The Minimum Distance Energy for Polygonal Unknots

By:Johanna Tam
Advisor: Rolland Trapp

Abstract

This paper investigates the energy U_{MD} of polygonal unknots. It provides equations for finding the energy for any planar regular n-gon and for any m-gon, where the vertices lie on the vertices of a regular $\frac{m}{2}$-gon and on the midpoints of each edge. In addition, we show that a regular 4-gon minimizes the energy for any quadrilateral. Finally, this paper includes a proof showing that if we have a regular n-gon, R_n, inscribed in a circle and can move only one vertex of R_n, v, along the circle between its two adjacent vertices, then the U_{MD} is minimized when v is a vertex on R_n.

1 Introduction

The minimum distance energy for polygonal knots, U_{MD}, was introduced by Jonathon Simon in [2]. This energy analyzes the minimum distances between the edges of a knot. Furthermore, Rawdon and Simon in [1] relates this minimum distance energy to the Möbius energy for smooth knots. They show that $U_{MD}(P_n) - U_{MD}(R_n)$ approaches the Möbius energy of the smooth knot K as P_n approaches K. It is conjectured that the regular n-gon minimizes U_{MD} for all knots with n-sticks. If this conjecture holds true, then $U_{MD}(P_n) - U_{MD}(R_n)$ will never be negative, and according to Jonathon Simon [3], that limiting number is the same as the minimum Möbius energy for the knot type K. My goal was to prove this conjecture. This paper provides an introduction to some of the observations that can be made about the energy of the regular n-gon.

Now we will briefly cover some important definitions that are used throughout this paper. To start with the basics, a **knot** is a continuous closed curve/loop in space. It can be thought of as a knotted strand of string that is connected at the ends, where the string has no thickness. A **polygonal knot** is a knot equivalent to a polygon in $\mathbb{R}^3$. It is depicted using straight edges (sticks) and vertices. This is the type of knot that we will be focusing on. The **unknot** is a knot that has a diagram with no crossings and therefore has a crossing number of zero. One type of unknot is the **regular n-gon**, which is a polygon that has n edges of equal length and corresponding angles of equal degree which will be denoted as R_n throughout this paper.

Finally, the main topic of this paper is the **minimum distance energy**. When given two non-adjacent edges X, Y of a polygonal knot K, we define the minimum distance energy, U_{MD}, by

$$\mathbf{U_{MD}(X,Y)} = \frac{\mathbf{l(X)*l(Y)}}{\mathbf{MD(X,Y)^2}}.$$

such that $l(X), l(Y)$ are the lengths of X and Y consecutively, and $MD(X,Y)$ is the minimum distance between X and Y. Now we define the minimum distance energy of K by

$$\mathbf{U_{MD}(K)} = \sum \mathbf{U_{MD}(X,Y)}.$$

where the sum is taken over all pairs of non-adjacent edges, X and Y, of polygon K.

Note: The energy U_{MD} is scale invariant. This holds true because if we increase the knot by a constant c, then the lengths of all the edges and the distances between them would all increase by c. Therefore, we have

$$U_{MD}(X,Y) = \frac{c*l(X)*c*l(Y)}{(c*MD(X,Y))^2} = \frac{c^2*l(X)*l(Y)}{c^2*(MD(X,Y))^2} = \frac{l(X)*l(Y)}{MD(X,Y)^2}.$$

2 Energy Equations for a Regular n-gon

This section discusses two general equations that can be used to compute the U_{MD} of a regular n-gon and the methods we used to generate them. One equation is for when n is odd and the other is for when n is even. The U_{MD} of any regular n-gon can be obtained easily by just plugging the equation into MAPLE and changing n accordingly.

For the first step, we decided to view the regular n-gon as inscribed inside a unit circle (see Figure 1) since U_{MD} is scale invariant. Every vertex of the n-gon lies on the following circle.

Figure 1 Unit Circle

Thus vertex $v_j = (cos(\frac{2*\pi*j}{n}), sin(\frac{2*\pi*j}{n}))$. For example, $v_1 = (cos(\frac{2*\pi}{n}), sin(\frac{2*\pi}{n}))$.

We then found the length of an edge, l_e, by finding the distance between two consecutive vertices of the polygon. Since the length of each edge is equal by definition, we can make it simple by just choosing to find the distance between v_0 and v_1, where $v_0 = (1,0)$. We have $l_e = \sqrt{(cos(\frac{2*\pi}{n}) - 1)^2 + (sin(\frac{2*\pi}{n}))^2}$, which can be simplified to

$$l_e = \sqrt{2*(1 - cos(\frac{2*\pi}{n}))}$$

thus, for the regular n-gon

$$l(X) * l(Y) = 2 * (1 - cos(\frac{2 * \pi}{n}))$$

where X and Y are non-adjacent edges.

