Investigation of 4-linear cutwidth critical tree graphs and complete cyclic cutwidth critical graphs

Ruffin Swain
with
Dr. Joseph Chavez and Dr. Rolland Trapp

August 24, 2006

Abstract

This paper was motivated by a thesis written by Dolores Chavez in 2006, *Investigation of 4-Cutwidth Critical Graphs*. In her thesis she provided a dictionary of 4-cutwidth critical graphs. In this paper we add three more trees to her list of 4-cutwidth critical tree graphs in an attempt to prove that this list is the complete set of 4-cutwidth critical tree graphs. We were also able to characterize the set of complete graphs that are cyclic cutwidth critical.

Chapter 1

Introduction

1.1 Introduction

A graph may be thought of as representing an electrical circuit, with vertices representing components and the edges representing wires that connect the components. Many problems in graph theory require that a complicated graph or circuit be embedded in a simpler host graph (a graph that is easier to analyze), while optimizing some important parameter. The cutwidth problem involves minimizing the highest number of edges (wires) running along any point.

The cutwidth of a graph is the minimum of the cutwidths over all possible embeddings into the host graph. When the host graph lies on a line the cutwidth is called linear cutwidth (lcw), and when the host graph lies on a circumference the cutwidth is called the cyclic cutwidth (ccw).

In this paper we will be investigating 4-cutwidth critical tree graphs and complete graphs that are cyclic cutwidth critical. This paper will be restricted to simple graphs (graphs that do not have loops, or more than one edge connecting two vertices).

Chapter 2

Further investigation 4-cutwidth critical tree graphs

2.1 Preliminaries

The following definitions and propositions will be used throughout this paper.

Definition 1 [1]A graph is a tree if it is a connected graph with no circuits, where a circuit is a simple closed path.

Definition 2 [3]A tree of diameter 4 has three levels the top in the root $\{v_0\}$, the intermediate level $\{v_{1,}v_{2,}..., v_m\}$, and the bottom level $\{v_{ij}(1 \leq i \leq m),(1 \leq j \leq n_1 - 1)\}$. (See Figure 2.1)

Definition 3 [3] A caterpillar is a tree which yields a path (the spine) when all its pendant vertices(vertices of degree one) are removed.

Definition 4[1]A labeling of a graph, $G = (V, E)$ with $|V| = n$, is a bijection

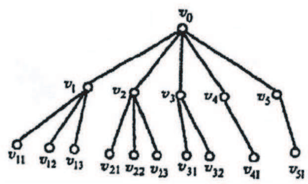

Figure 2.1: A tree of diameter 4

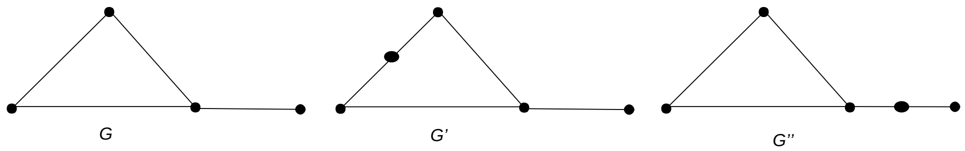

Figure 2.2: G and two subdivisions

$f : V \to \{1, \ldots, n\}$, which can be regarded as an embedding of G onto a path.

Definition 5[1] For a given labeling f of G, the cutwidth of G with respect to f is

$$c(G, f) = {}^{max}_{1 \leq i < n} \left| uv \in E : f(u) \leq i < f(v) \right|,$$

which represents the congestion of the linear embedding.

Definition 6[1] The cutwidth of G, c(G), is defined by

$$c(G) = {}^{min}_{f} c(G, f),$$

where the minimum is taken over all labelings of f.

Definition 7 By inserting new vertices of degree 2 onto G, the new graph G' is a subdivision of G. Figure 2.2 is an example of a subdivision.

Definition 8 Two or more graphs are homeomorphic if they are subdivisions of the same graph.

Definition 9 [1] A graph G is said to be k-cutwidth critical if:

1. c(G)=k;

2. for every proper subgraph G' of G, c(G')$< k$;

3. G is homemorphically minimal, that is, G is not a subdivision of any simple graph.

Proposition 1 [1] For any caterpillar T, $c(T) = \lceil \Delta(T)/2 \rceil$.

Proposition 2 [2] (1) If G' is a subgraph of G, then $c(G') \leq$ c(G).

(2) If G' is homeomorphic to G (i.e., they can both be obtained from the same graph by inserting new vertices of degree two into its edges, called a subdivision of the graph), then $c(G') = c(G)$.

