the largest cut that occurs on the embedding. The maximum cut of Figure 2 is 5. The cutwidth of a graph G is the minimum of all maximum cuts of all possible embeddings on G. So there is probably a different way to draw Figure 2 so that the cutwidth is less than 5. The difference between cyclic and linear embeddings is that with cyclic embeddings one can decide which way the edges go around the middle vertex. For example in Figure 2, some of the diagonals could be moved around the other side of the center to create a lower cyclic cutwidth.

2 Background

Graph theory has proven to be a useful tool for analyzing situations in which two sets of elements are joined by some sort of edge. These situations include electrical networks, telephone communication, road maps, oil pipelines, and subway systems. There are also other forms with which a graph can be used such as flow charts, organizational charts, computer data structures, evolutionary trees in biology, and the scheduling of tasks in a complex project (Tucker, 1995).

This paper is primarily concerned with finding the cyclic cutwidth of complete tripartite and n-partite graphs. It will build upon the works of others who have found the linear cutwidth of complete tripartite and n-partite graphs and the cyclic cutwidth of complete bipartite graphs. We will also explore ideas of how linear cutwidth of graphs relate to the cyclic cutwidth of graphs.

2.1 Some Known Cyclic Cutwidths

Many different people have worked with the problem of cyclic cutwidths. Fransisco Rios [6] developed a formula for the cyclic cutwidth of a complete graph K_n. The following results were proven by Rios:

For any complete graph K_n on n vertices,

$$lcw(K_n) = \begin{cases} \frac{n^2}{4}, & \text{n even} \\ \frac{n^2-1}{4}, & \text{n odd} \end{cases}$$

A two-dimensional mesh P_m x P_n is a rectangular graph that has dimensions m by n. Concerning the cyclic cutwidth of two-dimensional meshes Heiko Schroder [7] made some progress which Dwayne Clark [3] later amended. Dwayne Clark found that for a graph G which is a P_m x P_n mesh where $m \leq n \leq 3$,

$$ccw(G) = \begin{cases} n-1, & m = n \text{ even} \\ n, & m = n, n+1, \text{ and } n \text{ odd or } m = n+1,\ n+2, \text{ and } n \text{ even} \\ n+1, & \text{otherwise} \end{cases}$$

A tree (T) is a connected a-cyclic graph. Joe Chavez and Rolland Trapp [2] have proven that for any tree T,

$$lcw(T) = ccw(T).$$

2.2 Linear Cutwidth and Cyclic Cutwidth of Complete Bipartite Graphs

Matt Johnson [5] has proven the following three theorems:

Johnson's Theorem 1:

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2}, & mn \text{ even} \\ \frac{mn+1}{2}, & mn \text{ odd} \end{cases}$$

Johnson Theorem 2: For any graph G,

$$ccw(G) \geq \frac{lcw(G)}{2}.$$

Johnson Theorem 3:

$$ccw(K_{m,n}) = \begin{cases} \frac{mn}{4}, & m, n \text{ both even} \\ \frac{mn+3}{4}, & m = n \text{ odd} \end{cases}$$

Of these three theorems the one we will use the most is Theorem 2: For any graph G,

$$ccw(G) \geq \frac{lcw(G)}{2}.$$

Since this idea is used so much throughout this paper, I will show Johnson's proof of it.

Proof of Johnson's Theorem 2

Let y be the linear cutwidth of G. Consider any cyclic embedding of G with cyclic cutwidth x. Assume that $x < \frac{y}{2}$.

We shall number the vertices of the cyclic embedding clockwise from a_1 to a_n beginning at a region where the cutwidth x occurs. We also number the cuts of the graph such that the cut counterclockwise and adjacent to a_i will be α_i (Figure 3).

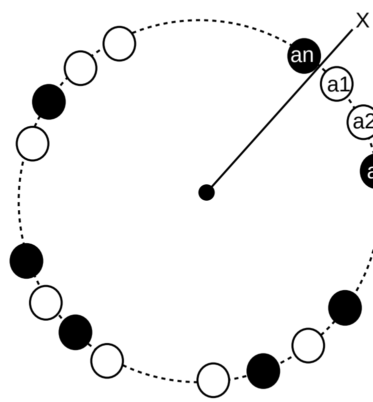

Figure 3

We now transform the cyclic embedding of G into a linear embedding. To do this we shall arrange each of the vertices, a_i in order on a linear embedding, connecting all of the vertices that were connected in the cyclic embedding. (Figure 4)

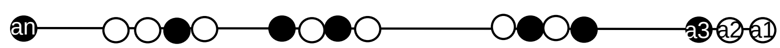

Figure 4

Let α_i be the cut on the cyclic embedding such that when transformed into a linear graph a maximum cut occurs at the cut α_i. Assume l is the number that the cut α_i increases by in the linear embedding. So the maximum linear cut is $\alpha_i + l$. We know

$$\begin{array}{rcll} \alpha_i + l & \leq & \alpha_i + x & \text{since } l \leq x \\ & \leq & 2x & \text{since } \alpha_i \leq x \\ & \leq & y & \text{by hypothesis} \end{array}$$

This is a contradiction since $\alpha_i + l$ is actually equal to y, but this chain of inequalities shows that $\alpha_i + l$ is less than or equal to y. y cannot be less than or equal to y.

