Two Dimensional Bandwidth
REU in Mathematics at CSUSB
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Abstract

The two dimensional bandwidth problem under the distances of L_∞-norm and L_1-norm has only been studied for a limited number of graphs. In this paper, I analyze the two dimensional bandwidth for a few different families of graphs.

1 Introduction

Below are some basic graph theory definitions that are necessary for understanding bandwidths.

A **graph G=(V,E)** is a set of V vertices and E edges. Each edge must start at a vertex and end at a different vertex. The **degree** of a vertex is defined as the number of edges incident to the vertex.
The **complete graph** K_n with $n \in Z^+$ is a graph in which every pair of vertices is connected by an edge. In a complete graph there are a total of n vertices, each of degree (n-1). Below is a representation of the graph K_4 with vertex set $\{1, 2, 3, 4\}$.

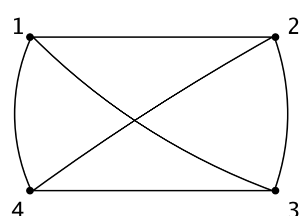

Figure 1.

The **product graph of two complete graphs,** $K_n \times K_m$, is simply a collection of m copies of K_n. Vertices in different copies of K_n are adjacent if and only if they are corresponding vertices. The product graph has a total

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of (nm) vertices, and all of the vertices have degree $(n+m$-2$)$. Figure 2 is a representation of $K_4 \times K_2$. As you can see there are two copies of K_4. Between the two copies, only the corresponding vertices are adjacent.

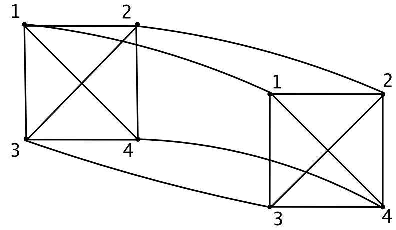
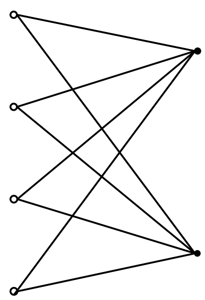


Figure 2.

The **complete bipartite graph**, $K_{n,m}$, consists of two sets of vertices. The first set has n vertices, and the second set has m vertices. There are a total of $(n+m)$ vertices with n vertices of degree m and m vertices of degree n. Every vertex in one set is connected by an edge to all the vertices of the other set. Also, no two vertices in the same set are connected by an edge. Below is $K_{2,4}$.



A **path,** P_m, is a graph of a sequence of distinct vertices $(x_1, x_2, x_3, ..., x_m)$ such that x_i is adjacent to x_{i+1} [6].

2 Background

2.1 B, B_2, and β_2

B, B_2, and β_2 represent the bandwidths under one-dimensional distance, two-dimensional L_1-norm distance, and two-dimensional L_∞-norm distance respectively.

2.1.1 B

Let $G = (V(G), E(G))$ be a connected graph with n vertices. An embedding of G into a path P_n can be represented by a bijection $f : V(G) \rightarrow \{1, 2, ..., n\}$ [4]. Now, let $a, b \in V(G)$, and $(a, b) \in E(G)$. The distance between a and b is defined as the difference between $f(a)$ and $f(b)$, $D(a, b) = |f(a) - f(b)|$.

For any nontrivial graph there are many linear embeddings. The bandwidth of the embedding f for graph G is [4]

$$B(G,f) = \max_{uv\epsilon E(G)} |f(u) - f(v)|.$$

The overall bandwidth of G is the minimum bandwidth over all embeddings, [4]

$$B(G) = \min_{f} B(G,f).$$

Bandwidth under the one-dimensional distance can be thought of as the longest edge between two adjacent vertices in a graph. The following is an example of the above bandwidth. In Figure 3, which is an optimal linear embedding of K_4, the longest edge is the edge between vertices 1 and 4. The differnce between vertex one and vertex four is three. Hence, $B(K_4) = 3$.