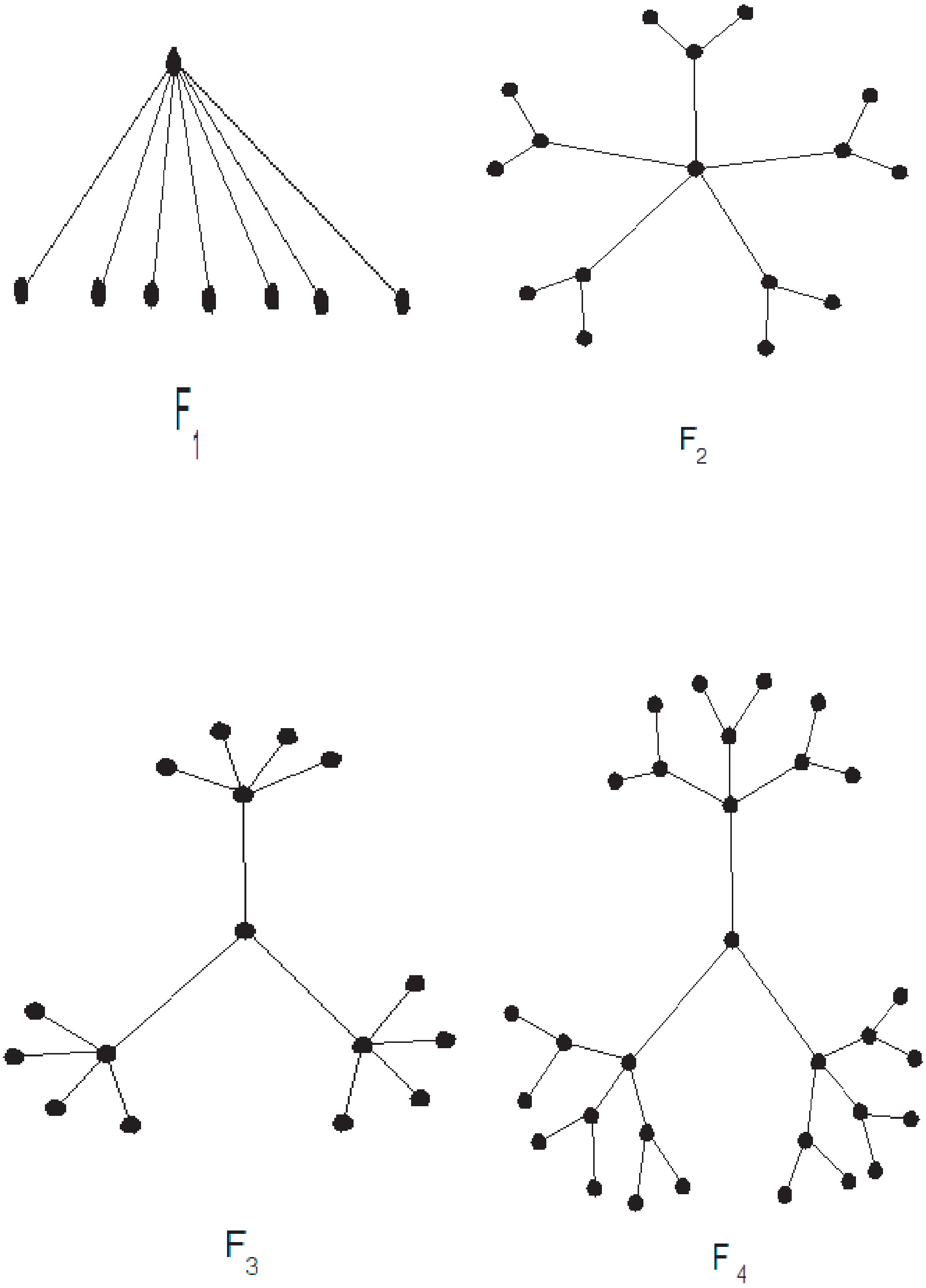

Figure 2.3: Dolores Chavez's list of 4-cutwidth critical trees

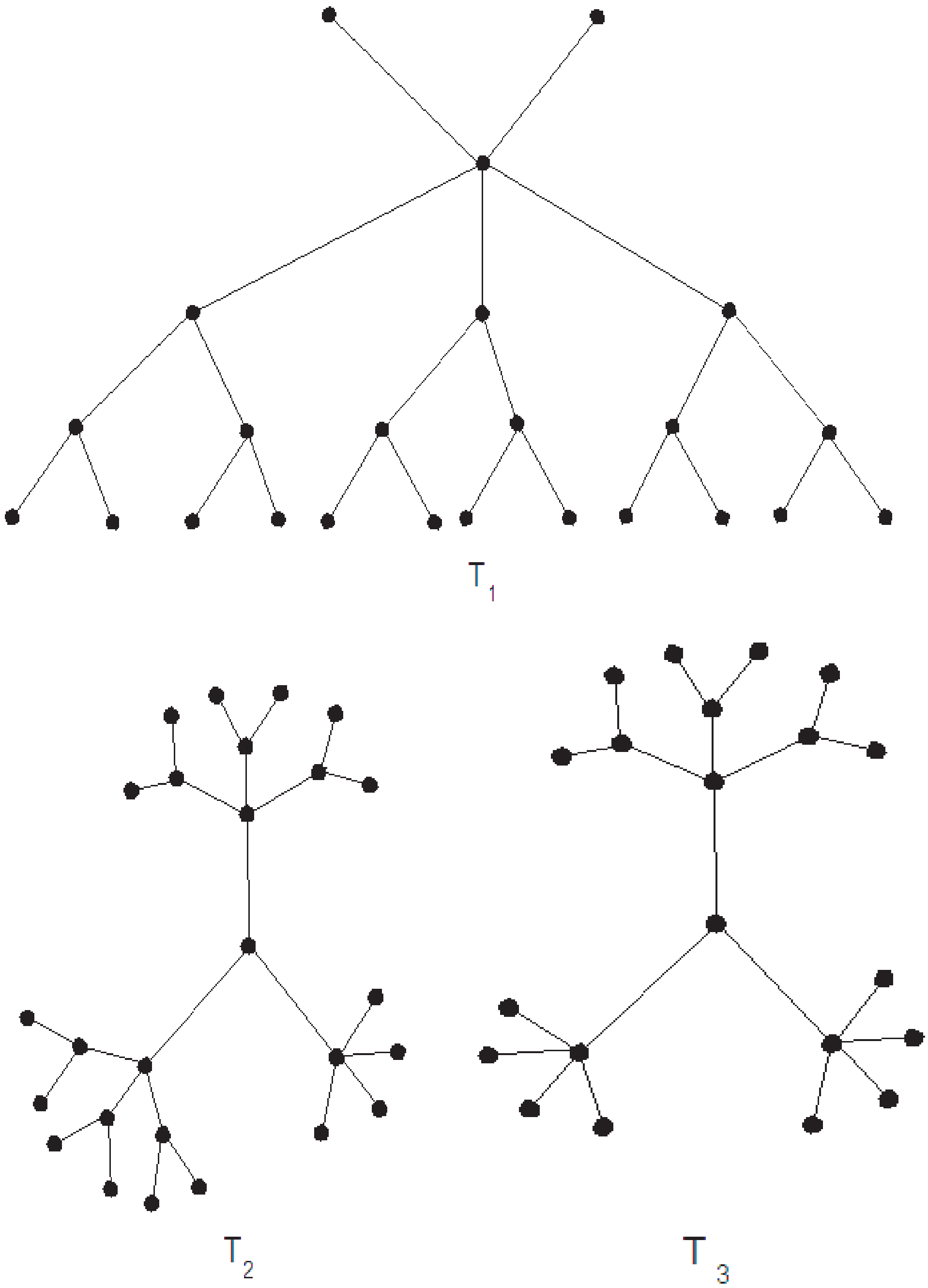

Figure 2.4: New 4-cutwidth critical trees

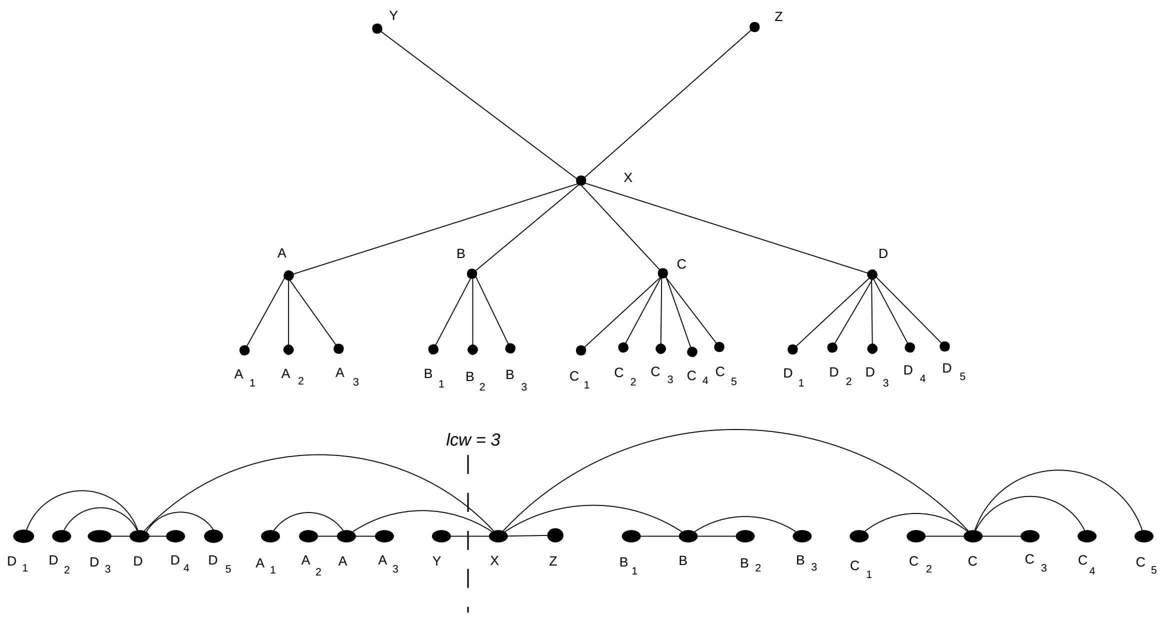

Figure 2.5: Linear embedding of a diameter-4 tree

2.2 4-cutwidth critical tree graphs

Lemma 2.1

A caterpillar is a 4-cutwidth critical graph if and only if it is F_1.

Proof. Note that F_1 is a star of degree 7. By Proposition 1 we have $c(F_1) = 4$. Any Proper subgraph of F_1 is homeomorphic to a caterpillar of $\Delta \leq 6$, whose cutwidth is at most 3.

Conversely let T be a 4-cutwidth critical caterpillar. If $\Delta T \leq 6$,then by Propostion 1 c(T)=3, which is a contradiction. Also If $\Delta T \geq 7$, then $F_1 \subseteq T$ and T=F_1 by the minimality of F_1 . □

Lemma 2.2

A tree T of Diameter 4 is 4-cutwidth critical if and only if T is either F_2 or F_3.

Proof. F_2 and F_3 are 4-cutwidth critical trees.[1]

Assume T' is a 4-cutwidth critical diameter 4 tree other then F_2 or F_3 . Then T' would either be a subgraph of a tree T (Figure 2.5), or contain the graphs of F_2 or F_3 as a subgraph. This tree graph in Figure 2.5 has a cutwidth of 3. By Proposition 2 any subgraph of this graph has cutwidth ≤ 3. This is a contradiction. Also if we place an extra edge anywhere on this tree then F_1, F_2, or F_3 will become a subgraph. Therefore F_2 and F_3 are the only 4-cutwidth critical diameter 4 trees. □

2.3 New 4 cutwidth critical tree graphs

The tree graphs T_1, T_2, T_3 as shown in Figure 2.4 are new trees that we added to the list of 4-linear critical cutwidth trees. We will present a proof that T_1 is a 4-cutwidth critical, since it is the most distinctive of the trees that we discovered.