Thus,

$$ccw(G) \geq \frac{lcw(G)}{2}.$$

2.3 Linear Cutwidth of Complete Tripartite Graphs

Stephanie Bowles [1] found the linear cutwidth of a complete tripartite graph.

Bowles' Theorem states:

Let $K_{r,s,t}$ be a complete tripartite graph. Then:

$$lcw(K_{r,s,t}) = \begin{cases} \frac{rs+rt+st}{2} & \text{for two or more } r,s,t \text{ even} \\ \frac{rs+rt+st+1}{2} & \text{otherwise} \end{cases}$$

Bowles found these results branching off of previous theorems about linear cutwidth of complete bipartite graphs. She especially built off of Matt Johnson's three theorems concerning linear and cyclic cutwidths of complete bipartite graphs.

2.4 Linear Cutwidth of Complete n-Partite Graphs

Chelsea Weitzel [9] proved the linear cutwidth of a complete n-partite graph.

The Weitzel-Chavez Theorem states:

$$lcw(K_{m_1,m_2,\ldots,m_n}) = \sum_{i=0}^{n-2} \left\lceil \frac{(m_n + \ldots + m_{n-i})m_{n-i-1}}{2} \right\rceil$$

Weitzel discovered this theorem from the results found by Bowles and then branching out to 4-partite graphs, before exploring n-partite graphs. This is an important result used later in this paper.

2.5 Cyclic Cutwidth of Complete Bipartite Graphs

Megan Holben [4] extends on the lower bound and upper bounds of the cyclic cutwidth of complete bipartite graphs by finding exact bounds for many different cases. In the end her findings came up with this final set of theorems, where the first equation is essentially drawn from Johnson's second theorem:

Holben's Theorem 1

$$ccw(K_{m,n}) = \begin{cases} \frac{mn}{4}, & m,n \text{ even} \\ \frac{mn+j}{4}, & m \text{ odd, } j = \frac{n}{m}, j \text{ even} \\ \frac{mn+j+2}{4}, & m \text{ odd, } j = \frac{n}{m}, j \text{ odd} \\ \frac{mn+2}{4}, & m \equiv 2(mod 4), n \text{ odd, } 2n \geq m \\ \frac{mn+4}{4}, & m \equiv 0(mod 4), n \text{ odd, } 2n \geq m \\ \frac{mn+l+2}{4}, & m \text{ even, } n \text{ odd, } 2n < m, l \text{ even,} \\ & m - ln < 2n, m - ln \equiv 2(mod 4) \\ \frac{mn+l+4}{4}, & m \text{ even, } n \text{ odd, } 2n < m, l \text{ even,} \\ & m - ln < 2n, m - ln \equiv 0(mod 4) \\ \frac{mn+l+4}{4}, & m,n \text{ odd, } m < n, l \text{ odd,} \\ & n - lm < 2m, n - lm \equiv 2(mod 4). \end{cases}$$

3 The Upper and Lower Bounds for Complete Cyclic Tripartite Graphs

The following lemmas give us and upper and lower bound for the cyclic cutwidth of any complete tripartite graph.

3.1 The lower bound for the Cyclic Cutwidth of a Complete Tripartite Graph

Lemma 1.
$$ccw(K_{r,s,t}) \geq \frac{rs + rt + st}{4}.$$

Proof The lower bound of the cyclic cutwidth of a complete tripartite graph can be found by combining Bowles's and Johnson's already proven theorems.

By Bowle's Theorem 4 we know that

$$lcw(K_{r,s,t}) \geq \frac{rs + rt + st}{2}$$

By Johnson's second theorem it has been established that whenever we have a graph, G, the ccw(G) $\geq \frac{lcw(g)}{2}$.

