3 and 4-Bandwidth Critical Graphs

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Abstract

This paper investigates 3 and 4-bandwidth critical graphs. It concludes Holly Westerfield's proof that only six types of 3-bandwidth critical graphs exist, including one infinite family. It classifies families of 4-bandwidth critical cyclic graphs, and examines other kinds of 4-bandwidth critical graphs. Also, all 4-bandwidth critical trees of height two are found.

1 Introduction

1.1 Background

Definition A graph $G = (V, E, \delta)$ consists of a set of vertices V, a set of edges E that connect pairs of vertices, and a function δ that identifies the vertices incident to an edge.

Holly Westerfield has attempted to categorize all 3-bandwidth critical graphs in her work entitled "On 3-Bandwidth Critical Graphs" [4]. This paper finishes Westerfield's proof that only six types of 3-bandwidth critical graphs exist (including one infinite family), and examines 4-bandwidth critical graphs.

1.2 Terminology

A *simple graph* contains no loops or multiple edges.

A *directed graph* denotes a graph where the edges have a specific direction.

The *degree* of a vertex is the number of edges incident to that vertex.

A graph is *complete* if and only if there is an edge between every pair of vertices. This is denoted by K_n for a complete graph with n vertices.

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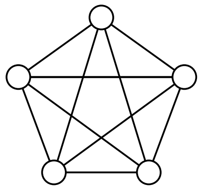
Example A K_5 graph:



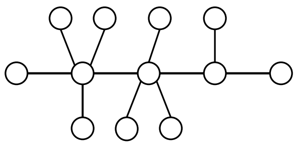
A *cyclic graph* C_n (for $n > 2$) has n vertices of degree two which form a single cycle.

A *tree* is an undirected, connected, simple graph containing no cycles.

A *leaf* or *pendant* is a vertex of degree one.

A *path* is a graph with two end vertices of degree one and all remaining vertices of degree two.

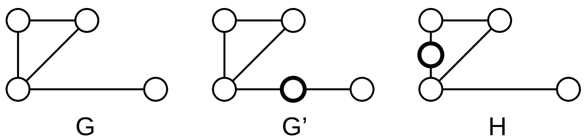
A *caterpillar* is a tree which becomes a path when all pendant vertices (leaves) are removed. That path is called the *spine* of the caterpillar.



A *linear embedding* of a graph G is a representation of G where all the vertices and edges of G are placed along a line.

The *bandwidth* of a linear embedding of graph G is the length of the longest edge in the embedding. Distance is measured by finding the number of vertices in between the edge's end vertices and adding 1. The bandwidth of G, denoted $bw(G)$, is the minimum bandwidth out of all possible linear embeddings.

A graph G' is a *subdivision* of G if new vertices of degree two can be inserted into G to obtain G'. Two graphs are *homeomorphic* if they are subdivisions of the same graph. G is *homeomorphcally minimal* if it is not a subdivision of any simple graph. For example, G' and H are homeomorphic because they are both subdivisions of G.



1.3 Critical bandwidth

One subcategory of the bandwidth problem investigates *critical bandwidth*.

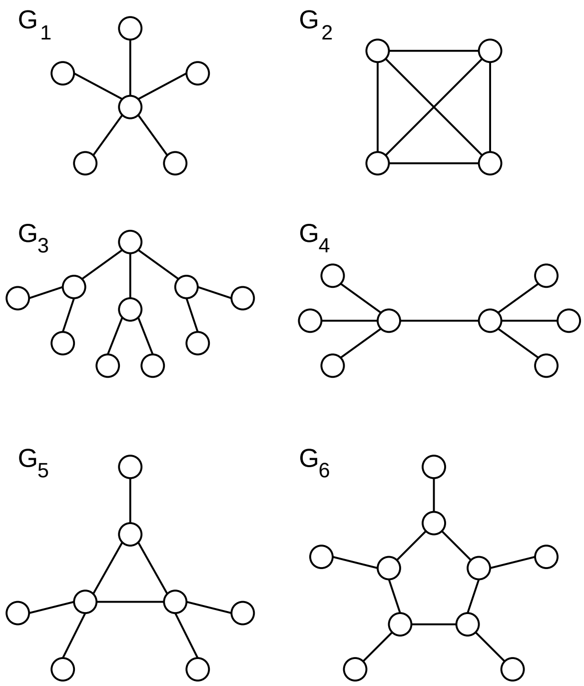
Definition A graph G is *n-bandwidth critical* if:

- G has bandwidth n
- G is homeomorphically minimal
- for every proper subgraph G' of G, $bw(G') < n$

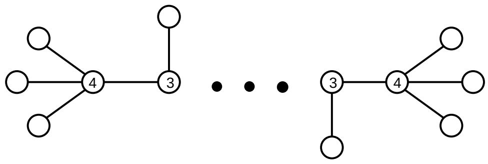
Proposition 1.1 *(Westerfield)*
If G' is a subgraph of a graph G, then $bw(G') \leq bw(G)$.

2 3-bandwidth critical graphs

Westerfield found six 3-bandwidth critical graphs, one of which contains an infinite family. The graphs are pictured below:



Additionally, the G_4 graph can be expanded into an infinite family by adding a caterpillar consisting of degree three vertices between the two degree four vertices.



Theorem 2.1 *(Westerfield and Kilzer) There are only six types of 3-bandwidth critical graphs, including one infinite family. They are G_1 through G_6 and the G_4 infinite family.*

Proof The following proof will break down all graphs into the three categories of trees, unicyclic graphs, and polycyclic graphs, examining all remaining cases of possible 3-bandwidth critical graphs.

2.1 Known Results: Trees and Unicyclic Graphs

Westerfield proved that the only 3-bandwidth critical trees are G_1, G_3, and G_4. She also showed that G_5 and G_6 are the only unicyclic graphs.

2.2 Polycyclic Graphs

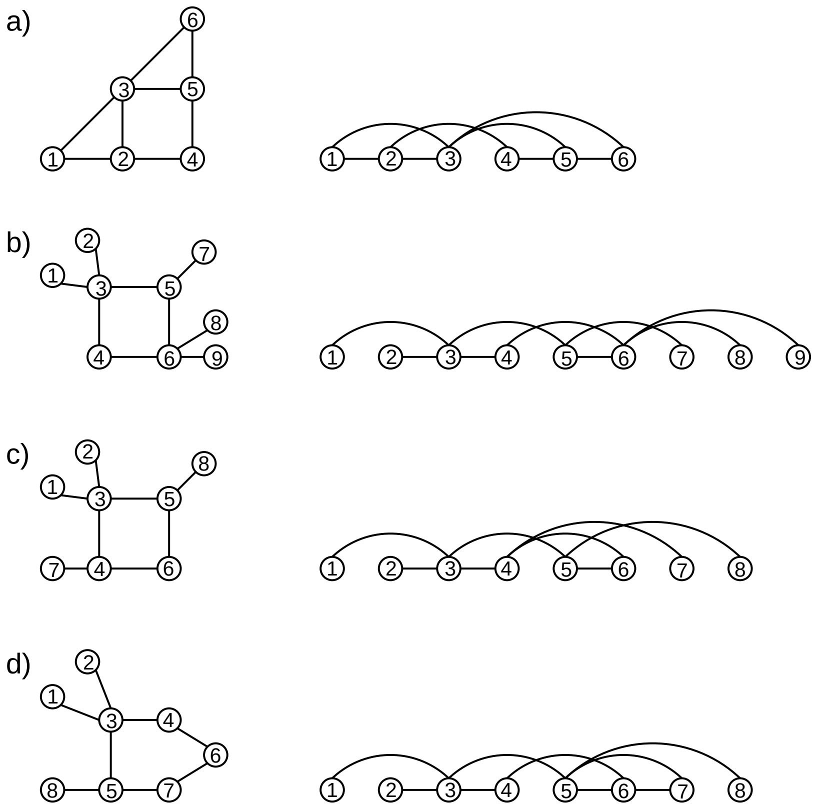
Westerfield suggests that K_4 (G_2) is the only polycyclic, 3-bandwidth critical graph, and makes several conjectures about different kinds of polycyclic graphs. This section will complete Westerfield's proof.