Lemma 2.3 T_1 is a 4-cutwidth critical tree

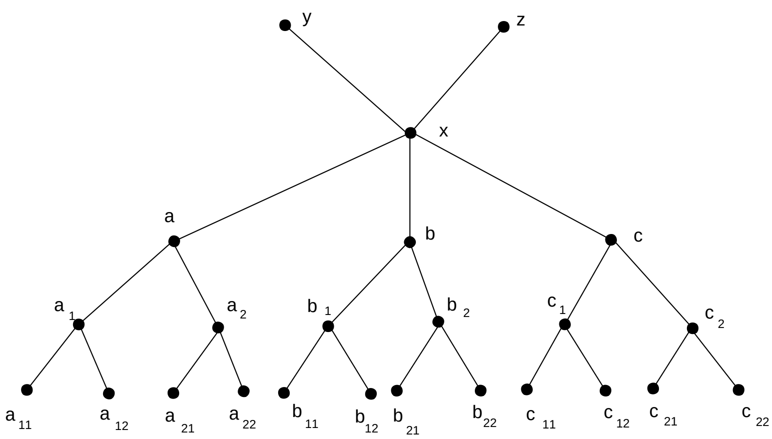

Figure 2.6: T_1

Proof. We will show the T_1 satifies all three conditions of 4-cutwidth critical graphs.

1. The first property we will verify is that $c(T_1) = 4$.The labeling of T_1 asserts that $c(T_1) \leq 4$. We will show that we cannot get a cut less than 4. Denote the vertex of degree 5 by x, and denote its neighbors by a, b ,c, y and z. For the labeling f of T_1 if $f(x)$ is not the median, then it is clear that $c(T_1, f) \geq 4$. Let $f(x)$ be the median of a labeling f of G, then there are two cases to consider.

Case 1: $\{f\,|f(a) < f(b) < f(y) < f(x) < f(z) < f(c)\}$. A linear embedding is shown in Figure 2.7. In this case the cutwidth is given by, $\{bb_2, xy, xb, xa\}$. So $c(T_1, f) \geq 4$.

Case 2: $\{f\,|f(a) < f(b) < f(x) < f(y) < f(z) < f(c)\}$. A linear embedding

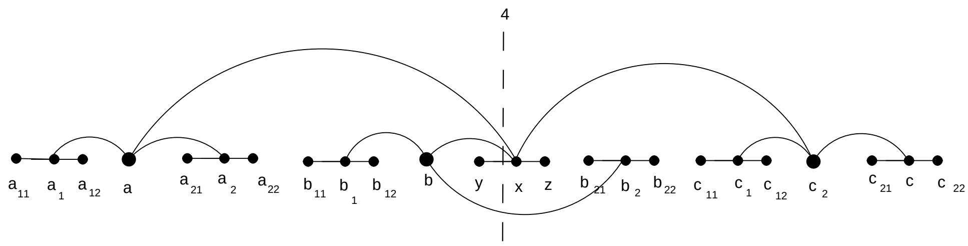

Figure 2.7: Linear embedding of T_1

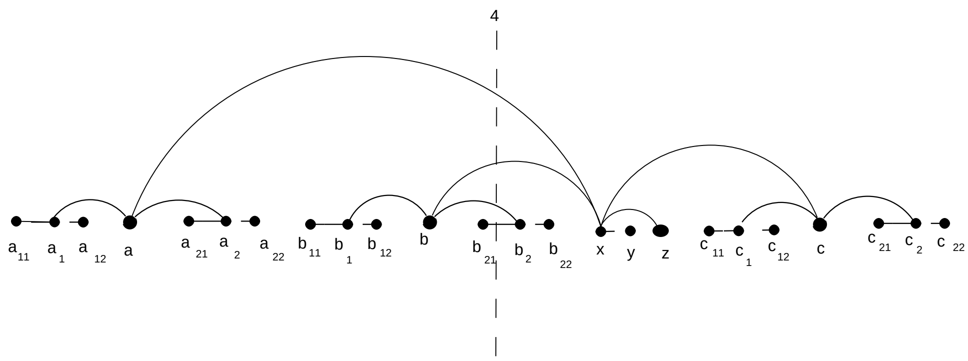

Figure 2.8: Linear embedding of T_1

in shown in Figure 2.8. In this case the cutwidth is given by $\{b_1b_{12}, bb_1, bd, ad\}$. So, $c(T_1, f) \geq 4$.

We have shown that the $c(T_1) \geq 4$ when x is the median. Therefore, $c(T_1) = 4$.

2. Now that we have shown that $c(T_1) = 4$, we now need to verify that every proper subgraph has cutwidth strictly less than 4. Due to the symmetry of T_1, removing edge (ax) is the same removing edge (bx), or edge(cx).

