The Twist Numbers of Graphs and the Tutte Polynomial

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Abstract

The twist number of a plane graph associated with a link's checkerboard coloring is defined. The twist number of any planar pseudograph is shown to be determined by the Tutte polynomial. This allows the definition of twist number to be extended to non-planar graphs, for which the twist number is also determined by the Tutte polynomial. As a corollary, the twist number of an alternating link is shown to be determined by the Jones polynomial.

1 Introduction

1.1 Checkerboard Graphs

It is well-known that every alternating link can be assigned a plane **checkerboard graph**. This pseudograph is constructed by first shading all regions in the link diagram that lie to the right of an overcrossing strand (as the strand approaches the crossing). Then a vertex is placed in every shaded region. Edges are placed between vertices where the regions touch at a crossing in the link diagram. The resulting diagram is called the **checkerboard graph** G^+. Another checkerboard graph G^- can be obtained by taking the dual of G^+, whose vertices lie in the regions in the link diagram which are *not* shaded. For sake of simplicity, however, only G^+ will be considered. Figure 1 shows the shaded regions in a trefoil. Figure 2 shows the resultant checkerboard graph:

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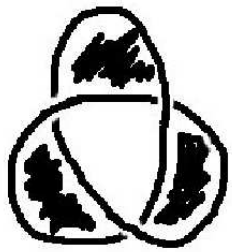

FIGURE 1

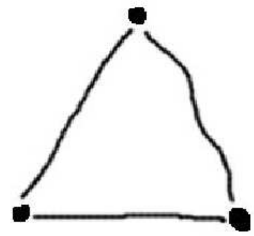

FIGURE 2

By reversing this process, that is, by replacing edges with crossings and vertices with regions, every plane connected pseudograph can be assigned to exactly one alternating link. This correspondence is useful because the Tutte polynomial of a checkerboard graph is related to the Jones polynomial of the associated link.

1.2 Equivalent Definitions of the Tutte Polynomial

Before we define the Tutte polynomial, we need to define several characteristics of graphs. An edge e in a pseudograph G is called a **loop** if it connects a vertex to itself. An edge f in a graph G is called an **isthmus** or **bridge** if its removal disconnects the graph.

The graph $G - e$ is obtained from G by deleting the edge e, and the graph G/e is obtained from G by contracting e, that is, by deleting e and identifying its two adjacent vertices.

The Tutte Polynomial $T(G)$ of a graph is a two-variable graph invariant. There are several equivalent definitions of the Tutte Polynomial which will be useful in later calculations. The Tutte Polynomial is defined by Tutte himself by the following relations [9]:
1. If G is edgeless, $T(G) = 1$.
2. If the edge set $E(G)$ consists of 1 edge that is an isthmus, then $T(G) = x$.
3. If $E(G)$ consists of 1 edge that is a loop, then $T(G) = y$.
4. If $G_1, G_2, \cdots, G_n$ are disconnected components of G, then $T(G) = T(G_1)T(G_2) \cdots T(G_n)$.
5. If $G_1, G_2, \cdots, G_n$ are the blocks of a connected but separable graph G, then $T(G) = T(G_1)T(G_2) \cdots T(G_n)$.
6. If e is an edge in G that is neither an isthmus nor a loop, then $T(G) = T(G - e) + T(G/e)$.

The Tutte polynomial can also be defined by a spanning tree expansion, which will be more convenient for most of our later results. [1, p. 351] Given a graph