There are three categories of polycyclic graphs; those containing:

1. Independent cycles connected by single caterpillars or trees
2. Cycles with shared edges
3. Cycles with a shared vertex

2.3 Bad subgraphs

We know that any graph containing a proper 3-bandwidth subgraph is not 3-bandwidth critical. Therefore, we can eliminate any graph containing any of G_1 through G_6. It is also useful to study other possible subgraphs which can never be a part of a 3-bandwidth critical graph. The following diagrams categorize several "bad subgraphs" which will help us reduce our search. All are bandwidth three, and all but j (an expansion of G_6) are not homeomorphically minimal.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b)
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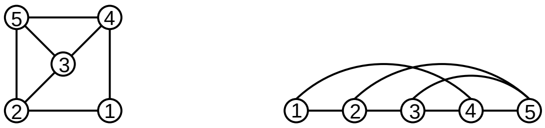
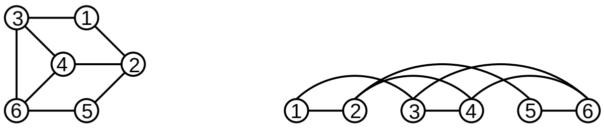
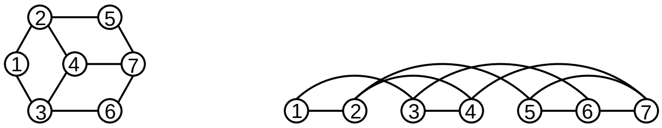
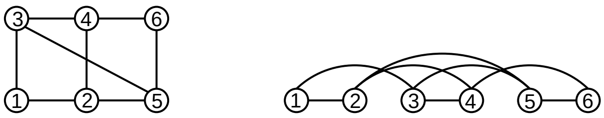
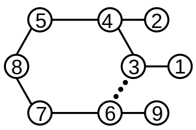
e) 

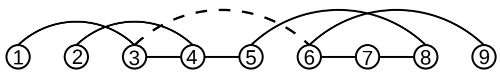
f) 

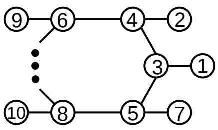
g) 

h) 

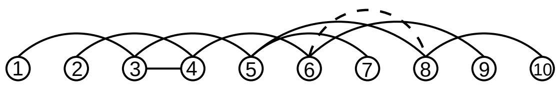
i) 

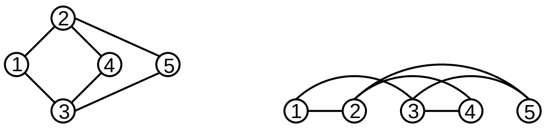
a path may be placed between vertex 3 and 6



j) 

a path may be placed between vertex 6 and 8



k) 

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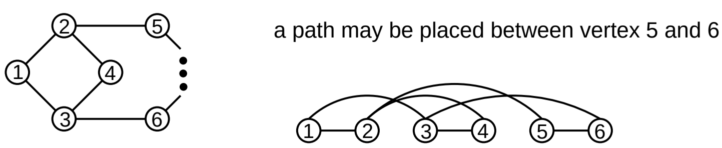

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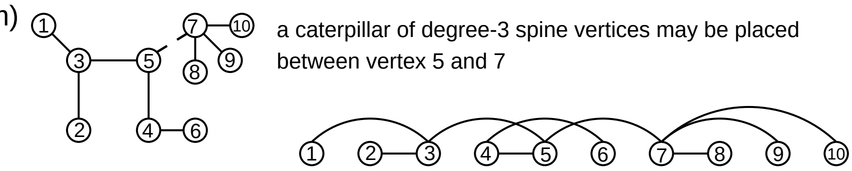
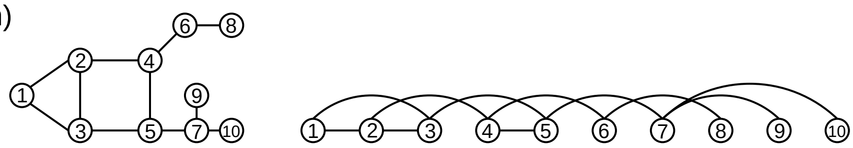


n)



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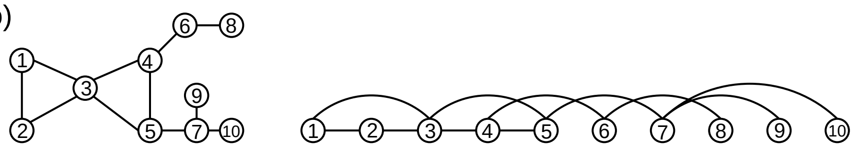


2.4 Independent cycles connected by single caterpillars or trees

Westerfield examined the case of independent cycles connected by single caterpillars, and determined that none are 3-bandwidth critical.

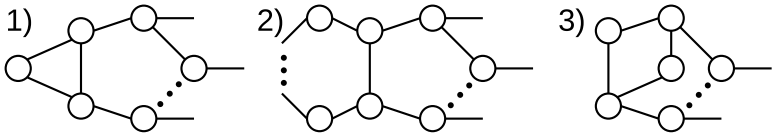
Any independent cycles connected by a non-caterpillar tree must contain a G_3 subgraph. Thus, there are no 3-bandwidth critical graphs in the category of independent cycles connected by single caterpillars or trees.

2.5 Cycles with shared edges

This category contains numerous cases to examine. We examine this problem using brute-force, and by eliminating larger cases by finding "bad subgraphs" of bandwidth three that are not critical. Thus any graph containing a "bad subgraph" cannot be 3-bandwidth critical.

Lemma 2.2 *Pentagons and larger cycles (not composed of triangles and rectangles) will not be included in any 3-bandwidth critical polycyclic graph.*

Proof Imagine a polycyclic graph contains a cycle C_n where $n > 4$ (Cycles formed by adjacent triangles and rectangles are excluded). If C_n shares one edge with another cycle, it must contain either subgraph 1 or 2:



Graph 1 contains bad subgraph i, while graph 2 contains bad subgraph j.

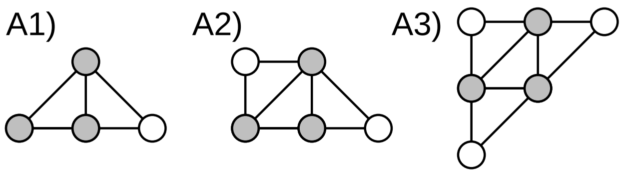
Suppose C_n shares multiple edges with another cycle. Attaching two sides of a triangle to the C_n cycle would divide it into a triangle connected to a C_{n-1} cycle at one edge. If $n > 5$, graph 1 must be a subgraph. If $n = 5$, then the cycle is really just a rectangle and triangle joined at one edge, which is covered in a later case. Now suppose a rectangle is connected at two edges to the C_n cycle. Graph 3 must be a subgraph, and it contains bad subgraph l. Attaching multiple edges of any larger cycle forces j to be a subgraph.

If an independent cycle C_n is connected only at a vertex, then it contains bad subgraph j. This is also true with a C_n cycle linked to another cycle via a caterpillar or tree. Therefore, there are no polycyclic graphs containing C_n cycles where $n > 4$ and C_n is not composed of triangular and rectangular cycles. ∎

Now we have narrowed down our search to polycyclic graphs composed of only triangular and rectangular subgraphs.