Next we investigated the minimum distances between the non-adjacent edges. Due to the symmetry of a regular n-gon, the U_{MD} for e_1 and its non-adjacent edges is the same as the U_{MD} for any other edge and its non-adjacent edges so we can multiply $\sum_{j=3}^{n-1} U_{MD}(e_1, e_j)$ by n. However, note that the U_{MD} of each pair of non-adjacent edges have been counted twice, and hence we must divide by 2. Therefore, the $U_{MD}(K) = \frac{n}{2} * \sum_{j=3}^{n-2} U_{MD}(e_1, e_j)$ where K is the unknot with n edges. The minimum distance between two edges is the distance between the 'head' of one edge and the 'tail' of the other where the 'head' is farther counterclockwise than the 'tail'.

2.1 The Minimum Distance Energy of R_n When n is Odd

We will now discuss how to find the equation for the $U_{MD}(R_n, n = odd)$. Note that each edge of a n-gon where n is odd, has an even number of non-adjacent edges. We can utilize this fact and the symmetry of R_n to make finding the equation more feasible. Throughout this subsection we can refer to the regular 5-gon in figure 2 as an example.

Figure 2

We noticed that $\sum_{j=3}^{\lceil \frac{n}{2} \rceil} U_{MD}(e_1, e_j) = \sum_{j=\lceil \frac{n}{2} \rceil + 1}^{n-2} U_{MD}(e_1, e_j)$ therefore we can just calculate $\sum_{j=3}^{\lceil \frac{n}{2} \rceil} U_{MD}(e_1, e_j)$ and multiply by two. So the first step is to find a formula for the $U_{MD}(e_1, e_j)$ where e_j is non-adjacent to e_1. We know that $l(e_1) * l(e_j) = 2 * (1 - cos(\frac{2*\pi}{n}))$ from earlier. Next we nee to find the minimum distance between the 'head' of e_1 and the 'tail' of e_j. Using trigonometry and geometry, we find that the 'head' of e_1 is at $(cos(\frac{2*\pi}{n}), sin(\frac{2*\pi}{n}))$ and that the 'tail' of e_j is at $(cos(\frac{2*\pi*(j-1)}{n}), sin(\frac{2*\pi*(j-1)}{n}))$. Thus the $MD(e_1, e_j) = \sqrt{(cos(\frac{2*\pi*i}{n}) - cos(\frac{2*\pi}{n}))^2 + (sin(\frac{2*\pi*i}{n}) - sin(\frac{2*\pi}{n}))^2}$, where $i = (j-1)$. When we put it all together, the $U_{MD}(R_n, n = odd) = n * \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \frac{2*(1-cos(\frac{2*\pi}{n}))}{(cos(\frac{2*\pi*i}{n}) - cos(\frac{2*\pi}{n}))^2 + (sin(\frac{2*\pi*i}{n}) - sin(\frac{2*\pi}{n}))^2}$. Then using some trigonometric identities, we can simplify the equation to the following.

Proposition 1.1-The minimum distance energy for R_n when n is odd is

$U_{MD}(R_n) =$

$$n * \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \frac{(sin(\frac{\pi}{n}))^2}{(sin(\frac{\pi*(i-1)}{n}))^2}$$

2.2 The Minimum Distance Energy of R_n When n is Even

Throughout this paper, finding a formula for the minimum distance energy is more complicated when the knot has an even number of sticks than when there is an odd number of sticks. In this subsection we can refer to the following 8-gon as an example, where the bolded edge is e_1 and the dotted lines represent the minimum distance between two edges (see Figure 3).

Figure 3

We then can follow the same procedures for when n is even for R_n with one addition. Unlike when n is odd, a n-gon with an even number of edges will have an odd number of non-adjacent edges for each edge as seen in the 8-gon above. However, notice that with the exception of the edge opposite of e_1, it follows the same pattern/symmetry as R_n when n is odd. Due to this difference, the U_{MD} of e_1 and all its non-adjacent edges minus the farthest edge is the same as when n is odd with different limits. So to complete the general equation for when n is even, we need to take the product of the lengths of e_1 and $e_{(\frac{n}{2}+1)}$ and divide it by their minimum distance squared. After that, we must multiply the resulting equation by n and divide by 2 to prevent double counting. When we put it all together the $U_{MD}(R_n, n = even) = n * \sum_{i=2}^{\frac{n}{2}-1} \frac{2*(1-cos(\frac{2*\pi}{n}))}{(cos(\frac{2*\pi*i}{n})-cos(\frac{2*\pi}{n}))^2+(sin(\frac{2*\pi*i}{n})-sin(\frac{2*\pi}{n}))^2} + \frac{n*(1-cos(\frac{2*\pi}{n}))}{2*(1+cos(\frac{2*\pi}{n}))}$. However, like the previous equation, we can simplify the equation using trigonometric identities to the following.