So by combining these two theorems, we find that:

$$ccw(K_{r,s,t}) \geq \frac{rs + rt + st}{4}.$$

3.2 The upper bound of the Cyclic Cutwidth of a Complete Tripartite Graph

Lemma 2.
$$ccw(K_{r,s,t}) \leq ccw(K_{r,s}) + ccw(K_{r+s,t})$$

In order to explore the upper bounds for a cyclic embedding of a complete tripartite graph we can build on the ideas already proven about the cyclic embedding of complete bipartite graphs. We will use Holben's first and second lemmas.

Holben Lemma 1. *Let A and B be two sets of vertices, such that $|A| = m$ and $|B| = n$.*

Given any cyclic embedding of $K_{m,n}$ where m is even, n is odd, and $m < n$, a line can always be drawn from between a pair of vertices to between a different pair of vertices such that there are exactly $\frac{m-2}{2}$ vertices from set A and exactly $\frac{n+1}{2}$ vertices from set B on one side of the line.

Holben Lemma 2. *Given any cyclic embedding of $K_{m,n}$, where m is odd, a line can always be drawn from between a pair of vertices to between a different pair of vertices such that there are exactly $\frac{m-1}{2}$ vertices from set A and $\lceil \frac{n+\frac{n}{m}}{2} \rceil$ vertices from set B on one side of the line.*

In order to arrange the vertices in such a way that this is possible we can use Holben's first and second lemmas to arrange the two seperate complete bipartite graphs. We will look at $K_{3,6,15}$ as an example of how to arrange the vertices before we investigate any upper and lower bounds.

When investigating $K_{3,6,15}$ we first look at $K_{3,6}$. We need to let set $|A| = 3$, $|B| = 6$, and $|C| = 15$. According to Holben's Lemma 2, since 3 is odd then the vertices can be split up so that there are $\frac{r-1}{2}$ vertices from set A and $\lceil \frac{s+\frac{s}{r}}{2} \rceil$ vertices from set B on one side of a line. So this means a line can be drawn through the graph so that there is 1 black vertex and 4 white vertices on one side of the line. This is illustrated in the first graph of Figure 5.

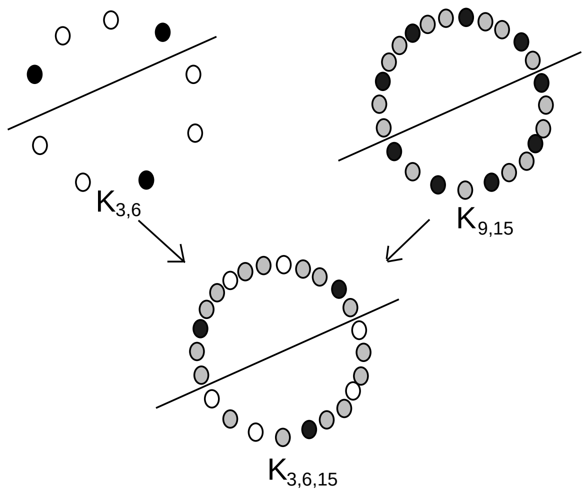

Figure 5

Now we need to look at $K_{3+6,15}$, which equals $K_{9,15}$. Since 9 is odd we will use Holben's second lemma again. The vertices can be split up so that there are 4 black vertices on one side of the line (in this case the black vertices represent the white and black vertices from the first graph) and 9 gray vertices on the same side of the line.

Now we can tell from the number of black vertices from the second graph how we should arrange the black and white vertices around the gray vertices. Since in the first diagram in Figure 5 there are 5 vertices total on one side of the line, we can line this up to the side of the line that has a total of 5 black vertices. So we need to change the black vertices to the corresponding white vertices. On the other side of the line we can also change the white vertices to the corresponding ones in the first picture. When we do this combination we should get the final picture which is below the first two graphs in Figure 5.

This illustration has shown how two complete bipartite graphs can be embedded to yield a complete tripartite graph. The cyclic cutwidths of these two graphs added together will indeed give us the upper bound for the tripartite graph it creates such that

$$ccw(K_{r,s,t}) \leq ccw(K_{r,s}) + ccw(K_{r+s,t})$$

This is definately an upper bound. For most cases it is probably not the best upper bound but it does give us a place to start to constrict the upper bound of the graphs we explore in this paper.

4 The cyclic cutwidth for $K_{r,s,t}$ for r, s, and t all even

Theorem 1. *$ccw(K_{r,s,t}) = \frac{rs+rt+st}{4}$, for r, s, and t all even.*

Proof

Let A, B, and C be sets of disjoint vertices such that $|A| = r$, $|B| = s$, and $|C| = t$. Assume $r \leq s \leq t$, with r, s, and t even.

To prove that the $ccw(K_{r,s,t}) = \frac{rs+rt+st}{4}$, it is sufficient to show that it is always possible to cyclically embed $K_{r,s,t}$ with a maximum cut of $\frac{rs+rt+st}{4}$ when r, s, and t are even because of Lemma 1.