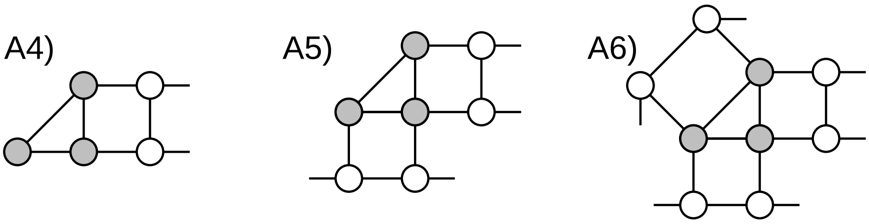
Case 1: No more than two cycles connnected at any one edge
A: Cycles added around a central triangle, added cycles not connnected

By adding only triangles and necessary edges to a central triangle, the following homeomorphically minimal candidate graphs are found:



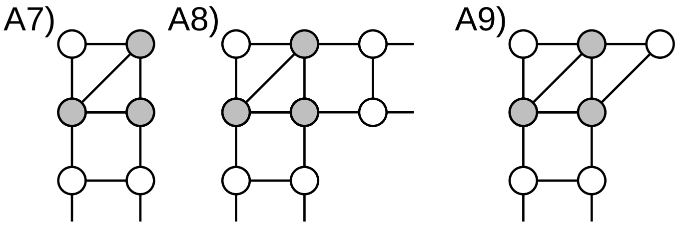
Graphs $A1$ and $A2$ are bandwidth two chains and are discussed in part E. $A3$ contains bad subgraph a.

By adding only rectangles and necessary edges to a central triangle, the following homeomorphically minimal candidate graphs are found:



Graphs $A4$ and $A5$ are bandwidth two chains. Graph $A6$ contains a G_5 subgraph.

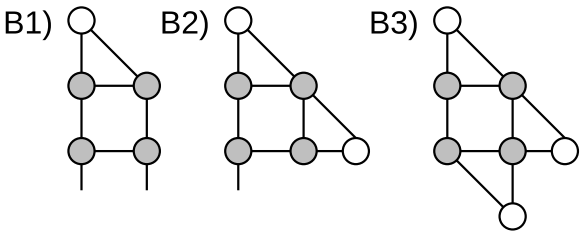
By adding both rectangles and triangles with necessary edges to a central triangle, the following homeomorphically minimal candidate graphs are found:

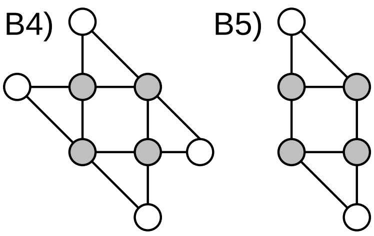


Graph $A7$ is a bandwidth two chain, discussed in part E. Graphs $A8$ and $A9$ contain bad subgraph c.

B: Cycles added around a central rectangle, added cycles not connected

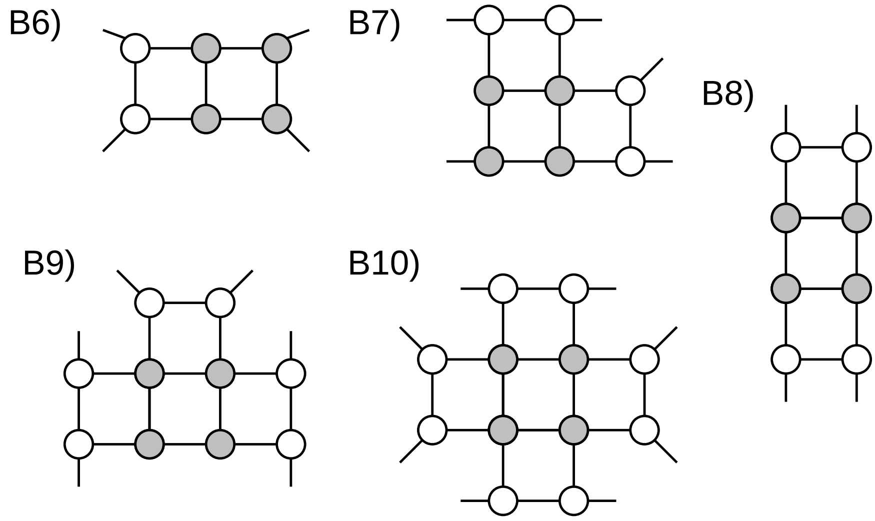
By adding only triangles and necessary edges to a central rectangle, the following homeomorphically minimal candidate graphs are found:





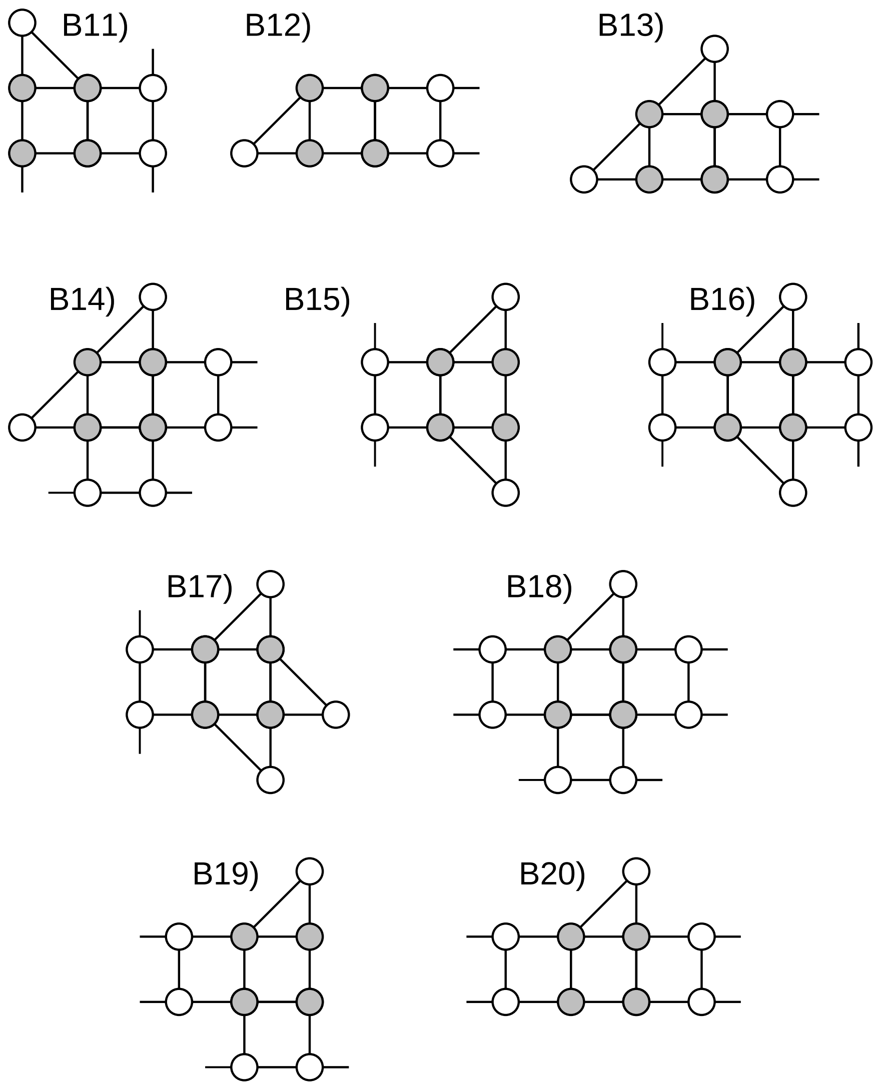
Graphs $B2$, $B3$, and $B4$ contain bad subgraph a. Graphs $B1$ and $B5$ are currently bandwidth two and fall into the category of chain graphs, which will be discussed in part E.

By adding only squares and necessary edges to a central rectangle, the following homeomorphically minimal candidate graphs are found:



Graphs $B7$, $B9$, and $B10$ contain bad subgraph c. Graphs $B6$ and $B8$ are bandwidth two and can be categorized as chain graphs. They will be discussed in part E.

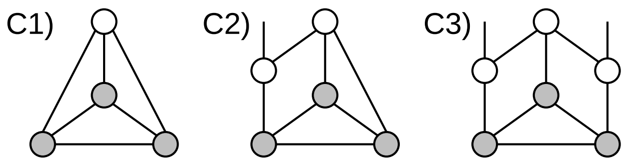
By adding both rectangles and triangles with necessary edges to a central rectangle, the following homeomorphically minimal candidate graphs are found:



Graphs $B12$ is a bandwidth two chain, discussed in part E. All other graphs ($B11$, $B13$-$B20$) contain bad subgraph c.