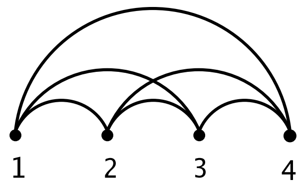


Figure 3.

The above figure is an optimal embedding of K_4 since any other embedding yields a bandwidth of at least 3.

2.1.2 B_2

Under L_1-norm, the distance between two vertices (i,j) and (i',j') in grid graph H is defined as [7]

$$d_H((i,j),(i',j')) = | i - i' | + | j - j' |,$$

called the rectilinear distance in the plane. Simply stated, the distance between two vertices under L_1-norm is the sum of the distances in the horizontal and vertical directions.

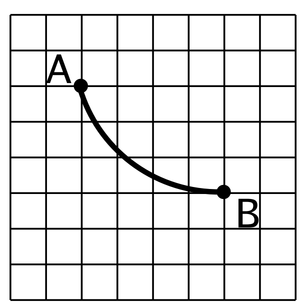


Figure 4.

In the above figure, the distance between vertex A and vertex B is 7, 4 in the horizontal direction and 3 in the vertical direction. The bandwidth of a labeling f of graph G is defined by [8]

$$B_H(G, f) = \max_{(u,v)\epsilon E(G)} d_H(f(u), f(v)).$$

The of graph G is [8]

$$B_H(G) = \min_f B_H(G, f)$$

where the minimum is taken over all labelings f.

2.1.3 β_2

In comparison, the distance between two vertices under the distance of L_∞-norm is defined below. Given two points in a grid graph H, (i, j) and (i', j'), the distance between them, ∂_H, is defined by [1]

$$\partial_H((i, j), (i', j')) = max\{|i - i'|, |j - j'|\}.$$

In other words, the distance between two vertices under L_∞-norm is the maximum of the distances between the vertices in the horizontal and vertical directions. Refering back to Figure 4, ∂_H(A,B)=max$\{4, 3\}$.

For any graph G, there are many different ways of embedding the graph into a grid. Each embedding, f, yields a bandwidth. Two-dimensional bandwidth of (G, f) under L_∞-norm is defined by [1]

$$\beta_2(G, f) = \max_{uv\epsilon E(G)} \partial_H(f(u), f(v)),$$

where f is a one-to-one mapping from $V(G)$ to $V(H)$, i.e, an injection $f : V(G) \to V(H)$. The overall bandwidth of the graph G is the minimum bandwidth of every embedding of G, defined by [1],

$$\beta_2(G) = \min_f \beta_2(G, f).$$

2.2

How do B, B_2, β_2 relate?

3 Previous Findings

3.1

There has been little work on two-dimensional bandwidth under the distance of L_∞-norm. The results are specialized, only classifying several families of graphs. For example, the two-dimensional bandwidth of a complete graph is given below.

Theorem 3.1 [4] *For a complete gragh K_n of n vertices,*

$$\beta_2(K_n) = \lceil \sqrt{n} - 1 \rceil.$$

The two-dimensional bandwidth of the product of a complete graph and a path, $(K_n \times P_m)$, is known:
Theorem 3.2 [1] *Let* $m, n \geq 2$, *denote* $d = \lceil \sqrt{n} - 1 \rceil$.

1. *If* $d^2 < n \leq d(d+2) = (d+1)^2 - 1$, *we have* $\beta_2(K_n \times P_m) = d$;
2. *If* $n = (d+1)^2$, *we have* $\beta_2(K_n \times P_m) = d + 1$.

Another family of graphs for which the two-dimensional bandwidth is known is the product graphs of any complete graph times any cycle graph of three or more edges, $(K_n \times C_m)$.
Theorem 3.3 [1] *Let* $n, m \geq 3$, *denote* $d = \lceil \sqrt{n} - 1 \rceil$.