Removing edge (bb_2) is the same as removing edge (aa_1),(aa_2),(bb_1),(cc_1), or (cc_2). Removing edge (b_1b_{12}) is the same as removing edge (a_1a_{11}), (a_1a_{12}), (a_2a_{21}), (a_2a_{22}), (b_1b_{11}), (b_2b_{21}), (b_2b_{22}), (c_1c_{11}), (c_1c_{12}), (c_2c_{21}), or (c_2c_{22}). Likewise, removing edge(xy) is the same as removing edge (yz). Removing edge (ax),(bb_2),(b_1b_{12}),or (xy) will result into a proper graph of T_1, and will decrease the cutwidth of the linear embeddings in Figure 2.7 or Figure 2.8 down to 3. Hence the cutwidth of every proper subgraph of T_1 is strictly less than 4.

3. We know T_1 is homeomorphically minimal because it does not contain any unnecessary vertex of degree 2.

The three conditions are satisfied, therefore, T_1 is 4-cutwidth critical.

□

T_2 and T_3, were discovered from the observation that both F_3 and F_4 contained a star of degree 3 as a subgraph. Using Dolores Chavez method 4 to create 4 cutwidth critical graphs, we took combinations of F_3 and F_4 to create T_2 and T_3.

2.4 Summary of 4-linear cutwidth critical tree graphs

So far we have proved that F_1 is the only caterpillar and that F_2 and F_3 are the only diameter-4 trees that are 4-linear cutwidth critical. At present, we do not have a proof to show F_4, T_1, T_2,T_3 are the only graphs of its kind in the set of 4-cutwidth critical trees, and that there are no more of 4-cutwidth critical tree graphs. Hopefully these proofs presented will give us some insight into completing this task.

Chapter 3

On complete graphs that are cyclic cutwidth critical

3.1 Preliminaries

The following definitions and propositions will be used throughout this chapter.

Definition 1 A complete graph with n vertices (denoted K_n) is a graph with n vertices in which each vertex is connected to each of the other(with one edge between each pair of vertices).Figure 3.1 shows the first five complete graphs.

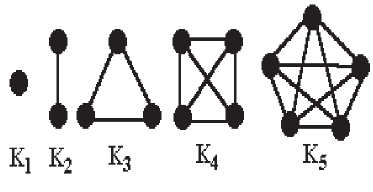

Figure 3.1: First five complete graphs

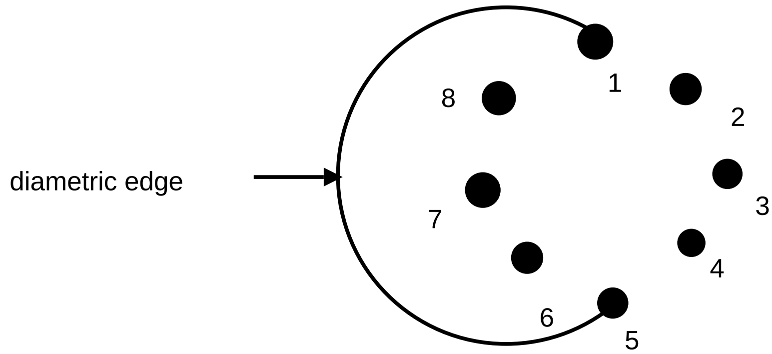

Figure 3.2: cutwidth sector of K_4

Definition 2 A cyclic embedding of K_n is a graph in which all of the vertices of K_n are embedded onto a cycle. Any edges that connect vertices in K_n will also connect vertices in the cyclic embedding of K_n, with edges running along the circumference.

Definition 3 An edge is considered diametric if it ends are attached to diametric opposed vertices. Figure 3.2 is and example of a diametric edge in K_8.

Definition 4 A sector along the circumference of a cyclic embedding of a graph where the cutwidth is maximized, is denoted as the cutwidth sector. The cutwidth sector of K_4 is shown in Figure 3.3.

Proposition 1 All complete graphs are homeomorphically minimal.

Proof. K_3 is the only complete graph with degree 2 vertices. If we remove a degree 2 vertex in K_3, the we will get a vertex with multiple edges. Hence

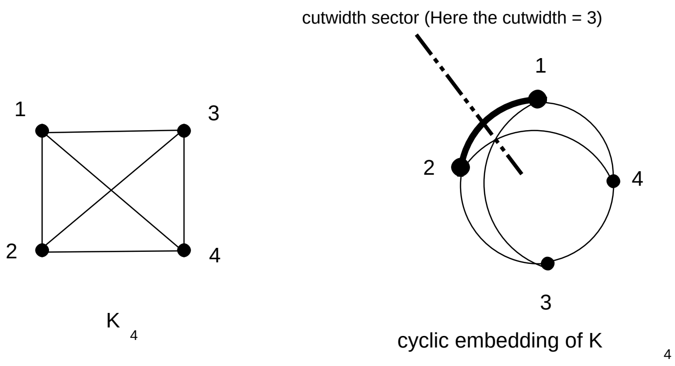

Figure 3.3: cutwidth sector of K_4

K_3 is homeomorphically minimal. Therefore all complete graphs are homeomorphically minimal. $\square$

Proposition 2[4]: For any complete graph K_n on n vertices,

$$ccw(K_n) = \begin{cases} \frac{n^2+8}{8} & \frac{n}{2} \quad even \\ \frac{n^2+4}{8} & \frac{n}{2} \quad odd \\ \frac{n^2-1}{8} & n \quad odd \end{cases}$$

Proposition 3 The cyclic cutwidth of a complete graph, K_n where n is equal to $2k+1$ for $k \in$ N. is equal to $\frac{k(k+1)}{2}$.