Proposition 1.2-The minimum distance energy for R_n when n is even is

$U_{MD}(n - gon) =$

$$n * \sum_{i=2}^{\frac{n}{2}-1} \frac{(sin(\frac{\pi}{n}))^2}{(sin(\frac{\pi*(i-1)}{n}))^2} + \frac{n}{2} * (tan(\frac{\pi}{n}))^2$$

3 Energy Equations for a Flattened n-gon

In this section, we discuss how to find the U_{MD} for a specific irregular n-gon. We let $K_{\frac{n}{2}}$ be a planar regular $\frac{n}{2}$-gon, and let L be the planar n-gon obtained by taking $K_{\frac{n}{2}}$ and adding a vertex on the midpoint of each edge. In other words, the flattened n-gon is when you take every other outer angle of a regular n-gon and flatten it to 180°. The equations for $U_{MD}(L)$ were found by following the same steps used to find the U_{MD} for R_n. Let e_1's 'head' be located on a midpoint of a edge in $K_{\frac{n}{2}}$. The minimum distance is still from the 'head' of one edge to the 'tail' of the other. Once again we chose to focus on the U_{MD} of one edge and its corresponding non-adjacent edges. Then we multiplied that quantity by $\frac{n}{2}$. The length of an edge in L is equivalent to the length of an edge in $R_{\frac{n}{2}}$ divided by two. We have $l_e = \frac{1}{2} * \sqrt{(cos(\frac{2*\pi}{\frac{n}{2}}) - 1)^2 + (sin(\frac{2*\pi}{\frac{n}{2}}))^2}$, which can be simplified to

$$l_e = \frac{1}{2} * \sqrt{2 * (1 - cos(\frac{4*\pi}{n}))}$$

therefore, the product of the lengths for two non-adjacent edges, X and Y, for L is given by

$$l(X) * l(Y) = \frac{1}{2} * (1 - cos(\frac{4*\pi}{n}))$$

As in the previous section, the $U_{MD}(L)$ requires two equations, one for when $\frac{n}{2} = even$ and one for when $\frac{n}{2} = odd$.

3.1 $U_{MD}(L, \frac{n}{2} = odd)$

This subsection focuses on L when $\frac{n}{2}$ is odd. However, the n-gon still has an even number of edges and thus L has an odd number of non-adjacent edges for each edge. Once again we can rely on the symmetry of the n-gon to simplify the problem.

Figure 4

Before we compute the equation, make note that the 'head' of e_1 is closest to the 'tails' of e_j when $3 \leq j \leq (\frac{n}{2} + 1)$. Also note that the 'tail' of e_1 is the closest to the 'heads' of e_j when $(\frac{n}{2} + 2) \leq j \leq (n - 1)$. We can now find $U_{MD}(L, n/2 = odd)$ by breaking up the problem into four sections by looking at the four possible cases of non-adjacent edge pairs.

First we find the U_{MD} of e_1 and e_j when $3 \leq j \leq (\frac{n}{2} + 1)$ where the 'tail' of e_j is located on a vertex of the regular $\frac{n}{2}$-gon, which equals
$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{(1-cos(\frac{4*\pi}{n}))}{2*((\frac{cos(\frac{4*\pi}{n})+1}{2} - cos(\frac{4*\pi*i}{n}))^2 + (\frac{sin(\frac{4*\pi}{n})}{2} - sin(\frac{4*\pi*i}{n}))^2)}$.

Secondly, we find the U_{MD} of e_1 and e_j when $3 \leq j \leq (\frac{n}{2} + 1)$ where the 'tail' of e_j is located on a midpoint of a edge in $K_{\frac{n}{2}}$, which equals
$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{(1-cos(\frac{4*\pi}{n}))}{2*((\frac{cos(\frac{4*\pi}{n})+1}{2} - (\frac{cos(\frac{4*\pi*i}{n})+cos(\frac{4*\pi*(i+1)}{n})}{2}))^2 + (\frac{sin(\frac{4*\pi}{n})}{2} - (\frac{sin(\frac{4*\pi*i}{n})+sin(\frac{4*\pi*(i+1)}{n})}{2}))^2)}$.