C: Cycles added around a central triangle, some added cycles are connnected

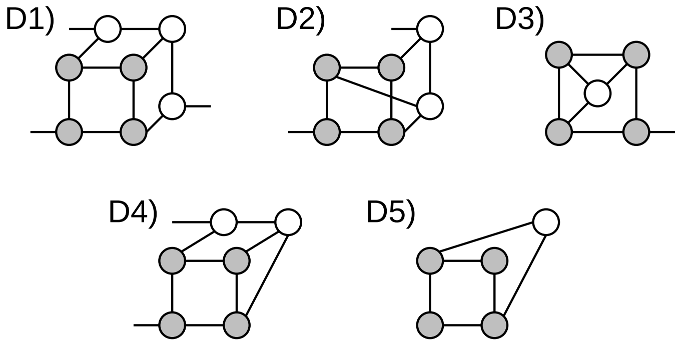
By adding rectangles and triangles around a central triangle, and by connecting at least two of the new cycles, one of the three following subgraphs must be generated:



$C1$ is the result of joining two new triangles attached to the central triangular cycle. This is the K_4 (G_2) 3-bandwidth critical graph. $C2$ will occur whenever a triangle is attached to a rectangle and both are attached to the central triangular cycle. This case contains bad subgraph e. Finally, $C3$ is the result of joining two rectangles attached to the central triangular cycle. Joining two sides of one rectangle to a triangle is equivalent to connecting two triangles in a chain. This contains bad subgraph f. Thus, only one graph in this category (G_2) is 3-bandwidth critical.

D: Cycles added around a central rectangle, some added cycles are connected

By adding rectangles and triangles around a central rectangle, and by connecting at least two of the new cycles, one of the five following subgraphs must be generated:

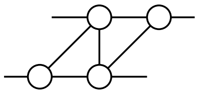


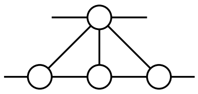
$D1$ and $D2$ are the two possible results of joining two new rectangles attached to the central rectangular cycle. They contain bad subgraphs g and h, respectively. $D3$ will occur whenever a triangle is attached to a rectangle and both are attached to the central cycle. This case contains bad subgraph f. Finally, $D4$ is the result of joining two triangles attached to the central cycle. This contains bad subgraph e. If one added square is attached to two sides of the central rectangle, $D5$ is a subgraph, which means bad subgraph k is also a subgraph. Thus, no graphs in this category may be 3-bandwidth critical.

E: Chains of squares and triangles

Lemma 2.3 *If a chain composed only of squares and triangles is bandwidth three, then it contains a 3-bandwidth critical subgraph.*

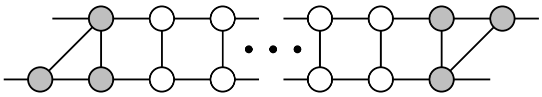
Proof Suppose the chain consists of only squares. Then the bandwidth of the chain is two. Now consider a chain with only one triangle and any number of squares. The bandwidth is still two. Finally, consider a chain with at least two triangles. The following arangements are possible. First, consider the case where two triangles are adjacent:

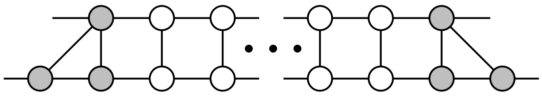




The uppermost graph is bandwidth two. However, the lower graph contains a G_1 subgraph and is bandwidth three.

Next, consider the case where the triangles are separated by any number of squares:



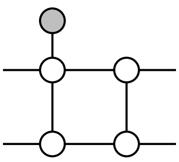


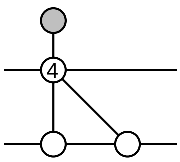
The uppermost graph is bandwidth two. However, the lower graph contains a G_4 infinite caterpillar subgraph and is bandwidth three.

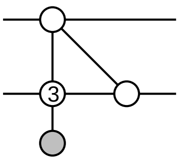
This exhausts all possible cases of chain graphs. Therefore, the only bandwidth three chains already contain 3-bandwidth critical subgraphs. ∎

Suppose the chain is bandwidth two. Then we must add pendants or trees to search for 3-bandwidth critical graphs.

Adding pendants to the interior:

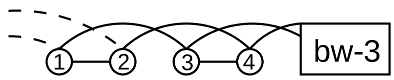






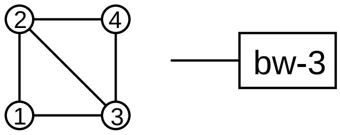
If one pendant is added to an interior rectangular cycle, bad subgraph c is created. If a single pendant is added to a degree four vertex of a triangular cycle, a G_1 subgraph is formed. Lastly, if a single pendant is added to a degree three vertex of a triangular cycle, a G_5 subgraph is formed. Therefore, we cannot add any pendant inside a chain to find a 3-bandwidth critical graph.

Adding pendants and trees to the ends: We now attempt to find 3-bandwidth critical graphs by adding pendants and trees to the ends of the chains. All bandwidth two chain graphs contain a rectangular (C_4) subgraph. Even if the chain is composed soley of triangles, any two adjacent triangles form a rectangle. The bandwidth of a rectangle is always two. Now suppose we add pendants and trees to the end of the chain to increase the bandwidth to three. We know that the bandwidth three portion of the embedding will only occur on one side of the chain, or the graph will not be critical. For example, the following diagram shows an arbitrary chain graph with a rectangular cycle. Without loss of generality, the bandwidth three portion is embedded to the right:



In order to be critical, we must be able to reduce the bandwidth three portion and embed at least one vertex toward the right by removing a single edge. If the rectangle is formed by two triangles (connect vertex 2 to 3), we can remove the diagonal edge of the rectangle and not reduce the graph's bandwidth, as in graph 1:

1)



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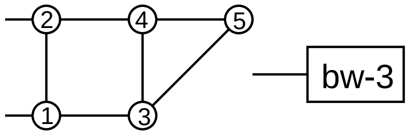

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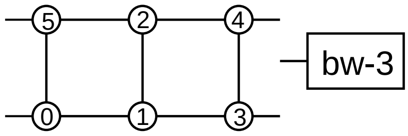

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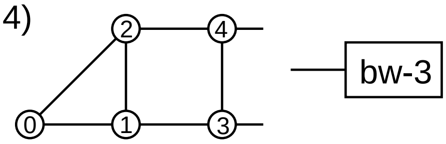


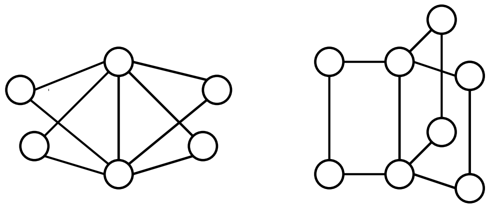
Now suppose we examine the rightmost rectangular cycle found in our embedding (WLOG). If the rectangle is not formed from triangles, as in case 1, three cases remain: 2) a triangular cycle exists to the right of the rectangle 3) a rectangular cycle lies to the left of the rectangle 4) A triangular cycle lies to the left of the rectangle, and no cycles lie to the right.

In Case 2, a triangle exists to the right of the rectangle. Removing the edge beginning at either vertex 1 or 2 and extending left will not decrease the bandwidth of the graph. In Case 3, removing the edge from 0 to 5 will not decrease the bandwidth of the graph. In Case 4, any combination of pendants or trees added to the rectangular cycle will create bad subgraph b, c, d, m, or n. Therefore, the graph cannot be critical.

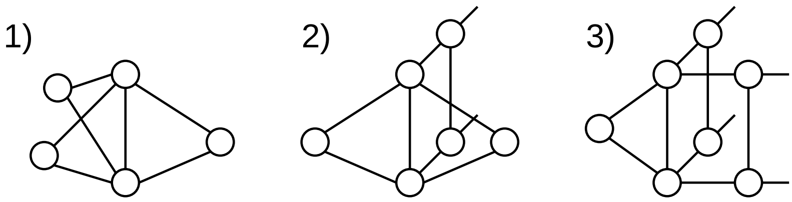
Thus, no 3-bandwidth critical graphs exist in the case of chains.