We now need to arrange the vertices of $K_{r,s,t}$ in such a way that each cut will be no more than $\frac{rs+rt+st}{4}$. In order to do this we can split the vertices up evenly into six hextants labeled I, II, III, IV, V, and VI so that there are $\frac{r}{2}$ vertices in each of the hextants I and IV, $\frac{s}{2}$ vertices in each of the hextants II and V and $\frac{t}{2}$ vertices in each of the hextants III and VI, as in Figure 6.

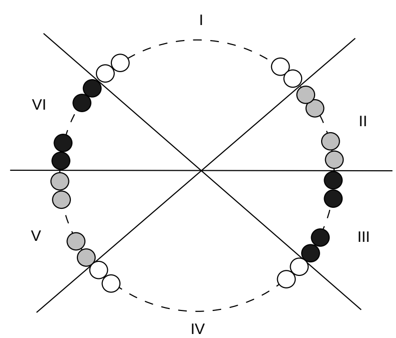

Figure 6

Then we can first look at the cuts between any two successive hextants. Since $\frac{r}{2}$, $\frac{s}{2}$, and $\frac{t}{2}$ vertices contribute to the cut from each side of the cut then the contributions from each side will look like Figure 7.

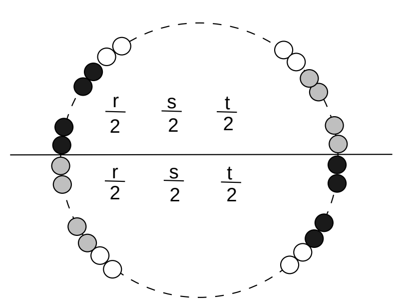

Figure 7

So the cut will be equivalent to:
$(\frac{r}{2})(\frac{s}{2} + \frac{t}{2}) + (\frac{s}{2})(\frac{r}{2} + \frac{t}{2}) + (\frac{t}{2})(\frac{r}{2} + \frac{s}{2})$

$$= \frac{rs + rt + st}{2}$$

But only half of these contribute since the vertices have to alternate which way they go around the middle so when then half to divide this in half again to get:

$$= \frac{rs + rt + st}{4}$$

Since this graph is symmetric the cutwidth will be the same between any two hextants and also the same between any other two vertices that divides the total vertices into two equal sets. In Figure 8 we can see that the if we move the cut from between any two hextants to any two vertices that the same number of vertices remain on each side of the diagonal. So the same equation applies no matter where the cut is made.

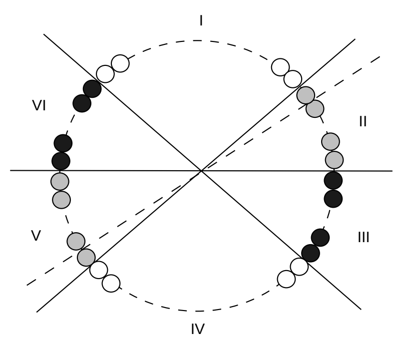

Figure 8

Therefore for a complete tripartite graph, with r, s, t even:

$$ccw(K_{r,s,t}) = \frac{rs + rt + st}{4}.$$

5 The cyclic cutwidth for an n-partite graph $K_{m_1,m_2,...,m_n}$ where $m_1, m_2, ...m_n$ are all even

The following theorem is an extension from Theorem 1.

Theorem 2. *Let an n-partite graph be cyclically embedded such that we have n sets, called A, B, C, N such that $|A| = m_1, |B| = m_2, ..., |N| = m_n$.*

Assume $m_1 \leq m_2 \leq ... \leq m_n$ and $m_1, m_2, ..., m_n$ are even. If this is the case then the

$$ccw(K_{m_1,m_2,...,m_n}) = \frac{m_1m2+m_1m3+...+m_1m_n+m_2m_3+...+m_2m_n+...+m_{n-1}m_n}{4}$$

Proof

Assume we have N sets labeled $A, B, C, ..., N$ such that $|A| = m_1, |B| = m_2, ..., |N| = m_n$.

Lower Bound

We know from Johnson's Theorem 2 that for any graph G

$$ccwG \geq \frac{lcw(G)}{2}$$

We also know from the Weitzel Chavez Theorem that

$$lcw(K_{m_1,m_2,...,m_n}) = \sum_{i=0}^{n-2} \left\lceil \frac{(m_n + ... + m_{n-i})m_{n-i-1}}{2} \right\rceil$$

.