Next we find the U_{MD} of e_1 and e_j when $(\frac{n}{2} + 2) \leq j \leq (n - 1)$ where the ‘head’ of e_j is located on a vertex from the regular $\frac{n}{2}$-gon, which equals $\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor -1} \frac{(1-cos(\frac{4*\pi}{n}))}{2*((cos(\frac{4*\pi}{n})-cos(\frac{4*\pi*(i+1)}{n}))^2+(sin(\frac{4*\pi}{n})-sin(\frac{4*\pi*(i+1)}{n}))^2)}$.

Lastly, we find the U_{MD} of e_1 and e_j when $(\frac{n}{2}+2) \leq j \leq (n-1)$ where the ‘head’ of e_j is located on a midpoint of a edge in $K_{\frac{n}{2}}$, which equals $\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{(1-cos(\frac{4*\pi}{n}))}{2*((cos(\frac{4*\pi}{n})-(\frac{cos(\frac{4*\pi*i}{n})+cos(\frac{4*\pi*(i+1)}{n})}{2}))^2+(sin(\frac{4*\pi}{n})-(\frac{sin(\frac{4*\pi*i}{n})+sin(\frac{4*\pi*(i+1)}{n})}{2}))^2)}$.

Now after all these computations, we only need to add up these summations, simplify and multiply the result by $\frac{n}{2}$. Therefore, we have

$$U_{MD}(L, \tfrac{n}{2} = odd) =$$

$$\frac{n}{2} * (\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{(sin(\frac{2*\pi}{n}))^2}{((\frac{cos(\frac{4*\pi}{n})+1}{2} - cos(\frac{4*\pi*i}{n}))^2 + (\frac{sin(\frac{4*\pi}{n})}{2} - sin(\frac{4*\pi*i}{n}))^2)} +$$
$$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{(sin(\frac{2*\pi}{n}))^2}{((\frac{cos(\frac{4*\pi}{n})+1}{2} - (\frac{cos(\frac{4*\pi*i}{n})+cos(\frac{4*\pi*(i+1)}{n})}{2}))^2 + (\frac{sin(\frac{4*\pi}{n})}{2} - (\frac{sin(\frac{4*\pi*i}{n})+sin(\frac{4*\pi*(i+1)}{n})}{2}))^2)} +$$
$$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor -1} \frac{(sin(\frac{2*\pi}{n}))^2}{((cos(\frac{4*\pi}{n}) - cos(\frac{4*\pi*(i+1)}{n}))^2 + (sin(\frac{4*\pi}{n}) - sin(\frac{4*\pi*(i+1)}{n}))^2)} +$$
$$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{(sin(\frac{2*\pi}{n}))^2}{((cos(\frac{4*\pi}{n}) - (\frac{cos(\frac{4*\pi*i}{n})+cos(\frac{4*\pi*(i+1)}{n})}{2}))^2 + (sin(\frac{4*\pi}{n}) - (\frac{sin(\frac{4*\pi*i}{n})+sin(\frac{4*\pi*(i+1)}{n})}{2}))^2)})$$

3.2 $U_{MD}(L, \frac{n}{2} = even)$

In this subsection we will show how we find a formula for the $U_{MD}(L, \frac{n}{2} = even)$. We follow the same procedures as when finding the $U_{MD}(R_n)$, where the equation for the case $\frac{n}{2}$ even builds from the equation used for the when $\frac{n}{2}$ odd case. Use the 16-gon below as an example for this subsection (see Figure 5). Once again the dotted lines represent the minimum distance between two non-adjacent edges.

Figure 5

For finding the equation for $U_{MD}(L, \frac{n}{2} = even)$, we can use the same method used above when $\frac{n}{2}$ is odd. Actually we can use the exact same summands with different bounds to represent the same cases of non-adjacent edge pairs of the unknot. However, when n is even, we have an odd number of non-adjacent edges for each edge. Similar to computing the $U_{MD}(R_n, n = even)$, the $U_{MD}(e_1, e_{(\frac{n}{2}+1)})$ requires an extra term added on to the formula for when $\frac{n}{2}$ is odd. The product of these two edges is the same as the rest of the non-adjacent pairs. The minimum distance is found by computing the minimum distance between the ‘head’ of e_1 and the ‘head’ of $e_{(\frac{n}{2}+1)}$, which is equal to $\frac{(sin(\frac{2*\pi}{n}))^2}{((cos(\frac{4*\pi}{n})+1)^2+(sin(\frac{4*\pi}{n}))^2}$. Therefore if we add this quantity to the summations found for $U_{MD}(L, \frac{n}{2} = odd)$ and change the limits appropriately, we end up with