Proof. Assume we have a cyclic graph of K_n where n is equal to $2k+1$ for $k \in$ N. Then by Proposition 2 the $ccw(K_n)$ is equal to $\frac{k(k+1)}{2}$. $\square$

Proposition 4 The cyclic cutwidth of a complete graph, K_n where n is even, is equal to

$$\left(\frac{n^2 - 2n}{8}\right) + \left(\left\lfloor\frac{n}{4}\right\rfloor + 1\right) \tag{3.1}$$

Proof. Two cases are considered

Case 1 When $\frac{n}{2}$ is even.

When $\frac{n}{2}$ is even $n = 4k$ for $k \in$ N. By Proposition 2 the $ccw(K_n)$ is equal to $\frac{n^2+8}{8}$. Hence $\frac{(4k)^2+8}{8} = \left(\frac{(4k)^2-2(4k)}{8}\right) + \left(\left\lfloor\frac{(4k)}{4}\right\rfloor + 1\right) = 2k^2 + 1$. Therefore Equation 3.1 holds when $\frac{n}{2}$ is even.

Case 2 When $\frac{n}{2}$ is odd.

When $\frac{n}{2}$ is odd, $n = 4k+2$ for $k \in$ N. By Proposition 1 the $ccw(K_n)$ is equal to $\frac{n^2+4}{8}$. Hence $\frac{(4k+2)^2+4}{8} = \left(\frac{(4k+2)^2-2(4k+2)}{8}\right) + \left(\left\lfloor\frac{(4k+2)}{4}\right\rfloor + 1\right) = 2k^2+2k+1$. Therefore Equation 3.1 holds when $\frac{n}{2}$ is odd.

Hence the $ccw(K_n)$ where n is even equals $\left(\frac{n^2-2n}{8}\right) + \left(\left\lfloor\frac{n}{4}\right\rfloor + 1\right)$. □

By separating Equation 3.1 into two parts, I will demonstrate that the critical cutwidth of a cyclic graph K_n where n is even, is dependent on the diametric edges overlapping the cutwidth sector.

Let Part A be the first half of Equation 3.1 $\left(\frac{n^2-2n}{8}\right)$, and Part B be the second half of Equation 3.1 $\left(\left\lfloor\frac{n}{4}\right\rfloor + 1\right)$. Part A represents the cutwidth number of every sector on the cyclic graph before diametric edges are attached

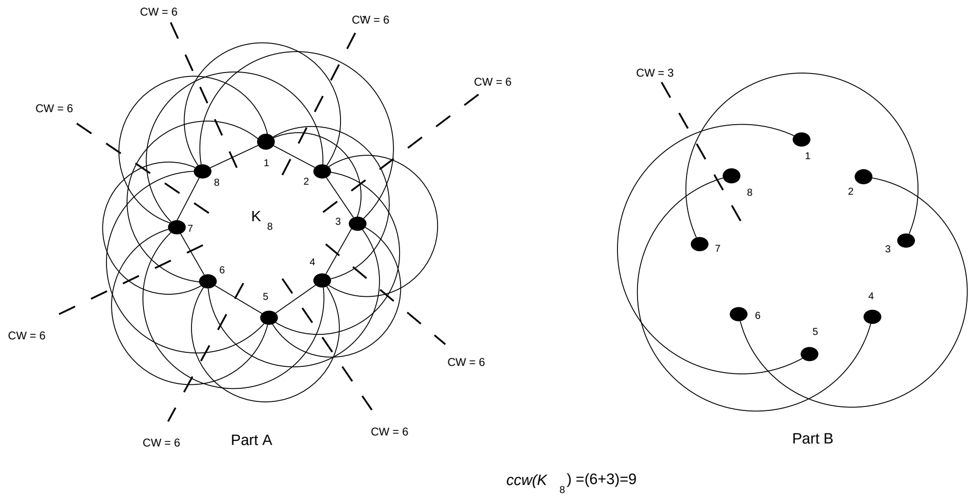

Figure 3.4: Demonstration of Equation 3.1 using K_8

to the vertices. Part B represents the number of diametric edges overlapping the cutwidth sector. Together their sum represents the cyclic cutwidth of K_n, where n is even. Part A of Equation 3.1 results because the cutwidth of any sector along the circumference is the same before the diametric edges are attached. Since the cutwidth is the same along every sector before attaching the diametric edges, observing a complete graph with just the diametric edges drawn will be sufficient enough in demonstrating whether or not a complete graph is cyclic cutwidth critical. Equation 3.1 is demonstrated in Figure 3.4 using K_8.

3.2 Optimal embedding of a complete graph, K_n where n is even, onto a cyclic host graph

This method will be used for all the proofs involving K_n, where n is even. This method of embedding is used to obtain the cyclic cutwidth of K_n, where n is even.

Let K_n be a complete graph on n vertices, where n is even. Lay its vertices evenly on a circumference and label them 1 through n in such a way that the labels of the vertices that are next to each other differ exactly by one (exception: vertex 1 and vertex n, whose labels differ by $n-1$). Then attach the diametric edges branching from vertices 1 on through $\left(\frac{n}{2}+1\right)$. The diametric edges branching from the odd vertices must have a counter clockwise orientation, and the even vertices must have a clockwise orientation. The cutwidth of this graph is equal to the largest number of diametric edges overlapping a cutwidth sector plus part A of Equation 3.1. Figure 3.5 is a cyclic embedding of K_8 using only the diametric edges.