So if we have vertices $m_1, m_2, ..., m_n$ such that each of these sets has an even number of vertices then

$$ccw(K_{m_1,m_2,...,m_n}) \geq \sum_{i=0}^{n-2} \left\lceil \frac{(m_n + ... + m_{n-i})m_{n-i-1}}{4} \right\rceil$$

.

Since each of the terms in the summation is even, we don't need the ceiling function, so we get

$$ccw(K_{m_1,m_2,...,m_n}) \geq \frac{m_1(m_2 + m_3 + ... + m_n) + m_2(m_3 + m_4 + ... + m_n) + ... + m_{n-1}m_n}{4}$$

$$= \frac{m_1m_2 + m_1m_3 + ... + m_1m_n + m_2m_3 + ... + m_2m_n + ... + m_{n-1}m_n}{4}.$$

Thus,

$$ccw(K_{m_1,m_2,...,m_n}) \geq \frac{m_1m_2+m_1m_3+...+m_1m_n+m_2m_3+...+m_2m_n+...+m_{n-1}m_n}{4}.$$

Upper bound

To find the upper bound we need to find the maximum cut at any point of the diagram. We can arrange the vertices so that we have $2n$-tants. It will look like Figure 9.

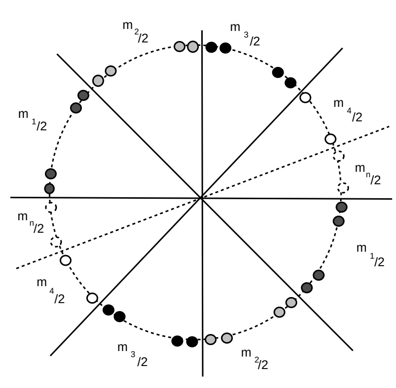

Figure 9

If we do this then at any point we have the same number of vertices from each group on either side of the cut (similar to the explanation for Figure 8). There will always be half of the vertices on one side and half on the other.

When we do this then the cut will be equivalent to

$$(\frac{m_1}{2})(\frac{m_2+m_3+...+m_n}{2})+(\frac{m_2}{2})(\frac{m_3+m_4+...+m_n}{4})+....+(\frac{m_{n-1}}{2})(\frac{m_n}{2})$$

$$=\frac{m_1m_2+m_1m_3+...+m_1m_n+m_2m_3+...+m_2m_n+...+m_{n-1}m_n}{4}.$$

Thus,

$$ccw(K_{m_1,m_2,...,m_n})\leq\frac{m_1m_2+m_1m_3+...+m_1m_n+m_2m_3+...+m_2m_n+...+m_{n-1}m_n}{4}.$$

Since this is the same as our lower bound we can conclude that for an n-partite graph where $m_1, m_2, ..., m_n$ even

$$ccw(K_{m_1,m_2,...,m_n})=\frac{m_1m_2+m_1m_3+...+m_1m_n+m_2m_3+...+m_2m_n+...+m_{n-1}m_n}{4}.$$

6 The cyclic cutwidth for $K_{r,r,r}$ for r odd

Theorem 3. *For r odd*

$$ccw(K_{r,r,r})=\frac{3r^2+1}{4}.$$

Proof

In order to prove that this is the cyclic cutwidth for this case we need to find the upper and lower bounds for the cyclic cutwidth of the graph and find that they are equal.

Lower bound

Let A, B, and C be sets of disjoint vertices such that $|A|=r$, $|B|=r$, and $|C|=r$. Let r be an odd integer.

We know by Lemma 1 that

$$ccw(K_{r,r,r})\geq\frac{3r^2}{4}.$$

$$=\frac{3r^2}{4}$$

which is not an integer so in order to round up to the next integer we have

$$=\frac{3r^2+1}{4}.$$

Therefore,

$$ccw(K_{r,r,r}) \geq \frac{3r^2+1}{4} \text{ when } r \text{ is odd.}$$

In order to arrange the vertices so that this holds true we can split the vertices up evenly so that they alternate every third vertex as in Figure 10. This figure will also be used to explore the upper bound of this case.

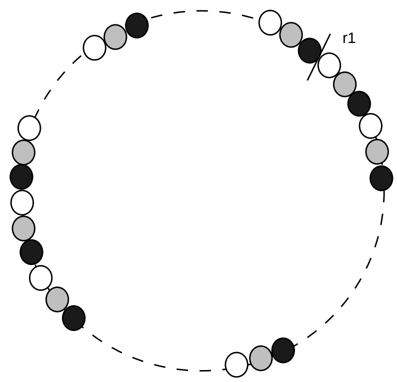

Figure 10

Upper bound

In order to find the upper bound of the cyclic cutwidth of this tripartite graph we need to find a maximum cut. In order to do this we can single out one cut that is the greatest. In this case the cuts are all going to be equal at any point since the layout of the graph is the same all around the circle. The vertices are spaced evenly throughout and since there are an equal number of each, an equal number of vertices will contribute to each cut.