$$U_{MD}(L, \tfrac{n}{2} = even) =$$

$$\frac{n}{2} * (\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor -1} \frac{(1-cos(\frac{4*\pi}{n}))}{2*((\frac{cos(\frac{4*\pi}{n})+1}{2}-cos(\frac{4*\pi*i}{n}))^2+(\frac{sin(\frac{4*\pi}{n})}{2}-sin(\frac{4*\pi*i}{n}))^2)} +$$

$$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor -1} \frac{(sin(\frac{2*\pi}{n}))^2}{((\frac{cos(\frac{4*\pi}{n})+1}{2}-(\frac{cos(\frac{4*\pi*i}{n})+cos(\frac{4*\pi*(i+1)}{n})}{2}))^2+(\frac{sin(\frac{4*\pi}{n})}{2}-(\frac{sin(\frac{4*\pi*i}{n})+sin(\frac{4*\pi*(i+1)}{n})}{2}))^2)} +$$

$$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor -1} \frac{(sin(\frac{2*\pi}{n}))^2}{((cos(\frac{4*\pi}{n})-cos(\frac{4*\pi*(i+1)}{n}))^2+(sin(\frac{4*\pi}{n})-sin(\frac{4*\pi*(i+1)}{n}))^2)} +$$

$$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor -1} \frac{(sin(\frac{2*\pi}{n}))^2}{((cos(\frac{4*\pi}{n})-(\frac{cos(\frac{4*\pi*i}{n})+cos(\frac{4*\pi*(i+1)}{n})}{2}))^2+(sin(\frac{4*\pi}{n})-(\frac{sin(\frac{4*\pi*i}{n})+sin(\frac{4*\pi*(i+1)}{n})}{2}))^2)} +$$

$$\frac{(sin(\frac{2*\pi}{n}))^2}{(cos(\frac{4*\pi}{n})+1)^2+(sin(\frac{4*\pi}{n}))^2})$$

4 Regular 4-gon Minimizes U_{MD} for all Quadrilaterals

This section provides a proof that the regular 4-gon minimizes the U_{MD} for all quadrilaterals, which includes irregular, not convex, and non-planar 4-gons. Before we begin the proof we must calculate the actual value of $U_{MD}(R_4)$. Let the edge lengths of R_4 equal a,b,c,d where the edges with lengths a and b are non-adjacent. WLOG, $U_{MD}(R_4)$ is given by

$$\frac{a*b}{d^2} + \frac{c*d}{a^2}.$$

Since we know that for a regular 4-gon, a=b=c=d,the above quantity can be written as

$$\frac{a*a}{a^2} + \frac{a*a}{a^2} = \frac{a^2}{a^2} + \frac{a^2}{a^2} = 1 + 1 = \mathbf{2}.$$

Theorem 1: *The regular 4-gon minimizes the minimal distance energy U_{MD} for all knots with four edges.*

Figure 6

Proof. Let F be any unknot with 4 edges, e_a, e_b, e_c, e_d, with edge lengths a, b, c, d respectively, where e_a and e_b are non-adjacent and $a < b$ and $d < c$.

Note that for each arbitrary 4-gon above the largest possible $MD(e_a, e_b)$ equals d, and the largest possible $MD(e_c, e_d)$ equals a. Since this maximizes the denominator, it minimizes the $U_{MD}(F)$. Then we have

$$U_{MD}(F) = \frac{a*b}{(MD(e_a,e_b))^2} + \frac{c*d}{(MD(e_c,e_d))^2} \geq \frac{a*b}{d^2} + \frac{c*d}{a^2} > \frac{a^2}{d^2} + \frac{d^2}{a^2} = \frac{a^4+d^4}{a^2*d^2}.$$

We know that $x^2+y^2 \geq 2*x*y$. Let $x = a^2$ and $y = d^2$. Thus $a^4+d^4 \geq 2*a^2*d^2$. It follows that

$$\frac{a^4+d^4}{a^2*d^2} \geq 2.$$

Therefore the $U_{MD}(4-gon)$ is minimized by a regular 4-gon since the $U_{MD}(R_4) = 2$. □

5 Movement of One Vertex Along the Circumference of a Circle

If we have a regular n-gon, R_n, inscribed in a circle, all of its vertices will lie on the circumference of the circle. Now if we are allowed to move only one of these vertices, v, along the circle between its two adjacent vertices, how does this affect the U_{MD} of the knot? In this section we will show that if v moves from its original position on R_n, it will only increase the minimum distance energy of the knot.

To start the investigation, let this altered R_n be denoted as A_n where one of its vertices have the coordinates $(1, 0)$, and let v be adjacent and located at $(cos(\theta), sin(\theta))$, where $0 < \theta < \frac{4*\pi}{n}$.