Case 2: Multiple cycles along a shared edge ("book" graphs)

Let a *book graph* denote a polycyclic graph where three or more cycles share one edge. The cycles form the *pages* of the book. All book graphs must have fewer than four pages to avoid containing a G_1 subgraph. Furthermore, every three-page book graph must contain at least one triangular cycle in order to avoid containing a G_4 subgraph. These cases are illustrated below:



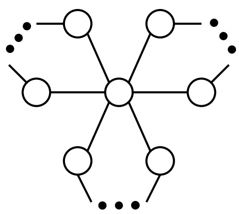
On the left is a four-page graph, which will always have a vertex of degree five and therefore contain G_1. On the right is a three-page graph composed of rectangular cycles, which has a G_4 subgraph. Therefore, all possible book graphs which are candidates for 3-bandwidth critical graphs must contain one of the following subgraphs:



Graph 1 contains bad subgraph k, graph 2 contains bad subgraph l, and graph 3 contains bad subgraph c. Therefore, no book graphs are 3-bandwidth critical.

2.6 Cycles with a shared vertex

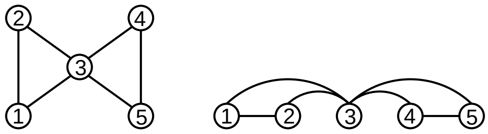
If three or more cycles share a vertex, a $K_{1,5}$ (G_1) subgraph will be formed:



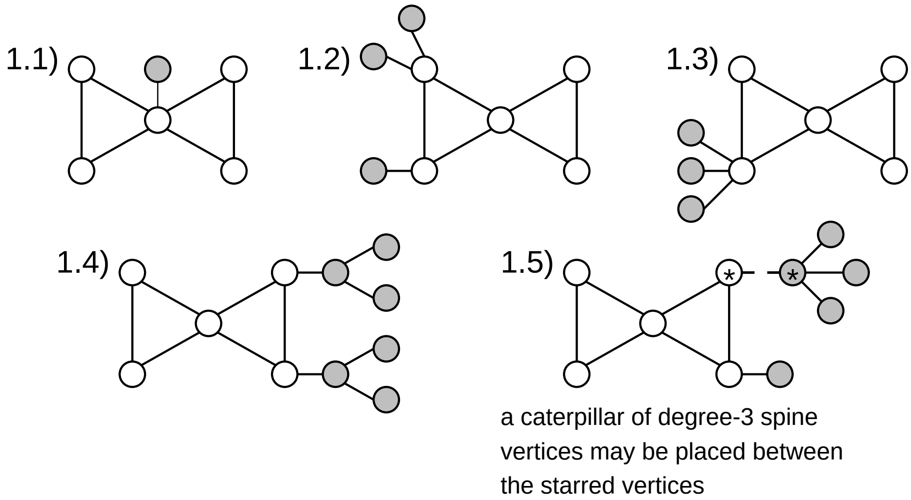
Therefore, we must only examine cases where two cycles share a vertex but no edges.

Case 1: Two triangles with a shared vertex

Two triangles with a shared vertex have bandwidth two. Suppose we add pendants and trees to the graph to find 3-bandwidth critical candidates. There is only one way to minimize the embedding of this two-triange formation, as shown below. The two triangles must be embedded to each side of the central vertex:

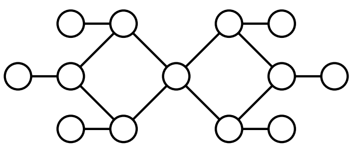


Therefore, in an optimal embedding, any pendants or trees attached to the right triangle must be embedded toward the right, and any pendants or trees attached to the left triangle must be embedded to the left (WLOG). What is attached to one triangle will have no effect on the embedding of anything attached to the other triangle. Thus, we must only consider cases where pendants and trees are added to one triangle:



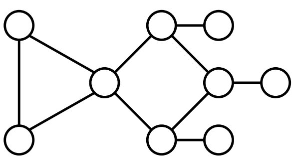
If we only add pendants to one triangle to obtain a minimal 3-bandwidth graph, graph 1.1, 1.2, or 1.3 must be a subgraph. Graphs 1.1 and 1.3 contain G_1 subgraphs, while graph 1.2 contains a G_5 subgraph. By adding trees and pendants to one triangle to obtain a minimal 3-bandwidth graph, graph 1.4, 1.5, or G_1 will always be a subgraph. However, graph 1.4 contains bad subgraph o, and graph 1.5 contains bad subgraph m.

Case 2: Two rectangles with a shared vertex



Any two connected rectangular cycles in this case contain bad subgraph c.

Case 3: A triangle and square with a shared vertex



Any connected rectangular and triangular cycles in this case contain bad subgraph c.

Therefore, there are no 3-bandwidth critical graphs in the case of cycles with a shared vertex.

2.7 Conclusion

The problem of finding 3-bandwidth critical polycyclic graphs has been broken down into three main categories: 1) independent cycles connected by caterpillars or trees 2) cycles with shared edges 3) cycles with a shared vertex only. No graphs excepting G_2 have been found. All cases have been examined, and Westerfield's claim has been proved: G_1 through G_6 and the G_4 infinite family are the only 3-bandwidth critical graphs. ■

3 Expansion of 3-bandwidth results

By observing characteristics of 3-bandwidth critical graphs, many similar 4-bandwidth critical results can be discovered.

3.1 Complete graphs

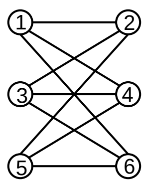
Lemma 3.1 *(Westerfield)*
For $n > 1$, the complete graph K_n is $(n-1)$-bandwidth critical.

Thus, K_4 (G_2) is 3-bandwidth critical, and K_5 is 4-bandwidth critical.

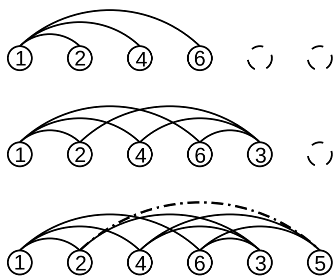
3.2 Stars and Bipartite Graphs

Lemma 3.2 *(Westerfield)*
For n odd and positive, the complete bipartite graph $K_{1,n}$ is $(\frac{n+1}{2})$-bandwidth critical.

By this lemma, $K_{1,5}$ (G_1) is 3-bandwidth critical, and $K_{1,7}$ is 4-bandwidth critical. Also, the $K_{3,3}$ graph is 4-bandwidth critical.



Proof $K_{3,3}$ only contains vertices of degree three, and thus is homeomorphically minimal. All edges and vertices are identical in this graph. Without loss of generality, we select vertex 1 to embed on the far left. Next, we place each connected vertex (2, 4, and 6) to the right of vertex 1 and draw in edges. The order is arbitrary.

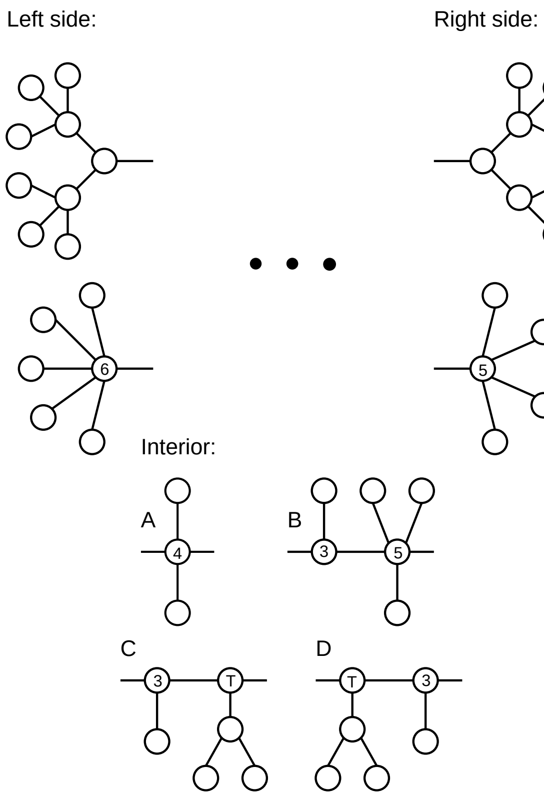


Then we add vertex three (WLOG) on the right of the embedding and connect vertices with edges. Vertex 5 remains. However, adding this final vertex increases the bandwidth of the graph to 4.