So in order to look at one cut we will position a graph so that a black vertex is on top. As an example we will look at $K_{7,7,7}$ (Figure 11).

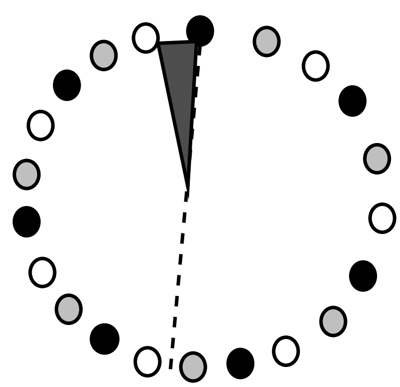

Figure 11

Since the number of vertices in all will be odd in any case, we do not need to worry about edges being drawn on either side of the middle since there will be no diameters. So there is no confusion about which way an edge contributes to a cut.

So if we look at our first cut as being to the left of the top black vertex we need to see what edges contribute to the cut. First we add the number of edges contributed by each black vertex to the white and gray vertices, and then from each white vertex to each gray vertex.

For any odd number r that summation will be as follows:

$$\left.\begin{array}{l} r \end{array}\right\} \text{edges contributed by top black vertex}$$

$$\left.\begin{array}{l} +r-2 \\ +r-4 \\ +r-6 \\ + \\ . \\ . \\ . \\ +1 \end{array}\right\} \text{edges contributed by the left black vertices from top to bottom}$$

$$\left.\begin{array}{l} +r-2 \\ +r-4 \\ +r-6 \\ + \\ . \\ . \\ . \\ +1 \end{array}\right\} \text{edges contributed by the right black vertices from top to bottom}$$

$$= (2\sum_{n=1}^{\frac{r+1}{2}} 2n-1) - r$$

Now we need to look at the edges contributed by each white vertex to each gray vertex:

$$\left.\begin{array}{l} \frac{r-1}{2} \\ +\frac{r-3}{2} \\ +\frac{r-5}{2} \\ + \\ . \\ . \\ +1 \end{array}\right\} \text{edges contributed by left white vertices from top to bottom}$$

$$\left.\begin{array}{l} +\frac{r-1}{2} \\ +\frac{r-3}{2} \\ +\frac{r-5}{2}+ \\ . \\ . \\ . \\ +1 \end{array}\right\} \text{edges contributed by right white vertices from top to bottom}$$

$= (2\sum_{n=1}^{\frac{r-1}{2}} n)$

So finally we need to add these edges together to get our final upper bound:

$\leq (2\sum_{n=1}^{\frac{r+1}{2}} 2n-1) - r + (2\sum_{n=1}^{\frac{r-1}{2}} n)$

$\leq 2(\frac{r+1}{2})^2 - r + 2(\frac{(\frac{r-1}{2})(\frac{r-1}{2}+1)}{2}))$

$\leq \frac{3r^2+1}{4}$

Thus,

$$ccw(K_{r,r,r}) \leq \frac{3r^2+1}{4} \text{ when } r \text{ is odd.}$$

Since the upper bound is the same as the lower bound we have found the cyclic cutwidth for this case.

Therefore,

$$ccw(K_{r,r,r}) = \frac{3r^2+1}{4} \text{ when } r \text{ is odd.}$$

It is also useful to note that the

$$ccw(K_{r,r,r}) = \frac{3r^2}{4} \text{ when } r \text{ is even.}$$

but this case is simply implied by Theorem 1 which proves the cyclic cutwidth for when r, s, and t are all even.

7 The Cyclic Cutwidth for n-Partite Graphs $K_{r,r,r,\ldots,r}$, when r is odd

Theorem 4. *For n odd,*

$$ccw(K_{r,r,r,\ldots,r}) = \left\lceil \frac{\binom{n}{2}r^2+1}{4} \right\rceil$$

For n even

$$\frac{\binom{n}{2}r^2}{4} \leq ccw(K_{r,r,r,\ldots,r}) \leq \frac{\binom{n}{2}r^2+\frac{n}{2}}{4}$$

Proof

Lower bound

Recall from the Weitzel-Chavez Theorem that

$$lcw(K_{m_1,m_2,\ldots,m_n}) = \sum_{i=0}^{n-2} \left\lceil \frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{2} \right\rceil$$

if $m_1 = m_2 = ... = m_n$ and $m_1 = r$ when r is odd then,

$lcw(K_{r,r,...,r}) = \sum_{i=0}^{n-2} \left\lceil \frac{(r + ... + r)r}{2} \right\rceil$