Before computing the minimum distance energy, notice that only two of the edges, e_1 and e_2, from R_n will be altered. Therefore, U_{MD} of those two edges and their corresponding non-adjacent edges will change, while the contributions to U_{MD} of the remaining edge pairs will remain the same. Let e_1 have the endpoints $(1, 0)$ and $(cos(\theta), sin(\theta))$, and e_2 have the endpoints $(cos(\theta), sin(\theta))$ and $(cos(\frac{4*\pi}{n}), sin(\frac{4*\pi}{n}))$. Therefore we have $l(e_1) = 2 * sin(\frac{\theta}{2})$ and $l(e_2) = 2 * sin(\frac{\frac{4*\pi}{n}-\theta}{2})$.

5.1 $U_{MD}(A_n, n = odd)$

To find the $U_{MD}(e_1)$ we need to find $U_{MD}(e_1, e_i)$, where e_i is non-adjacent to e_1, there are two cases to consider: when $(1, 0)$ is the point closest to the non-adjacent edge and when $(cos(\theta), sin(\theta))$ is the closest. The energy for when $(1, 0)$ is the point nearest to e_i equals $\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} 2 * \frac{sin(\frac{\pi}{n})*sin(\frac{\theta}{2})}{sin(\frac{\pi*(i-1)}{n})^2}$. The energy for when v is the closest to e_i is equal to $\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} 2 * \frac{sin(\frac{\pi}{n})*sin(\frac{\theta}{2})}{sin(\frac{\theta}{2} - \frac{\pi*i}{n})^2}$. The sum used for computing the $U_{MD}(e_2)$ is similar to the $U_{MD}(e_1)$, but instead of $2 * sin(\frac{\pi}{n}) * sin(\frac{\theta}{2})$, the numerator changes to $2 * sin(\frac{\pi}{n}) * sin(\frac{\frac{4*\pi}{n} - \theta}{2})$. Therefore, the $U_{MD}(e_2)$ equals to $\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} 2 * \frac{(sin(\frac{\pi}{n}))*sin(\frac{\frac{4*\pi}{n}-\theta}{2})}{sin(\frac{\pi*(i-1)}{n}))^2} + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} 2 * \frac{(sin(\frac{\pi}{n}))*sin(\frac{\frac{4*\pi}{n}-\theta}{2})}{sin(\frac{\theta}{2} - \frac{\pi*i}{n}))^2}$. The remaining edge pairs have the same U_{MD} as if the n-gon were regular, and therefore equal $\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor}(n-4) * \frac{(sin(\frac{\pi}{n}))^2}{(sin(\frac{\pi*(i-1)}{n}))^2}$. We can now add up these summations to get the minimal distance energy of the knot. When simplified,

$U_{MD}(A_n, n = odd) =$

$$\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor}(2 * \frac{sin(\frac{\pi}{n})*(sin(\frac{\theta}{2})+sin(\frac{\frac{4*\pi}{n}-\theta}{2}))}{(sin(\frac{\pi*(i-1)}{n}))^2} + 2 * \frac{sin(\frac{\pi}{n})*(sin(\frac{\theta}{2})+sin(\frac{\frac{4*\pi}{n}-\theta}{2}))}{(sin(\frac{\theta}{2}-\frac{\pi*i}{n}))^2} + (n-4) * \frac{(sin(\frac{\pi}{n}))^2}{(sin(\frac{\pi*(i-1)}{n}))^2}).$$

Figure 7

5.2 $U_{MD}(A_n, n = even)$

Once again when finding the equation for $U_{MD}(n-gon, n = even)$, we can employ the same method used above when n is odd, which will give us the same summands, but with different limits and some extra terms. The summands from $U_{MD}(A_n, n = odd)$ do not include the $U_{MD}(e_i, e_{(i+\frac{n}{2})})$ where e_i is any edge in the n-gon. When e_i and $e_{(i+\frac{n}{2})}$ are not e_1 or e_2, then we know from the $U_{MD}(R_n)$ that the $U_{MD}(e_i, e_{(i+\frac{n}{2})}) = (\frac{n}{2} - 2) * (tan(\frac{\pi}{n}))^2$. We need to find the $U_{MD}(e_1, e_{(\frac{n}{2}+1)})$ and the $U_{MD}(e_2, e_{(\frac{n}{2}+2)})$. The $MD(e_1, e_{(\frac{n}{2}+1)})$ is the distance between $(cos(\theta), sin(\theta))$ and $(-1, 0)$. The $MD(e_2, e_{(\frac{n}{2}+2)})$ is the distance between $(cos(\frac{4*\pi}{n}), sin(\frac{4*\pi}{n}))$ and $(cos(\frac{\pi*(n+2)}{n}), sin(\frac{\pi*(n+2)}{n}))$. Therefore the $U_{MD}(e_1, e_{(\frac{n}{2}+1)}) = \frac{(sin(\frac{\pi}{n}))*sin(\frac{\theta}{2})}{(cos(\frac{\theta}{2}))^2}$ and the $U_{MD}(e_2, e_{\frac{n}{2}+2}) = \frac{(sin(\frac{\pi}{n}))*sin(\frac{\frac{4*\pi}{n}-\theta}{2})}{(sin(\frac{\pi*(n-2)}{2*n}))^2}$. We can now add up these equations and simplify to get the following equation.