1. *If* $n = d^2 + 1$, *we have* $\beta_2(K_{d^2+1} \times C_3) = d$.
   If $d^2 + 2 \leq n \leq (d+1)^2$, *we have* $\beta_2(K_n \times C_3) = d + 1$.
2. *If* $m \geq 4$, *we have* $d \leq \beta_2(K_n \times C_m) \leq d + 1$.

One other previous finding that's interesting when studying two-dimensional bandwidth under L_∞-norm is $K_{1,n}$. The theorem says [4], $\beta_2(K_{1,n}) = \left\lceil \frac{\sqrt{n+1}-1}{2} \right\rceil$.

3.2

Under the distance of L_1-norm, the bandwidth B_2 is known for many graphs. Some of the bandwidths that are known are $B_2(K_n)$ and $B_2(K_{n,m})$. However, stating their theorems here is not important to explaining my work.

4 β_2 of Product Graphs of Complete Graphs

4.1 $\beta_2(K_{d^2} \times K_4)$

Theorem 4.1 *Let* $d \geq 1$.

$$\beta_2(K_{d^2} \times K_4) = d$$

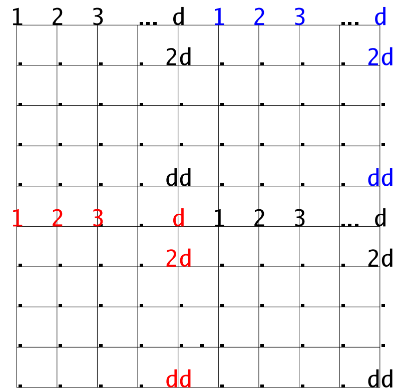
Proof: First, we know $\beta_2(K_{d^2} \times K_2) = \beta_2(K_{d^2} \times P_2)$ since $K_2 = P_2$. By Theorem 3.2, $\beta_2(K_{d^2} \times P_2) = d$, which implies that

$$\beta_2(K_{d^2} \times K_2) = d.$$

From the above equation and the fact that $(K_{d^2} \times K_2)$ is a subgraph of $(K_{d^2} \times K_4)$,

$$\beta_2(K_{d^2} \times K_4) \geq d.$$

From this equation we know that the two-dimensional bandwidth of K_4 times any K_{d^2} is at least d. Now, I will show that the upper bound is also d. Consider the embedding of $(K_{d^2} \times K_4)$ in the grid as shown below. The vertices in each of the four copies of K_{d^2} are numbered. Each copy of K_4 is in its own $d \times d$ square. For the sake of clarity, there are no edges drawn. However, within each copy of K_4 all vertices are adjacent, and between the four copies the 1's vertices are adjacent, 2's vertices are adjacent, etc.



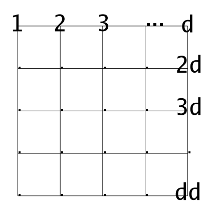
By Theorem 3.1, $\beta_2(K_{d^2}) = d-1$. Therefore, within each copy of K_{d^2} the bandwidth is d-1. Looking at the 1's vertices, all edges between the 1's vertices have size d. Likewise, all edges between the 2's vertices have size d. This pattern continues for all sets of adjacent vertices. Hence, the longest edge in the entire graph of $(K_{d^2} \times K_4)$ has size d. For this embedding the bandwidth is d. Overall, we now have an upper bound of d, which is also the lower bound. Therefore,

$$\beta_2(K_{d^2} \times K_4) = d.$$

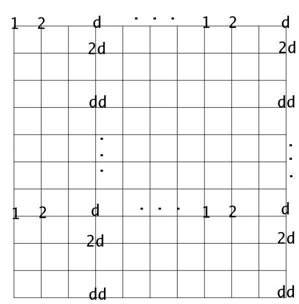
4.2 $\beta_2(K_{d^2} \times K_{c^2})$

Conjecture: *If* $2 \le c \le d$, *then* $\beta_2(K_{d^2} \times K_{c^2}) = d(c-1)$.