$= \frac{r(r + r + ... + r) + r(r + r + ... + r) + ... + r(r)}{2}$

$= \frac{(n-1)r^2 + (n-2)r^2 + ... + (n-(n-1))r^2}{2}$

$= \frac{\binom{n}{2} r^2}{2}$

This holds true if n is even. If n is odd then this number isn't an integer. So we can define the linear cutwidth to be

$$= \left\lceil \frac{\binom{n}{2} r^2 + 1}{2} \right\rceil$$

Now recall from Johnson's Theorem 2 that for any graph G

$$ccw(G) \geq \tfrac{lcw(G)}{2}.$$

Now by combining these two theorems we get that the:

$$ccw(K_{r,r,r,...,r}) \geq \left\lceil \frac{\binom{n}{2} r^2 + 1}{2} \right\rceil$$

where n is the number of sets of vertices we have.

Upper bound

When we look at the cut of the graph $K_{r,r,r,...,r}$ we will consider a diagram similar to the one in Figure 12. For the maximum cut and optimal arrangement we can arrange the vertices so that the vertices alternate from set 1 through set n until all the vertices are used as in Figure 12. To find the upper bound of this graph we must find a line that gives the maximum cut. Since at every cut the same number of vertices contribute to the cut (similar to the graph in section 5) the maximum cut will occur between any two consecutive vertices.

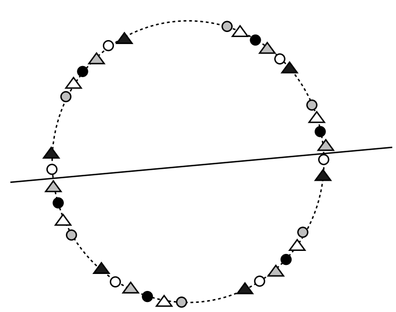

Figure 12

For the upper bound we have to look at two different cases, when n is even and when n is odd.

Case 1 n even where n is the number of sets of vertices.

In this case we have a diagram like Figure 9 where there are an even number of vertices, since r is odd and n is even. If that is the case then we have $\frac{nr}{2}$

vertices on both sides of the cut. But since r is odd the number of vertices from each set on each side differ by 1. So we have a cut like Figure 13.

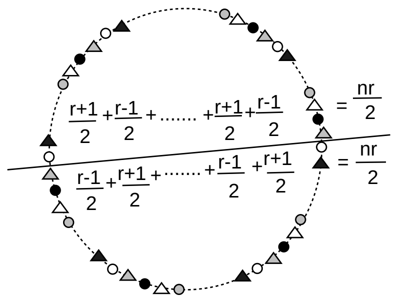

Figure 13

So to figure the cutwidth from this diagram we have the following equation:

$$(\frac{r+1}{2})(\frac{nr}{2}-\frac{r-1}{2})+(\frac{r-1}{2})(\frac{nr}{2}-\frac{r+1}{2})+....+(\frac{r+1}{2})(\frac{nr}{2}-\frac{r-1}{2})+(\frac{r-1}{2})(\frac{nr}{2}-\frac{r+1}{2})$$

$= \frac{n(n-1)r^2+n}{4}$
$= \frac{\frac{n(n-1)}{2}r^2+\frac{n}{2}}{2}$
$= \frac{\binom{n}{2}r^2+\frac{n}{2}}{2}.$

We must then divide this in half again since only half of these vertices actually contribute to the cut since the edges go different ways around the middle vertex. So we get

$$ccw(K_{r,r,r,\ldots,r}) \leq \frac{\binom{n}{2}r^2+\frac{n}{2}}{4}$$

Case 2 n is odd, where n is the number of sets of vertices.

To find the upper bound where n is odd is similar to the previous case, where n is even. But now our diagram will look like Figure 14.

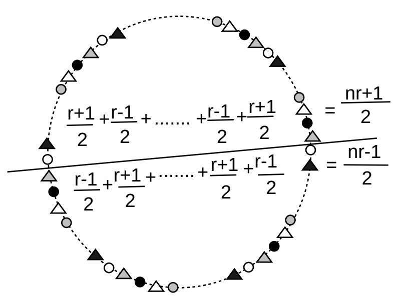

Figure 14

So now the equation we look at to find the upper bound for this case is as follows:

$$(\frac{r+1}{2})(\frac{nr-1}{2}-\frac{r-1}{2})+(\frac{r-1}{2})(\frac{nr-1}{2}-\frac{r+1}{2})+....+(\frac{r+1}{2})(\frac{nr-1}{2}-\frac{r-1}{2})$$