$U_{MD}(A_n, n = even) =$

$$\sum_{i=2}^{\frac{n}{2}-1}(2 * \frac{(sin(\frac{\pi}{n}))*(sin(\frac{\theta}{2})+sin(\frac{\frac{4*\pi}{n}-\theta}{2}))}{(sin(\frac{\pi*(i-1)}{n}))^2} + 2 * \frac{(sin(\frac{\pi}{n}))*(sin(\frac{\theta}{2})+sin(\frac{\frac{4*\pi}{n}-\theta}{2}))}{(sin(\frac{\theta}{2}-\frac{\pi*i}{n}))^2} + (n-4) * \frac{(sin(\frac{\pi}{n}))^2}{(sin(\frac{\pi*(i-1)}{n}))^2}) + (\frac{n}{2}-2) * (tan(\frac{\pi}{n}))^2 + \frac{(sin(\frac{\pi}{n}))*sin(\frac{\theta}{2})}{(cos(\frac{\theta}{2}))^2} + \frac{(sin(\frac{\pi}{n}))*sin(\frac{\frac{4*\pi}{n}-\theta}{2})}{(sin(\frac{\pi*(n-2)}{2*n}))^2}.$$

Figure 8

5.3 $U_{MD}(A_n)$ is Minimized When $\theta = \frac{2*\pi}{n}$

To analyze the $U_{MD}(A_n)$ further, we will look at the rate of change of θ. The following proof will use calculus to show that $U_{MD}(A_n)$ is minimized when $\theta = \frac{2*\pi}{n}$. However, before the proof we need to find the derivative of $U_{MD}(A_n)$.

$$\frac{d}{d\theta}(U_{MD}(A_n, n = odd)(\theta)) = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor}(2 * \frac{(sin(\frac{\pi}{n}))*(\frac{1}{2}*cos(\frac{\theta}{2})-\frac{1}{2}*cos(-\frac{2*\pi}{n}+\frac{\theta}{2}))}{(sin(\frac{\pi*(i-1)}{n}))^2} + 2 * \frac{(sin(\frac{\pi}{n}))*(\frac{1}{2}*cos(\frac{\theta}{2})-\frac{1}{2}*cos(-\frac{2*\pi}{n}+\frac{\theta}{2}))}{(sin(\frac{\theta}{2}-\frac{\pi*i}{n}))^2} - 2 * \frac{(sin(\frac{\pi}{n}))*(sin(\frac{\theta}{2})-sin(-\frac{2*\pi}{n}+\frac{\theta}{2}))*cos(\frac{\theta}{2}-\frac{\pi*i}{n})}{(sin(\frac{\theta}{2}-\frac{\pi*i}{n}))^3})$$

$$\frac{d}{d\theta}(U_{MD}(A_n, n = even)(\theta) = \sum_{i=2}^{\frac{n}{2}-1}(2 * \frac{(sin(\frac{\pi}{n}))*(\frac{1}{2}*cos(\frac{\theta}{2})-\frac{1}{2}*cos(-\frac{2*\pi}{n}+\frac{\theta}{2}))}{(sin(\frac{\pi*(i-1)}{n}))^2} + 2 * \frac{(sin(\frac{\pi}{n}))*(\frac{1}{2}*cos(\frac{\theta}{2})-\frac{1}{2}*cos(-\frac{2*\pi}{n}+\frac{\theta}{2}))}{(sin(\frac{\theta}{2}-\frac{\pi*i}{n}))^2} - 2 * \frac{(sin(\frac{\pi}{n}))*(sin(\frac{\theta}{2})-sin(-\frac{2*\pi}{n}+\frac{\theta}{2}))*cos(\frac{\theta}{2}-\frac{\pi*i}{n})}{(sin(\frac{\theta}{2}-\frac{\pi*i}{n}))^3}) + \frac{sin(\frac{\pi}{n})}{2*cos(\frac{\theta}{2})} + \frac{sin(\frac{\pi}{n})*(sin(\frac{\theta}{2}))^2}{(cos(\frac{\theta}{2}))^3} - \frac{sin(\frac{\pi}{n})*cos(-\frac{2*\pi}{n}+\frac{\theta}{2})}{2*(sin(\frac{\pi*(n-2)}{2*n}))^2}$$