G and an *arbitrary* ordering of its edges, we consider each spanning tree (or forest) of the graph, i.e., every acyclic subgraph connecting all the vertices of G that are connected in G. An edge in one of these trees is considered **internally active** if it precedes all edges outside of the tree with which it forms a cycle in G. Likewise, an edge not contained in a tree is considered **externally active** if it precedes all edges in the tree with which it forms a cycle in G. A tree has internal activity i and external activity j if it has i internally active edges and j externally active edges. Let t_{ij} be the number of spanning trees with internal activity i and external activity j. Then the Tutte polynomial of G is given by the following formula:

$$T_G(x,y) = \sum_{i,j} t_{ij} x^i y^j. \tag{1}$$

Thirdly, the Tutte polynomial can be defined as a function of the rank-generating polynomial of a graph [1, p. 339], where a graph and its subgraphs are defined by their edge sets. For a graph $< E >= G(V,E)$ and an arbitrary spanning subgraph $< F >= H(V,F)$, where $F \subseteq E$, let $k < F >$ be the number of components of F. Let the rank of F be $r < F >= |V| - k < F >$, and let the nullity of F be $n < F >= |F| - r < F >$. Then:

$$T_G(x,y) = \sum_{F \subseteq E} (x-1)^{r<E>-r<F>} (y-1)^{n<F>}. \tag{2}$$

The Jones Polynomial $V_L(t)$ of a link L can be obtained from the Tutte Polynomial $T_G(x,y)$ of the link's checkerboard graph G^+ by the following equation, where a is the number of shaded regions and b is the number of unshaded regions in the checkerboard diagram, and w is the writhe of the link [1, p. 370]:

$$V_L(t) = (-1)^w t^{\frac{b-a+3w}{4}} T_{G^+}(-t, -t^{-1}) \tag{3}$$

This equation is used to relate properties of links (specifically twist numbers) to graphs.

1.3 Twist Number

A **twist** in a link is an **integral tangle**, a section of its diagram that is a sequence of simple crossings between only two strands.
(Note that there exist twists consisting of one crossing.) The **twist number** of a link is the minimum number of twists taken over all minimum-crossing diagrams of the link. The twist number of Figure 3 is 3.

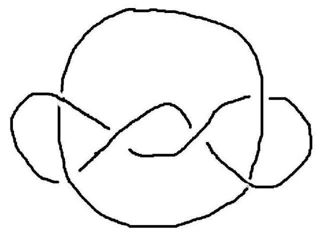

FIGURE 3

Let $a_1,\ a_{p-1}$ represent the coefficients of the second-lowest power of x and the second-highest power of x, respectively, of the Jones Polynomial of an alternating knot K. Dasbach and Lin show that the twist number $\tau(K)$ is given by the following formula: [2]

$$\tau(K) = \mid a_1 \mid + \mid a_{p-1} \mid . \tag{4}$$

Dasbach and Lin start with the following definition for the twist number, where $E(G)$ is the edge set of a checkerboard graph of an alternating knot K, $\tilde{E}$ is the edge set of the *reduced graph* of G, where multi-edges have been replaced by single edges, and $\tilde{E}^*$ is the edge set of the *reduced graph* of the dual of G [2]:

$$\tau(K) = \mid \tilde{E} \mid + \mid \tilde{E}^* \mid - \mid E \mid . \tag{5}$$

This would be a convenient formula for exploring the relationship between twist numbers and graphs, but because the formula requires the dual of a graph, it is restricted to planar graphs.

1.4 Flyping Knot Diagrams

A **flype** is a particular homeomorphism on a knot that may reduce the twist number. If R and S are sections of the following knot diagram that encompass all crossings except for one, a flype is represented by the rotation of R in space that moves the remaining crossing from one side of R to the other.

To find the minimum twist number of an alternating knot, then, one must look at all possible projections of a knot that differ by a sequence of flypes [4]. A **twist-reduced** knot diagram is a diagram which contains the fewest number of twists.

To obtain the minimal twist number, one must also undo nugatory crossings, which correspond to isthmi or loops in the checkerboard graph.

2 The Twist Number of Links and Graphs

2.1 Definitions of the Twist Number

Let a **multi-edge** be a set of edges between two vertices, and the **weight** of the multi-edge be the number of edges in that set. The **reduced graph** of a graph G is identical to G, except multiple edges are replaced by single edges. Let $\tilde{E}$ denote the reduced graph throughout this paper.