From Theorem 3.1 we know that $\beta_2(K_{d^2}) = d-1$. To acheive this bandwidth, K_{d^2} could be embedded into a d $\times$ d square grid as shown below.



Now, in the graph $(K_{d^2} \times K_{c^2})$ there are c^2 copies of K_{d^2}. These c^2 copies should be arranged in a c $\times$ c square as shown below.



If we look at the 1's vertices, the distance from the upper left 1 vertex and the upper right 1 vertex is d(c-1). Likewise, the distance from the upper left 1 vertex to the lower right 1 vertex is d(c-1). This pattern continues for each vertex 1 thru d. Therefore, in this embedding the bandwidth is d(c-1). Hence, this is an upper bound for $(K_{d^2} \times K_{c^2})$.

Upper Bound:

$$\beta_2(K_{d^2} \times K_{c^2}) \leq d(c-1)$$

The lower bound follows directly from this theorem [4], $\beta_2(G) \geq \left\lceil \frac{\sqrt{n}-1}{D(G)} \right\rceil$ where $D(G)$ is the diameter of the graph G.

Lower Bound:

$$\beta_2(K_{d^2} \times K_{c^2}) \geq \left\lceil \frac{dc-1}{2} \right\rceil$$

Therefore,

$$\left\lceil \frac{dc-1}{2} \right\rceil \leq \beta_2(K_{d^2} \times K_{c^2}) \leq d(c-1)$$

.

4.3 $\beta_2(K_{n,m})$

Theorem 5: Let $1 \leq n \leq m$.

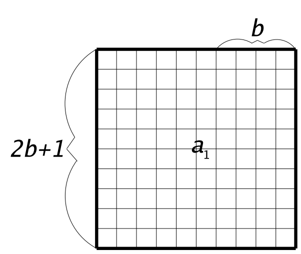
1. *If* $(d+1)^2 - d \leq n \leq (d+1)^2$ *for some* $d \epsilon \mathbf{N}$*, then*

$$\beta_2(K_{n,m}) = \left\lceil \frac{\sqrt{m+n} + \lceil \sqrt{n} - 2 \rceil}{2} \right\rceil$$

2. *Otherwise,*

$$\beta_2(K_{n,m}) = \left\lceil \frac{\sqrt{4(m+n)+1} + \lceil \sqrt{n} - 2 \rceil + \lfloor \sqrt{n} - 2 \rfloor}{4} \right\rceil$$

Proof: Let $\beta_2(K_{n,m}) = b$ such that $1 \leq n \leq m$.
Let $A = \{a_1, a_2, ..., a_n\}$ denote the set of n vertices, and let $B = \{b_1, b_2, ..., b_m\}$ denote the set of m vertices.
Let $d(r) = \{x \ \epsilon \ H \mid \partial_H(x, r) \leq b\}$ where H is a grid graph. $d(r)$ is simply a closed ball of radius b.
Given n and b, we want to maximize $|B|$ such that the bandwidth b is maintained.
Let's look at the example of $n = 1$, that is $A = \{a_1\}$. Since the bandwidth b is fixed there is a limit to the number of vertices in the set B.



The grid above has a total of $d(a_1) = (2n+1)^2$ grid points. There are $d(a_1) - 1$ grid points available for the vertices of set B to be embedded into such that the bandwidth b is maintained. 1 is subtracted from $d(a_1)$ because the vertex a_1 is already occupying one of the grid points.
The next example to look at is n=2. The set $A = \{a_1, a_2\}$. We want to embed a_1 and a_2 in such a way that maximizes $|B|$, which is equivalent to trying to maximize $|d(a_1) \cap d(a_2)|$.