Figure 9

Theorem 2: *The $U_{MD}(A_n)$ is minimized when $A_n = R_n$.*

Proof. Given $U_{MD}(A_n)$ and $\frac{d}{d\theta}(U_{MD}(A_n))$, we need to show that the function $U_{MD}(A_n)(\theta)$ increases if θ starts at $\frac{2*\pi}{n}$ and decreases towards 0 or increases towards $\frac{4*\pi}{n}$. Due to symmetry (see Figure 9), we can analyze the one-sided derivative of $U_{MD}(A_n)$ where $\theta \geq \frac{2*\pi}{n}$. We want to show that $\frac{d}{d\theta}(U_{MD}(A_n))(\frac{2*\pi}{n}) > 0$. Using trigonometric identities we are able to simplify $\frac{d}{d\theta}(U_{MD}(A_n))(\frac{2*\pi}{n})$ where

$$\frac{d}{d\theta}(U_{MD}(A_n, n = odd)(\frac{2*\pi}{n})) = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \frac{4*(sin(\frac{\pi}{n}))^2*cot(\frac{(i-1)*\pi}{n})}{(sin(\frac{(i-1)*\pi}{n}))^2}$$

$$\frac{d}{d\theta}(U_{MD}(A_n, n = even)(\frac{2*\pi}{n})) = \sum_{i=2}^{\frac{n}{2}-1} \frac{4*(sin(\frac{\pi}{n}))^2*cot(\frac{(i-1)*\pi}{n})}{(sin(\frac{(i-1)*\pi}{n}))^2}) + (tan(\frac{\pi}{n}))^3$$

Since we proved that the regular 4-gon minimizes the energy U_{MD} for all 4-gons, we can let $n \geq 5$. When we plug in the limits, 2 and $\lfloor \frac{n}{2} \rfloor$, we find that $\frac{\pi}{n} \leq \frac{(i-1)*\pi}{n} < \frac{(n-2)*\pi}{2*n}$. Thus, $0 < \frac{(i-1)*\pi}{n} < \frac{\pi}{2}$. Hence, $\frac{4*(sin(\frac{\pi}{n}))^2*cot(\frac{(i-1)*\pi}{n})}{(sin(\frac{(i-1)*\pi}{n}))^2} > 0$ and $(tan(\frac{\pi}{n}))^3 > 0$ since we know that $cot(\theta) > 0$ and $tan(\theta) > 0$ when $0 < \theta < \frac{\pi}{2}$. So it follows that $U_{MD}(A_n)$ is increasing as θ is increasing from $\frac{2*\pi}{n}$. Therefore, $U_{MD}(A_n)$ is minimized when $\theta = \frac{2*\pi}{n}$. □

Remark: Since $\frac{d}{d\theta}(U_{MD}(A_n))(\frac{2*\pi}{n}) \neq 0$, we know that $U_{MD}(A_n)$ is not differentiable at $\theta = \frac{2*\pi}{n}$.

6 Conclusion

In conclusion, we have found some equations for computing the minimum distance energy for the regular n-gon and other types of polygonal unknots. In addition, we proved that the regular 4-gon minimizes U_{MD} for all unknots with 4 sticks. We also proved that if we have a regular n-gon, R_n, inscribed in a circle and can move only one vertex of R_n, v, along the circle between its two adjacent vertices, the U_{MD} of that knot is minimized when the knot is a regular n-gon. Overall, we employed different methods to analyze the U_{MD} of the unknot. We focused on the minimum distance between edges in section 2 and section 3. Then in section 4 we focused on edge lengths. Finally, for section 5 we focused on the angles between the endpoints.

The minimum distance energy of a regular n-gon has yet be proven to minimize the U_{MD} for all unknots with n sticks. There are definitely different ways to investigate this conjecture. One direction to go is to find equations for the regular n-gon and other polygonal unknots that can be compared effectively. This can maybe be done by investigating whether the unknot is easier to analyze when we are focusing on the edge lengths, minimum distances, angles, etc. Since this paper discusses mainly planar, non-convex n-gons, maybe we can analyze convex or non-planar n-gons.

If the regular n-gon is proven to minimize the U_{MD} for all n-gons, then maybe U_{MD} of other knot types with higher crossing numbers can be minimized by equalizing the edges and expanding the angles between the edges evenly. Overall, it will give us a better understanding of the minimum distance energy U_{MD} and its properties.

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