Let a **pairwise-disconnecting set** be a set of edges, any two of which disconnect the graph when deleted. Often a pairwise-disconnecting set forms what Read and Whitehead [6] call a **chain**, that is, the edges connecting a sequence of two-degree vertices, for example, edges a, b, and c in Figure 4. The edges in a pairwise-disconnecting set do not need to be adjacent to one another (such as edges a', b', and c' in Figure 5), but they always belong to the same cycle. Note that a sequence of flypes in the associated link can juxtapose two non-adjacent edges in a pairwise-disconnecting set, so this definition eliminates the need to look at graphs obtained from flype-reduced diagrams.

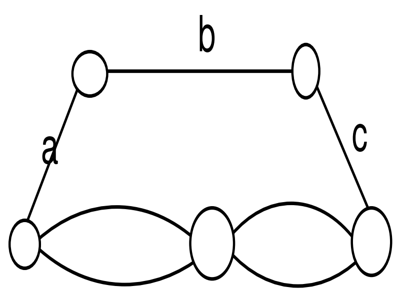

Figure 4

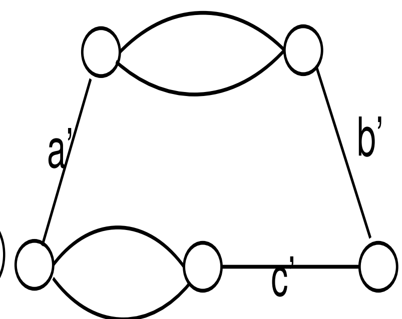

Figure 5

The only pairwise-disconnecting set that contains a multi-edge is itself a multi-edge of weight 2 that is a block of the graph (edges a and b in Figure 6). For convenience, this block will be referred to as a **Hopf block** because it corresponds to a connect sum factor of a Hopf link (Fig. 7). Let s denote the number of Hopf blocks in a graph. Also let E' denote the partition of the edge set of a graph such that each pairwise-disconnecting set is one set of the partition, and each remaining edge is one set. Then each pairwise-disconnecting set contributes 1 to the value of $|E'|$, while each remaining edge contributes 1.

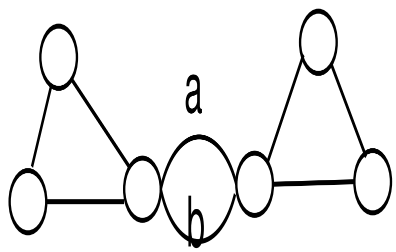

Figure 6

FIGURE 7

We are now ready to define the twist number for alternating links.

Definition Let p be the number of multi-edges in a checkerboard graph G of an alternating link L. Let q be the number of pairwise-disconnecting sets in G. Let r be the number of edges in G not belonging to a multi-edge or a pairwise-disconnecting set, and s the number of Hopf blocks in G. Further suppose that G contains no isthmi or loops. Then let the twist number of an alternating link L be given by the following formula:

$$\tau(L) = p + q + r - s. \tag{6}$$

Proposition 2.1 demonstrates that this definition includes the twist definition of Dasbach and Lin for aternating knots (5) as a special case.

This definition is motivated by the fact that crossings in a single twist become either multi-edges or edges in sequences of 2-degree vertices in the checkerboard graph.

Of course, if the link diagram is not twist-reduced, then the edges in the 2-degree "chains" may not be adjacent. In this case, the edges still belong to a pairwise-disconnecting set.

All single crossings become "single edges" that do not belong to a pairwise-disconnecting set or to a multi-edge. To add the twist number, then, one only need add the number of pairwise-disconnecting sets, the number of multi-edges, and the number of remaining single crossings. Since Hopf blocks are both multi-edges and pairwise-disconnecting sets, they were double counted. Subtracting the number of Hopf blocks from the sum yields the twist number of the link.