As all edges and vertices are identical in this complete bipartite graph, we must only show that removing one edge results in a bandwidth less than four. In this example, if we remove the edge between vertices 2 and 5, the embedding is reduced to a bandwidth of three. Thus, every proper subgraph has a bandwidth of less than four. Therefore, $K_{3,3}$ is 4-bandwidth critical. ∎

3.3 Expansion of G_4 and the infinite tree family

By adding pendants to G_4, a 4-bandwidth critical graph and infinite tree family can be obtained. Furthermore, a similar infinite family can be formed between any combination of the left and right subtrees pictured below:



Infinite caterpillars and trees can be formed by adding blocks of

(A) degree four vertices

(B) degree three followed by degree five vertices

(C) degree three followed by a tree T: a degree three root vertex with two leaves attached to the pendant

(D) a tree T: a degree three root vertex with two leaves attached to the pendant followed by a degree three vertex

in any combination between any left side and right side subtree. Without loss of generality, these blocks must be assembled from left to right as pictured above.

4 Trees

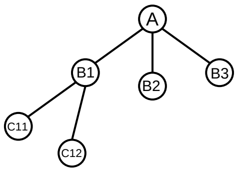
This section explores 4-bandwidth critical trees.

Remark In order to remain homeomorphically minimal, no tree may have a vertex of degree two. Therefore, the root vertex of a tree must have degree of at least three. Additionally, the maximum degree of the tree Δ must be less than seven or the graph will contain a $K_{1,7}$ subgraph.

4.1 4-bandwidth critical trees of height two

Theorem 4.1 *There exist twelve 4-bandwidth critical trees of height two.*

Proof Let T_i denote the ith 4-bandwidth critical tree with height two. T_i must contain the following subtree:



Let A, B, and C denote the three different levels of the tree. The B and C level vertices are numbered for clarification. The tree shown above has a bandwidth of two. By adding leaves to the subtree, we can search for all height two, 4-bandwidth critical trees which may be candidates for T_i.

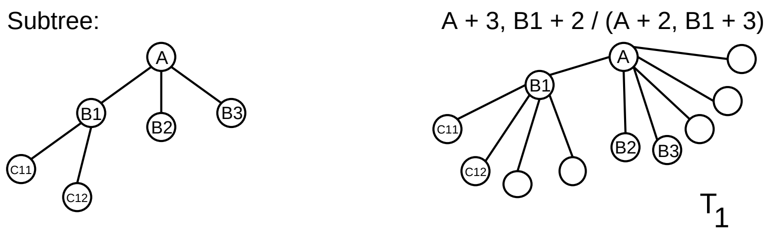
Case 1: Consider the set of candidates for T_i created by exclusively adding leaves to vertex A. In order to create a 4-bandwidth graph, four vertices must be added to vertex A. However, the degree of A is now seven, meaning that A has a $K_{1,7}$ subgraph and thus is not 4-bandwidth critical.

Case 2: Consider the set of candidates for T_i created by adding leaves only to vertices A and $B1$. We will not yet consider adding leaves to newly formed vertices on level B, as these will be covered in Cases 3 and 4.

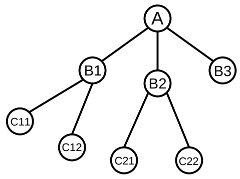
Pictured below is a layout for an optimal embedding of any member of this set. We can add up to four vertices while maintaining bandwidth three. The dotted circles denote possible locations for the new vertices. However, the addition of a fifth vertex results in a 4-bandwidth graph and a possible candidate for T_i.

Note: no more than three vertices may be added to vertex A or $B1$ without obtaining the 4-bandwidth critical $K_{1,7}$ subgraph.

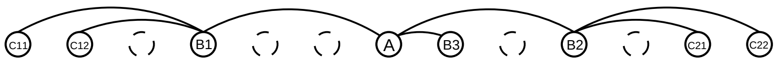
Let $X + n$ for some integer n denote the addition of n leaves to vertex X. There is only one candidate for T_i, which turns out to be 4-bandwidth critical. Note that T_1 is the same graph as the G_4 expansion.



Case 3: Consider the set of 4-bandwidth trees generated by adding leaves to A, $B1$, and $B2$. The following tree must be a subgraph:

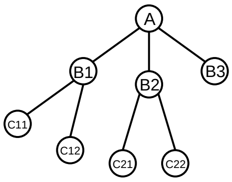


The optimal embedding layout of bandwidth three is pictured below, with dotted circles denoting the five possible locations for new vertices while maintaining bandwidth three. The addition of a sixth vertex will cause the tree to have bandwith four, and will create a candidate for T_i.

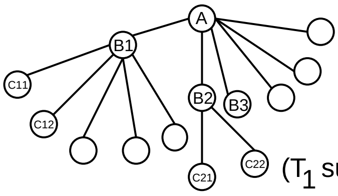


Through a stepwise process of adding six leaves to the subtree, we can obtain seven candidates for 4-bandwidth critical trees with fourteen vertices. Of these, only one (T_2) is found to be 4-bandwidth critical. The other graphs contain T_1 or its infinite caterpillar family as a subgraph, and thus are not 4-bandwidth critical.

Subtree:

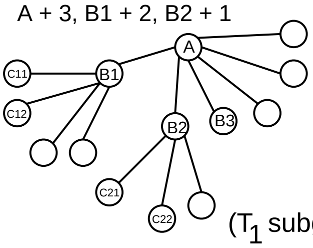


A + 3, B1 + 3



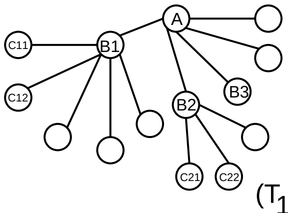
(T_1 subgraph)

A + 3, B1 + 2, B2 + 1



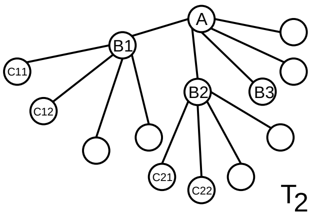
(T_1 subgraph)

A + 2, B1 + 3, B2 + 1



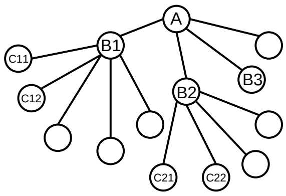
(T_1 subgraph)

A + 2, B1 + 2, B2 + 2



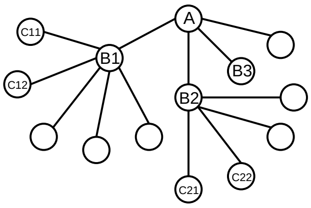
T_2

A + 1, B1 + 3, B2 + 2



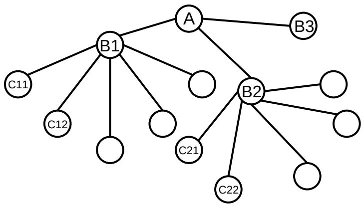
(T_1 subgraph)

A + 1, B1 + 3, B2 + 2



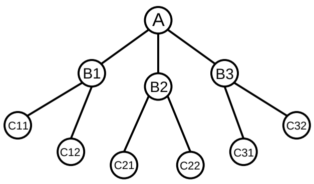
(T_1 subgraph)

B1 + 3, B2 + 3



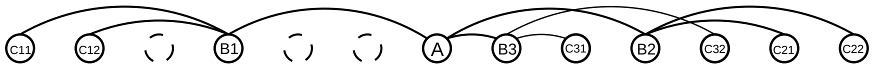
(T_1 subgraph)