$= \frac{n(n-1)r^2+2}{4}$

$$= \frac{\frac{n(n-1)}{2}r^2+1}{2}$$
$$= \frac{\binom{n}{2}r^2+1}{2}$$
we must then divide this in half again since only half of these vertices actually contribute to the cut. So we get

$$ccw(K_{r,r,r,\ldots,r}) \leq \frac{\binom{n}{2}r^2+1}{4}$$

Now since this matches the lower bound we can conclude that for n being odd,

$$ccw(K_{r,r,r,\ldots,r}) = \left\lceil \frac{\binom{n}{2}r^2+1}{4} \right\rceil$$

We have to include the ceiling function here since for different values $\binom{n}{2}$ and r we will get numbers that are not integers, so the ceiling function allows us to round up to an integer value.

We cannot conclude the cyclic cutwidth for n being even, but we do have the bounds,

$$\frac{\binom{n}{2}r^2}{4} \leq ccw(K_{r,r,r,\ldots,r}) \leq \frac{\binom{n}{2}r^2+\frac{n}{2}}{4}.$$

8 The Lower Bound of the Cyclic Cutwidth of the Complete n-Partite Graph

Theorem 5.

$$ccw(K_{m_1,m_2,\ldots,m_n}) \geq \sum_{i=0}^{n-2} \left\lceil \frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{4} \right\rceil$$

Proof
By Johnson's second theorem we know,
For any graph G,

$$ccw(G) \geq \frac{lcw(G)}{2}.$$

By the Weitzel-Chavez theorem we know,

$$lcw(K_{m_1,m_2,\ldots,m_n}) = \sum_{i=0}^{n-2} \left\lceil \frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{2} \right\rceil$$

Thus we can conclude,

$$ccw(K_{m_1,m_2,\ldots,m_n}) \geq \sum_{i=0}^{n-2} \left\lceil \frac{(m_n+\ldots+m_{n-i})m_{n-i-1}}{4}. \right\rceil$$

9 Conjectures

This paper does not cover every case for the cyclic cutwidth of tripartite and n-partite graphs. But we do have some conjectures on other cases.

9.1 An Upper bound for Tripartite Graphs

An upper bound can be found for any tripartite graph by splitting a graph $K_{r,s,t}$ into two complete bipartite graphs, $K_{r,s}$ and $K_{(r+s),t}$, and adding their cyclic cutwidths together. We can find an upper bound for any graph by using Holben's results to find the cyclic cutwidth of these bipartite graphs since she found the results for all bipartite graphs. The sum of these graphs does give an upper bound but we believe it is not the best upper bound. It should be able to be improved for each case.

9.2 The Cyclic Cutwidth of the Complete Tripartite Graph $K_{r,jr,kr}$, where r is odd

We found the cutwidth of many examples of graphs when the numbers are all multiples of the first number. A definite upperbound seems to be:

$$ccw(K_{r,jr,kr}) = \frac{jr^2 + kr^2 + jkr^2 + j + k}{4}$$

This matches the lower bound for some cases, which leads us to believe it may be the cyclic cutwidth for the multiple case. The cases where the lower bound does not match this upper bound are $K_{1,3,4}$, $K_{1,3,5}$, $K_{1,3,6}$, $K_{1,5,6}$, $K_{1,5,7}$, $K_{3,9,12}$, $K_{3,9,15}$, and $K_{3,9,18}$. For cases higher than $K_{3,9,18}$ the upper bound and lower bound were equivalent to the conjectured cyclic cutwidth. So maybe these cases are special or we need to find a different lower bound.

10 Conclusion

We found the upper and lower bounds for any complete tripartite graph. We also found the lower bound of an n-partite graph. We were also able to find the cyclic cutwidth of $K_{r,s,t}$ when r, s, t are all even and when they are all equal and odd. We then extended these cases to the n-partite graph, but we were only able to find an upper and lower bound for the cyclic cutwidth when n is even for the equal and odd case.

There are still many unknown cases, especially since there are many more cases for a complete tripartite graph compared to the cases of a complete bipartite graph. The next step would seem to be to try and find the cyclic cutwidth of the complete tripartite graph $K_{r,s,t}$ where s and t are multiples of r. Does this case have to be further split into different cases?

We also should try to find an upper bound for the n-partite case in general. Maybe the upper bound can be found by combining graphs similar to the method we used for finding the upper bound of complete tripartite graphs. Maybe an induction proof would be useful to find this.

We also need to figure out exactly what all of the different cases are for the tripartite graph and try to extend those cases to the n-partite graph.

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