Since a plane graph corresponds to some alternating link diagram, we are now ready to define the twist number of a graph in terms of the twist number of its corresponding alternating link.

Definition Given an alternating link L and the checkerboard graph G assigned to it by the process mentioned in Section 1.1, let the **twist number** of G be the twist number of L.

2.2 Equivalent Definitions of the Twist Number

Proposition 2.1 *The twist number of a planar pseudograph G with no isthmi or loops is given by the following formula:*

$$\tau(G) = |\tilde{E}| + |E'| - |E| + s. \tag{7}$$

Proof Let p, q, r, and s be defined as in (6). Let p' be the number of edges in G that belong to some multi-edge. Let q' be the number of edges that belong to some pairwise-disconnecting set in G, and let s' be the number of edges that belong to some Hopf block in G. Then the edge set is merely the sum of these numbers of edges, after subtracting the double-counted Hopf block edges:

$$|E| = p' + q' + r - s'.$$

The reduced graph $\tilde{E}$ is the sum of the edges, counting 1 for each multi-edge:

$$|\tilde{E}| = p + q' + r - s'.$$

Recall that each edge contributes 1 to $|E'|$, except for edges in pairwise-disconnecting sets, which contribute 1 to $|E'|$:

$$|E'| = p' + q + r - s'.$$

Adding the equations yields:

$$|\tilde{E}| + |E'| - |E| + s = p + q + r - s' + s.$$

Note that $s' = 2s$, since there are 2 edges in every Hopf block:

$$|\tilde{E}| + |E'| - |E| + s = p + q + r - s.$$

Observe that for all knots, $s = 0$, since no knot has a Hopf link as a connect sum factor. Also, $|E'| = |\tilde{E}^*|$, because any pairwise-disconnecting set but the Hopf block contributes 1 to $|E'|$, and its dual, a multi-edge, contributes 1 to $|\tilde{E}^*|$. Thus equation (5), used by Dasbach and Lin for alternating knots, agrees with (6).

It will also be useful to recognize the following incarnation of the twist number. Recall from Section 1.2 that for a graph $G(V, E)$, $k < E >$ is the number of components of G, $r < E >= |V| - k < E >$, and $n < E >= |E| - r < E >$.

Proposition 2.2 *The twist number of any planar pseudograph G is given by the following formula:*

$$\tau(G) = -n < E > -r < E > +|\tilde{E}| + |E'| + S. \tag{8}$$

It can be quickly checked that $-n < E > -r < E > = -|E|$. Then use (7).

3 Determining Twist Numbers from Polynomials

3.1 Coefficients of the Tutte Polynomial

Lemma 3.1 *The Tutte Polynomial of any graph G can be written in the form:*

$$T_G(x,y) = x^c y^d (b_0 x^{n-1-c} + b_1 x^{n-2-c} + b_2 x^{n-2-c} y + \cdots$$

$$+ b_{p-2} x^2 y^{m-n-d} + b_{p-1} y^{m-n-d} + b_p y^{m-n+1-d}), \tag{9}$$

where m is the number of edges in G; n is the number of vertices in G; c is the number of isthmi in G; d is the number of loops in G; and $a_0, a_p = 1$ if G contains any edges that are not isthmi or loops.