Case 4: Consider the set of 4-bandwidth trees generated by adding leaves to A and all B level vertices. The following tree must be a subgraph:



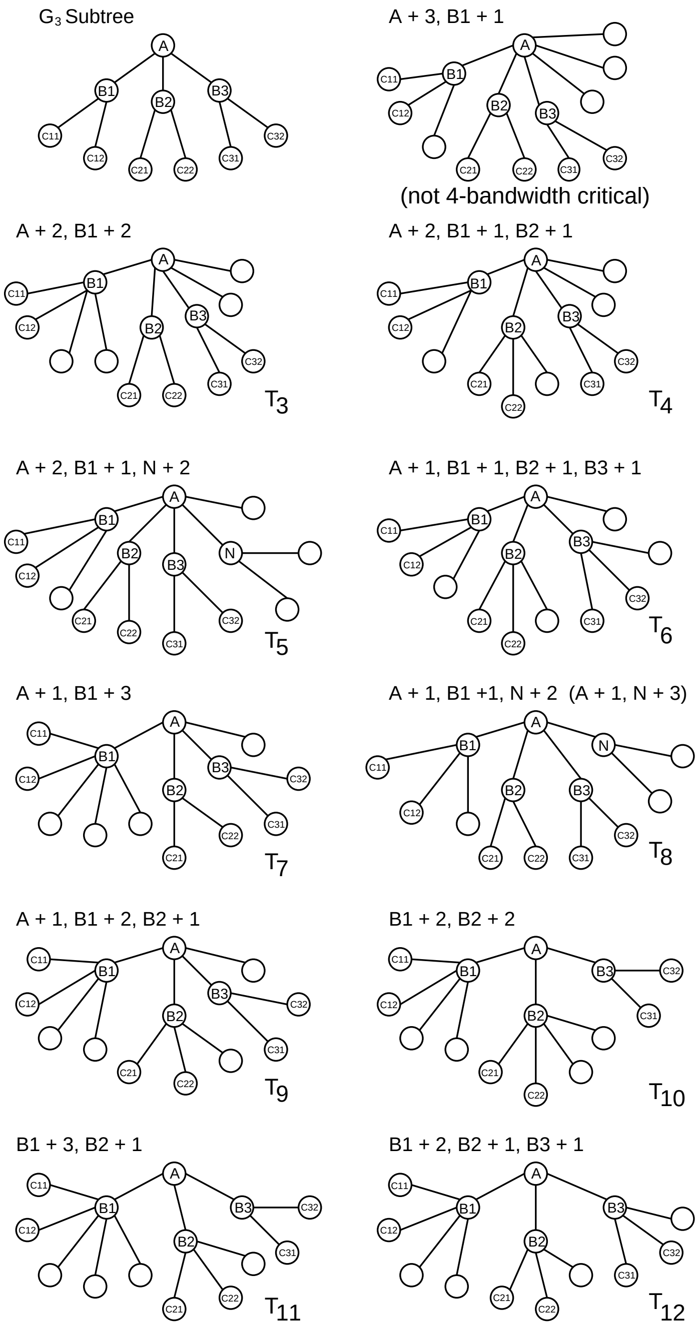
It is notable that Westerfield found this tree, known as G_3, to be 3-bandwidth critical.

A 4-bandwidth critical tree must have a 3-bandwidth critical subgraph. The G_3 tree can be embedded while maintaining bandwidth three by placing vertices in their widest arrangement. Adding three vertices will create the largest 3-bandwidth graph within this arangement. At least four leaves must be added in order to obtain a 4-bandwidth graph. If any fewer are added, the graph will remain at bandwidth three. An optimal embedding is shown below, with space for three more vertices while maintaining bandwidth three:



If we add five vertices without widening the embedding, there will be at least two edges of bandwidth four in an optimal embedding. Removing one will leave a 4-bandwidth graph, and thus the graph will not be 4-bandwidth critical. Thus, the only way to add more than four vertices to the G_3 tree and obtain a candidate for T_i would be to place new vertices to either side of the embedding. First, we want to find a "full" bandwidth three tree. This means that we will find a tree such that we can obtain a 4-bandwidth graph by adding one more vertex. New vertices added to the left or right of the embedding must be attached to a C vertex in order to remain bandwidth three. However, the height of the resulting tree would exceed two. Thus, the only remaining candidates for T_i contain the subtree G_3, and are examined in this final case.

Through a stepwise process of adding four leaves to the 3-bandwidth critical tree, we can obtain eleven candidates for 4-bandwidth critical trees with fourteen vertices. Of these, only ten are found to be 4-bandwidth critical:



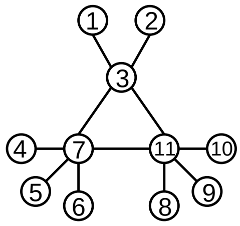
Ten of the eleven trees found by the expansion of G_3 are 4-bandwidth critical.

As we cannot widen the embedding without increasing the height of the tree, this exhausts all cases. Only the twelve trees $T_1 ... T_{12}$ of height two are 4-bandwidth critical. ∎

5 Cyclic graphs

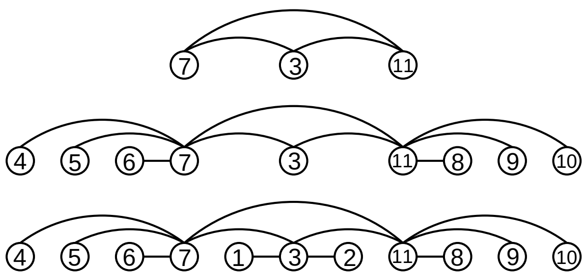
5.1 Expansion of 3-bandwidth critical cyclic graphs

The G_5 graph can be expanded by adding leaves to the C_3 subgraph to form 4 and 5-bandwidth critical graphs. It might be worth looking into whether this pattern can be continued to form graphs with higher bandwidth criticallity.



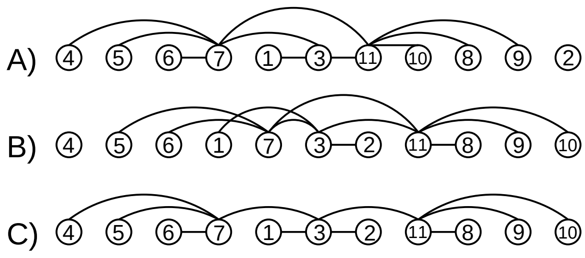
The graph pictured above is 4-bandwidth critical.

Proof The graph is homeomorphically minimal, containing only vertices of degrees one, three, and four. To embed the graph optimally, first place the vertices of the cycle (3, 7, 11), such that the vertices of highest degree (7, 11) are to the outside.

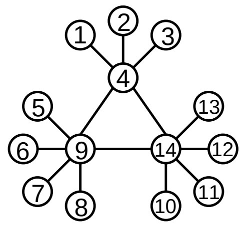


Next, embed the leaves of vertices 7 and 11 to the outside of the cycle. Finally, embed the leaves of vertex 3. Any placement of these final vertices will increase the bandwidth to at least four.