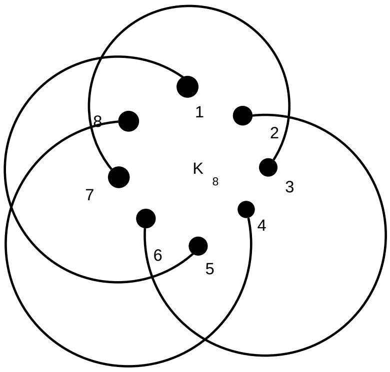

Figure 3.5: Cyclic embedding of K_8 using only the diametric edges

3.3 K_2 and K_n, where n is a multiple of 4 are the only complete graphs that are cyclic cutwidth critical.

Lemma 1 K_2 and K_n where n is a multiple of 4 are cyclic cutwidth critical.

Proof. Two cases are considered

Case 1 K_2 is cyclic cutwidth critical.

Since the $ccw(K_2) = 1$ and removing an edge from K_2 yield $ccw(K_2) < 1$, then K_2 is cyclic cutwidth critical.

Case 2 K_n where n is a multiple of 4 is cyclic cutwidth critical.

Let k equal the number of odd vertices in between vertex 1 and vertex $\left(\frac{n}{2}+1\right)$.Let m equal the number of even vertices in between vertex 1 and vertex $\left(\frac{n}{2}+1\right)$. It is always the case that m= k+1. Hence there are $k+1$ diametric edges directed in the clockwise direction, and k edges directed in the counterclockwise direction. The region of this cyclic graph overlapped by the diametric edges stemming from the even vertices between vertex 1 and vertex $\left(\frac{n}{2}+1\right)$ will generate a cutwidth of one higher then the region of this cyclic graph overlapped by the diametric edges stemming from the odd vertices between vertex 1 and vertex $\left(\frac{n}{2}+1\right)$.If we remove the diametric edge attached to even vertex labeled $\frac{n}{2}$, this will decrease the cutwidth by one. Hence K_n where n is a multiple of 4 is cyclic cutwidth critical.

Therefore K_2 and K_n where n is a multiple of 4 are cyclic cutwidth critical. $\square$

An example of this proof is demonstrated in Figure 3.6. Notice that when Edge (4,8) is removed, the *ccw* is decreased by one.

Lemma 2 All complete graphs K_n where n is odd, are not cyclic cutwidth critical.

Proof. Assume we have a complete graph K_n where n is equal to $2k+1$ for $k \in$ N. Then if a line is drawn through the median of the graph, the number of edges crossing that line is equal to $(k)(k+1)$. Let $(k)(k+1)$ equal $2m$ for $m \in$ N, since $(k)(k+1)$ is even. By Proposition 2, the cutwidth is equal to m. Hence there are two sectors along the circumference, where each sector

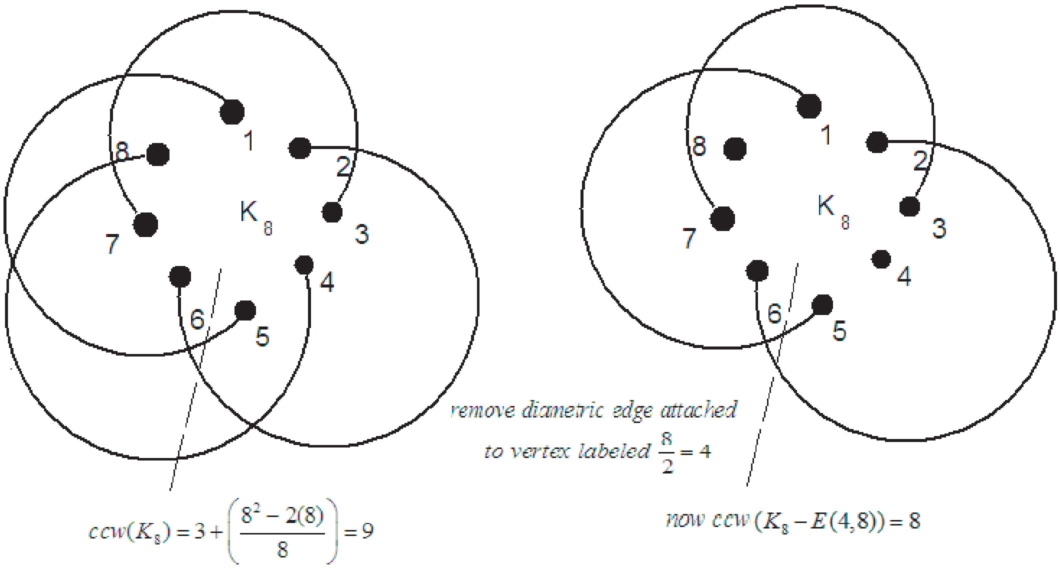

Figure 3.6: Demonstrating the $ccw(K_8)$

is overlapped by a different set of edges, and the cutwidth of both sectors are equal to m. So removing one edge will only affect the cutwidth at one sector, leaving another sector along the circumference of this graph with a cutwidth still equal to m. Hence the cutwidth of this graph is not less than m. Therefore K_n where n is odd, is not cyclic cutwidth critical. $\square$

An example of this proof is demonstrated in Figure 3.7. Notice that when Edge (3,5) is removed, the ccw does not change.