Proof (a) By the spanning tree expansion of the Tutte polynomial, each coefficient counts the number of spanning trees of a given activity of the graph, which are maximal connected trees. If a graph G contains an isthmus e, then e belongs to each spanning tree of G. Therefore the contraction of e or the insertion of e does not change the number of spanning trees of G, so the coefficients of the Tutte polynomial do not change. Each isthmus *does* contribute a power of x to every term in the polynomial so these powers can be factored out.
(b) In the spanning tree expansion of the Tutte polynomial, the number of edges in any tree is $n-1$. The highest power of x is given by spanning trees with all edges *internally active*, i.e., $n-1$ internally active edges, where every edge in the tree precedes every edge in its cut. Choose an arbitrary numbering of $E(G)$. Then take the tree that contains all of the lowest-numbered edges in each cycle. This is the one tree in which every internal edge is active. This implies that $b_0 = 1$. Also, there are no externally active edges in this tree, unless they are loops, which were factored out initially. Therefore the power of y is d.
(c) For the second term, we choose to look at the term with one less power of x but the same power of y. Its relevancy will become apparent in Theorem 3.2. The coefficient may be zero.
(d) The number of edges external to any spanning tree is $m-(n-1) = m-n+1$. The highest power of y, therefore, is $m-n+1$, which occurs in the tree that contains all highest-numbered edges, though isthmi may have lower numbers. In this tree, every external edge is active. There is precisely one of these trees, so $b_p = 1$.
(e) For the second-to-last term, we choose to look at the term with one less power of y but the same power of x. Again, the reason will become apparent. The coefficient may be zero.
Note that if G is the trivial graph (a single vertex), then $c, d = 0$, $n = 1$, $b_0 x^{n-1-c} = b_p y^{m-n+1-d}$ and $T_G = 1$, as expected. ∎

3.2 Determing the Twist Number from the Tutte Polynomial

Theorem 3.2 *Suppose a connected, nonseparable, planar multigraph G has no isthmi or loops. If b_1, b_{p-1} are two coefficients of the Tutte polynomial T_G given by Lemma 3.1, and s is the number of Hopf blocks in G (see Section 2.1), then the twist number $\tau(G)$ is given by the following formula:*

$$\tau(G) = b_1 + b_{p-1} + s. \tag{10}$$

Proof The rank-generating form of the Tutte polynomial is used here. We prove that the sum of the coefficients $b_0 + b_1 = p + q + r$. The highest power of x occurs when $r < E > -r < F >= r < E >$. This is given by the term $(x-1)^{r<E>}$. (Note that when $r < F >= 0$, that is, when the subgraph given by F consists of only vertices and no edges, then $n < F >= 0$. So the term is $(x-1)^{r<E>}(y-1)^0$.) To find the coefficient of the second highest power of x, that is, of $x^{r<E>-1}$, we expand $(x-1)^{r<E>} = x^{r<E>} + (-1)r < E > x^{r<E>-1} + \cdots$ We must also look at the terms $(x-1)^{r<E>-1}(y-1)^{n<F>}$. These terms are given by graphs where $r < F >= 1$. This implies F contains one more vertex than its number of components. In other words, there exists one isthmus (or multi-edge) between one pair of vertices. There are r subgraphs that contain only one edge. Note that $n < F >= 0$. Then the coefficient of the sum of these terms is r.

Given that there are q multi-edges in the graph, suppose there are n edges in one of these multi-edges. Then there are $\binom{n}{1}$ subgraphs that contain one of these edges, $\binom{n}{2}$ that contain 2, $\cdots$, and $\binom{n}{n}$ that contain n. The term in $(x-1)^{r<E>}(y-1)^{n<F>}$ that we want is $(x-1)^{r<E>}(-1)^{n<F>}$, which is negative when F contains an even number of edges. So the sum of all the subgraphs whose edge set is a subset of a single multi-edge is:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} = 1. \tag{11}$$

Thus the first twist coefficient is given by the following formula, where $|\tilde{E}|$ is the number of edges in the reduced graph of E:

$$b_1 = -r < E > +|\tilde{E}|. \tag{12}$$

To find the coefficient b_{p-1}, we must look at all terms $(y-1)^{n<E>-1}$, which come from the terms $(y-1)^{n<E>}$ and $(y-1)^{n<E>-1}(x-1)^{r<E>-r<F>}$. The only subgraph that gives $n < F >= n < E >$ is that where $F = E$. The coefficient of $(y-1)^{n<E>-1}$ in $(y-1)^{n<E>}$ is $-n < E >$. Since no subgraphs have fewer vertices than E, the only graphs that give $n < F >= n < E > -1$ are graphs with one fewer edge or one fewer component than E. (Recall that