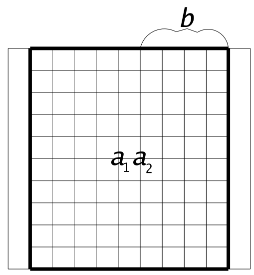
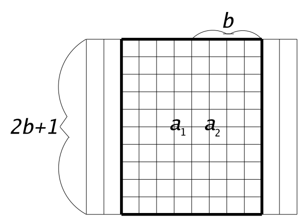
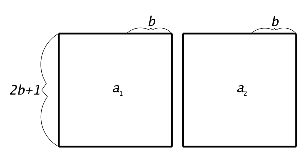


Figure 5.

The highlighted lines in the above figure represent $d(a_1) \cap d(a_2)$. As one can see, the vertices lost in the transition from $n = 1$ to $n = 2$ total $2b + 1$, one column. $|B| \leq |d(a_1) \cap d(a_2)| - 2$. In other words, $m \leq (2b)(2b+1) - 2$. Placing a_1 as close as possible to a_2 maximizes the number of vertices in the intersection of the two closed balls of radius b. For clarification, consider the non-optimal embedding below.

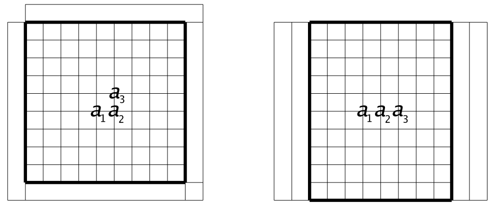


This embedding is non-optimal because $|d(a_1) \cap d(a_2)|$ is not maximized. In the above figure we lose two columns of $d(a_1)$ as opposed to losing just one column in Figure 5.
There is also the extreme non-optimal case of $|d(a_1) \cap d(a_2)| = \phi$. This example is below.



The closed ball of a_1 never intersects the closed ball of a_2. Thus, in this embedding there is no positive value for m that would maintain the bandwidth b. For example, consider $B = \{b_1\}$. Recall that in a compete bipatite graph, $K_{n,m}$, every vertex in the set of size n is adjacent to all the vertices in the set of size m. In this case, a_1 and a_2 are both adjacent to b_1. If b_1 is embedded inside the closed ball of a_1, then the distance from a_2 to b_1 would be greater than b. Therefore, the above arrangement is another example of an embedding of the vertices of a_1 and a_2 that does not maximize $|B|$.

Now, here is one further example of an embedding of A that does maximize $|B|$. Let's look at the case of $A = \{a_1, a_2, a_3\}$.



Optimal embedding Non-optimal

As seen above, $|d(a_1) \cap d(a_2) \cap d(a_3)|$ is maximized when a_3 is embedded above a_2 instead of embedding it to the right of a_2.

References

[1] J.X. Hao, Two-Dimensional Bandwidth of Graphs, *Ars Combinatoria*, 74(2005), pp.77-88.

[2] F.R.K Chung, Labelings of Graphs, in: L.W.Beineke and R.J. Wilson(ed), *Selected Topics in Graph Theory*, 3(1988), 151-168.

[3] S.N. Bhatt and F.T. Leighton, A framework for solving VLSI graph layout problems, *J.Computer and System Sciences*, 28(2)(1981), 300-313.

[4] Y.X. Lin, J.X. Hao and X.L. Li, Two dimensional bandwidth problem under Distance of L_∞ -Norm, *OR Transactions*, 4(3)(2000), 8-12.

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory With Applications*, London, Macmillan Press Ltd, 1976.

[6] Alan Tucker, *Applied Combinatorics*, Third Edition, John Wiley & Sons, Inc., 1995.

[7] Y.X. Lin, On Density Lower Bounds of Two Dimensional Bandwidth, *J.Mathematical Research & Exposition*, 16(3)(1996),343-349.

[8] Y.X. Lin, Two Dimensional Bandwidth Problem, in: Y.Alavi et al.(ed.), Combinatorics, Graph Theory, Algorithms and Applications, World Scientific, 1994. 223-232.