Next, we must show that every proper subgraph has a smaller bandwidth than four. There are three cases for removing a single edge: A) removing a leaf edge from vertex 3 B) removing a leaf edge from vertex 7 or 11 C) removing an edge of the cycle. The optimal embeddings of each case are pictured below:

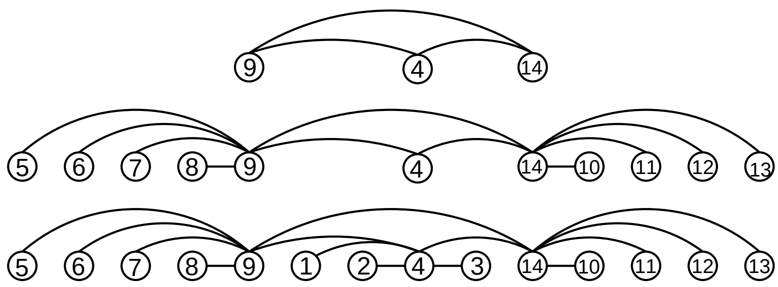


Because all three cases have bandwidth three, every possible subgraph has bandwidth smaller than four. Thus, the graph is 4-bandwidth critical. ∎



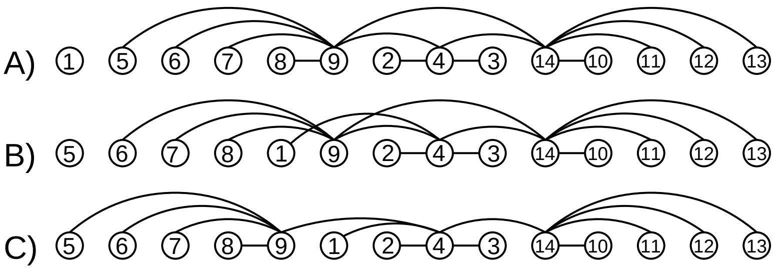
By adding more pendants to G_5, we can create the 5-bandwidth critical graph pictured above.

Proof The graph is homeomorphically minimal, containing only vertices of degrees one, four, and five. To embed the graph optimally, first place the vertices of the cycle (4, 9, 14), such that the vertices of highest degree (9, 14) are to the outside.



Next, embed the leaves of vertices 9 and 14 to the outside of the cycle. Finally, embed the leaves of vertex 4. Any placement of these final vertices will increase the bandwidth to at least five.

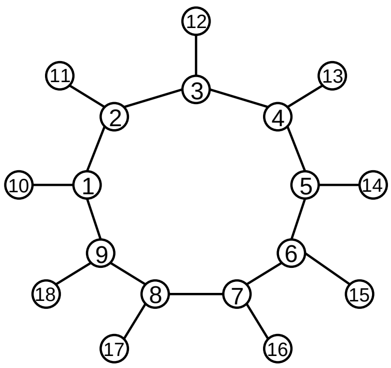
Next, we must show that every proper subgraph has a smaller bandwidth than five. There are three cases for removing a single edge: A) removing a leaf edge from vertex 4 B) removing a leaf edge from vertex 9 or 14 C) removing an edge of the cycle. The optimal embeddings of each case are pictured below:



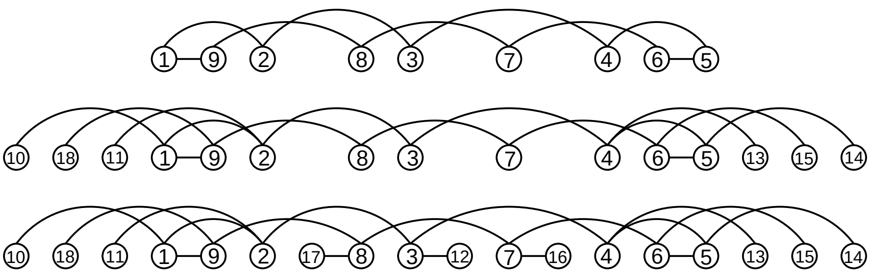
Because all three cases have bandwidth four, every possible subgraph has bandwidth smaller than five. Thus, the graph is 5-bandwidth critical. ∎

More work is left to be done to see if these G_5 expansion graphs form an infinite family of n-bandwidth critical graphs.

The following graph, modeled after G_6, is 4-bandwidth critical. This graph consists of a C_9 subgraph with a leaf attached to each vertex in the cycle.

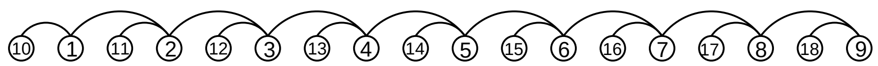


Proof This graph contains only vertices of degree three and degree one, and thus is homeomorphically minimal. All pendant edges and vertices are identical in this graph. Furthermore, all edges and vertices on the cycle are identical. Without loss of generality, we first embed the cycle in a bandwidth two arrangement. Next, we place six pendants (10, 11, 13, 14, 15, 18) on both sides of the embedded cycle. This increases the bandwidth to 3.



Three pendant vertices (12, 16, 17) remain. We can safely place 16 and 17 and maintain bandwidth three. However, adding vertex 12 will raise the graph's bandwidth to four, no matter what arrangement the three vertices are placed. Thus, the graph has bandwidth four.

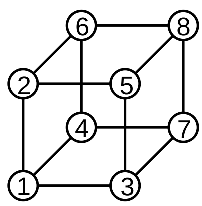
As pendant edges are identical and cycle edges are identical, we must demonstrate that two cases result in a bandwidth less than four: A) removing one pendant edge B) removing one cycle edge. Without loss of generality, we remove the pendant edge from vertex 3 to 12, and the resulting subgraph has bandwidth three. In the second case, the edge from vertex 1 to 9 is removed. The cycle is broken and now forms a path with pendants, or a caterpillar. The resulting caterpillar is embedded below with bandwidth two:



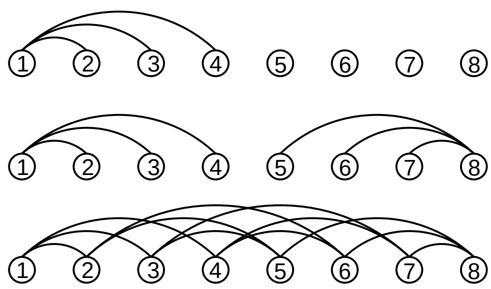
Thus, every proper subgraph has a bandwidth of less than four. Therefore, the G_6 expansion is 4-bandwidth critical. ∎

5.2 Cubes

The cubic graph Q_3 is 4-bandwidth critical.

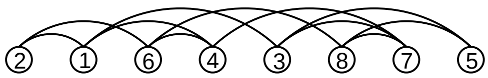


Proof Q_3 only contains vertices of degree three, and thus is homeomorphically minimal. All edges and vertices are identical in this graph. Without loss of generality, we select vertex 1 to embed on the far left. Next, we place each connected vertex (2, 3, and 4) to the right of vertex 1 and draw in edges. The order is arbitrary.



Then we add vertex 8, the most distant vertex from 1, to the far right of the embedding. Connected vertices (5, 6, and 7) and their edges are placed to the left of vertex 8. Finally, the remaining edges are added, creating a 4-bandwidth graph. No matter what order vertices 2 through 7 are placed, the embedding cannot be improved.

As all edges and vertices are identical in this cubic graph, we must only show that removing one edge results in a bandwidth less than four. In this example, if we remove the edge between vertices 1 and 2, the embedding is reduced to a bandwidth of three.



Thus, every proper subgraph has a bandwidth of less than four. Therefore, Q_3 is 4-bandwidth critical. ∎

6 Conclusion

This paper concludes Westerfield's work on 3-bandwidth critical graphs, demonstrating that G_1 through G_6 and the G_4 infinite family are the only 3-bandwidth critical graphs. The paper also categorizes many types of 4-bandwidth critical graphs, including trees of height two. New infinite families have been explored. It may be of interest to continue studying 4-bandwidth critical graphs. More work needs to be done in classifying unicyclic and polycyclic graphs, as well as larger trees. However, this problem is NP-complete, and the set of all 4-bandwidth critical graphs is likely much larger than that of 3-bandwidth critical graphs.

Acknowledgements
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References

[1] F.R.K. Chung, "Labelings of Graphs," Selected Topics in Graph Theory, vol. 3, pp. 151-167, 1988.

[2] Yixun Lin, Aifeng Yang, "On 3-cutwidth critical graphs," Discrete Mathematics, vol. 275, pp. 339-346, June 2004.

[3] Maciej M. Syslo and Jerzy Žak, "The Bandwidth Problem: Critical Subgraphs and the Solution for Caterpillars," Annals of Discrete Mathematics, vol. 16, pp. 281-286, 1982.

[4] Holly Westerfield, "On 3-Bandwidth Critical Graphs," Project for CSUSB REU, August 2005.