$n < E >= |E| - |V| + k < E >$, where $k < E >$ is the number of components of G.) We must also consider subgraphs where there are j fewer edges but $j+1$ more components than E. First of all, there is no way to decrease the number of components. Every subgraph with one edge missing (except for edges in pairwise-disconnecting sets, which will be dealt with momentarily) contributes 1 to the coefficient of $(y-1)^{n<E>-1}(x-1)^0$. Finally there is the case where two edges are removed. Since none of the edges of E are isthmi, the only way to increase the number of components by one while decreasing the number of edges by two is to remove two of the edges in a pairwise-disconnecting set. Then the remaining edges in the set become isthmi. Thus the only way to increase the number of components by two while decreasing the number of edges by three is to remove three of the edges in such a set, and similarly for decreasing more edges. The number of ways to remove n edges from a pairwise-disconnecting set is given by (11). This includes the cases where just one edge is removed. Thus the sum of the coefficients so far is:

$$b_{p-1} = -n < E > +|E'|, \tag{13}$$

where $|E'|$ is the cardinality of the special edge partition described in Section 2.1. Thus adding (12) and (13), and using (8) yields:

$$b_1 + b_{p-1} = -n < E > -r < E > +|E'| + |\tilde{E}| = \tau(G) - s. \quad \blacksquare$$

Corollary 3.3 *Let G be any planar pseudograph with any number of isthmi or loops. If b_1, b_{p-1} are two coefficients of the Tutte polynomial T_G given by Lemma 3.1, and s is the number of Hopf blocks (see (6)), then the twist number $\tau(G)$ is given by:*

$$\tau(G) = b_1 + b_{p-1} + s. \tag{14}$$

Proof First suppose G is disconnected or separable, consisting of blocks G_1, G_2, $\ldots G_k$, and does not contain any bridges or loops. Then the Tutte polynomial T_G is the product of the Tutte polynomials of its blocks (see Section 1.2). The term x^{n-2-c} in T_G is given by the products of all the first terms x^{n_i-1-c} in T_{G_i} and the first term x^{n_j-2-c} in *one* of T_{G_j}. By (9), the coefficient b_0 of each T_{G_i} is 1, so the coefficient of x^{n-2-c} is $b_{1_i} + b_{1_2}$.

$$T_G = T_{G_1}T_{G_2}\cdots T_{G_k} = (b_{0_1}x^{n-1}y^d + b_{1_1}x^{n-2}y^d + \cdots + b_{p_1-1}y^{m-n} + b_{p_1}y^{m-n+1})$$

$$\cdot(b_{0_2}x^{n-1} + b_{1_2}x^{n-2} + \cdots + b_{p_2-1}y^{m-n} + b_{p_2}y^{m-n+1})$$

$$\cdots(b_{0_k}x^{n-1} + b_{1_k}x^{n-2} + \cdots + b_{p_k-1}y^{m-n} + b_{p_k}y^{m-n+1}).$$

$$T_G = x^c y^d(b_0x^{n-1} + (b_{1_1} + b_{1_2} + \cdots + b_{1_k})x^{n-2} + \cdots$$

$$+(b_{p_1-1} + b_{p_2-1} + \cdots + b_{p_k-1})y^{m-n+1} + b_p y^{m-n}).$$

Since the twist number of a graph merely counts subsets of edges, then we can add the twist numbers of its blocks:

$$\tau(G) = \tau(G_1) + \tau(G_2) + \cdots + \tau(G_k)$$

$$= (b_{1_1} + b_{p_1-1} + s_1) + (b_{1_2} + b_{p_2-1} + s_2) + \cdots + (b_{1_k} + b_{p_k-1}) + s_k)$$

$$= (b_{1_1} + b_{1_2} + \cdots + b_{1_k}) + (b_{p_1-1} + b_{p_2-1} + \cdots + b_{p_k-1}) + (s_1 + s_2 + \cdots + s_k)$$

$$= b_1 + b_{p-1} + s.$$

Isthmi and loops simply multiply the Tutte polynomial by powers of x and y, respectively, and do not change the coefficients [1, p. 339]. If e is a bridge, then:

$$T_G = xT_{G-e}.$$

(Note that by relations 4 and 5 in Section 1.2, $T_{G-e} = T_{G/e}$).

Also, if e is a loop, then:

$$T_G = yT_{G-e}. \quad \blacksquare$$

Corollary 3.4 *If s is the number of Hopf links that are connected sum factors or split components of a link L, and if a_1 and a_{p-1} are the coefficients of the Jones polynomial of L used in (4), then the twist number of a link is:*

$$\tau(L) = |a_1| + |a_{p-1}| + s \tag{15}$$

Proof This follows from equation (3) and Corollary 3.3. Note that the coefficients of the Jones polynomial of an alternating link are multiples by ± 1 of the Tutte coefficients of the checkerboard graph G^+. Thus one only needs to take the absolute values of the Jones coefficients to get the Tutte coefficients. Specifically, note that the second highest power of t in the Jones polynomial depends on the second-highest power of x in the Tutte polynomial. Also, notice that the second-lowest power of the Jones polynomial depends on the second-highest power of y in the Tutte polynomial. We can therefore identify the coefficients in this manner:

$$|a_1| = b_1$$

$$|a_{p-1}| = b_{p-1} \quad \blacksquare$$

4 Extending the Twist Number to Non-Planar Graphs

Note that we now have two definitions of the twist number of a graph, both as a sum of Tutte coefficients and as a sum of subsets of edges. These two statements are equivalent for all non-planar pseudographs, since no conditions for planarity were required to prove their equivalence. (Furthermore, for all planar graphs, the statements give the twist numbers for the corresponding links, so the twist number of the graph is well-defined.) The definition of **twist number** therefore extends to *non-planar* pseudographs.

Definition Let p be the number of multi-edges in G. Let q be the number of pairwise-disconnecting sets in G. Let r be the number of edges in G not belonging to a multi-edge or a pairwise-disconnecting set, and s the number of Hopf blocks in G (see (6)). Then let the **twist number** of any pseudograph G be defined by the following formula:

$$\tau(G) = p + q + r - s. \tag{16}$$

Note that Corollary 3.3 now holds for non-planar pseudographs.

The twist number of a graph is, of course, an invariant by virtue of the Tutte polynomial's invariance. There are two noteworthy cases where the twist number is invariant. If we use Read and Whitehead's term **amallomorphism** to mean the adding of edges parallel to a given edge [6], then we can say that the twist number of a graph is invariant under homeomorphism on an edge that is amallomorphically minimal. In other words, subdividing an edge does not change the twist number if that edge is *not* part of a multi-edge. Figure 12 shows two homeomorphic graphs with the same twist number. On the other hand, the twist number of a graph is invariant under amallomorphism only on an edge that is homeomorphically minimal. Figure 13 shows two amallomorphic graphs with the same twist number.

5 Open Questions

The twist number of a graph has a knot-theoretic interpretation for planar graphs, but what about non-planar graphs? Any non-planar graph can be embedded on a surface of some genus. Then a link diagram can be obtained in a manner similar to the planar case. Checkerboard graphs of link diagrams appear to have the same *genus* as the link diagram itself. Does it make sense to speak of the twist number of the higher-genus link diagram, and is this twist number equal to the twist number of its checkerboard graph?

What do the third and third-to-last coefficients of the Tutte polynomial count? These are harder to see, and there have been partial results with the Jones polynomial (see [2] and [5]).

We have a formula for the twist numbers of alternating links, but is there a formula for non-alternating links that can be derived from the Jones or Tutte polynomials? This could involve assigning to each edge in the checkerboard graph a positive or negative sign, (see [1, p. 368] and [8]).

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