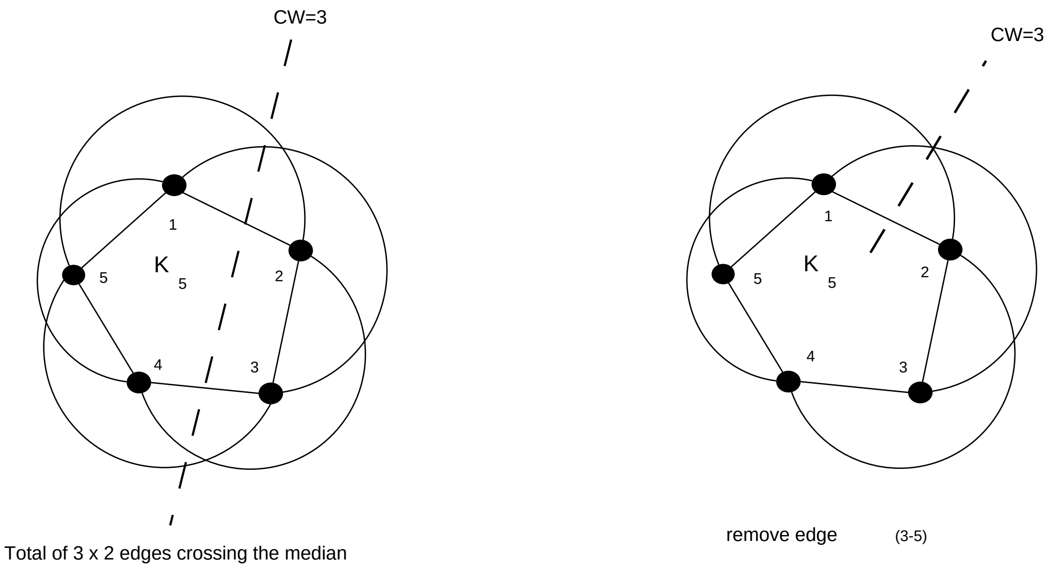

Figure 3.7: Demonstrating the $ccw(K_5)$

Lemma 3 K_n where n=4p+2 for $p \in$ N, is not cyclic cutwidth critical.

Proof. Let k equal the number of odd vertices in between vertex 1 and vertex $\left(\frac{n}{2}+1\right)$. Let m equal the number of even vertices in between vertex 1 and vertex $\left(\frac{n}{2}+1\right)$. It is always the case that m is equal to k. Hence the number of diametric edges attached to the even vertices that are directed clockwise are equal to the number of diametric edges attached to the odd vertices, that are directed counterclockwise. Due to this symmetry there will be at least two sectors along the circumference of this cyclic graph with an equal cutwidth composed of different diametric edges. Hence removing any diametric edge

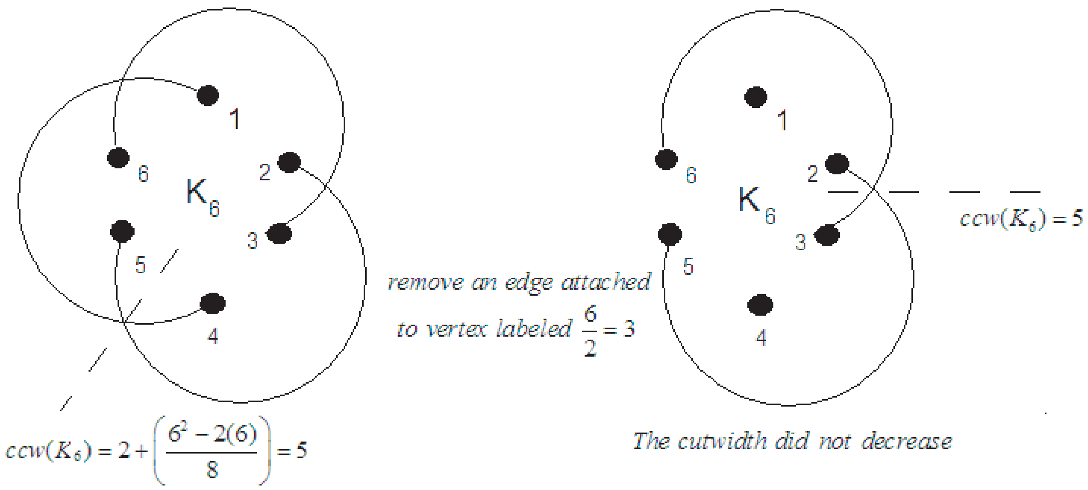

Figure 3.8: Demonstrating the $ccw(K_6)$

overlapping one of those sectors will not decrease the cutwidth of all sectors. Hence the cutwidth cannot be reduced by removing on edge. Therefore K_n where n=4p+2 for $p \in$ N, is not critical. □

An example of this proof is illustrated in Figure 3.8 using K_6.

Theorem 1 K_2 and K_n, where n is a multiple of 4 are the only graphs that are cyclic cutwidth critical.

Proof. Two parts

Part 1 **Existence**

By Lemma 2, K_2 and K_n, where n is a multiple of 4 are cyclic cutwidth critical.

Part 2 **Uniqueness**

Suppose there exist a complete graph that is cyclic cutwidth critical other than K_2 and K_n, where n is a multiple of 4. Then this graph would belong to the set of complete graphs K_n where n is odd, or K_{4n+2} where $n \in \mathrm{N}_1$. This is a contradiction, since both cases of K_n are not critical by lemma 1 and lemma 3.

Therefore K_2 and K_n, where n is a multiple of 4 are the only graphs that are cyclic cutwidth critical. □

3.4 Summary on complete graphs that are cyclic cutwidth critical

We have described and proved the set of complete graphs that are cyclic cutwidth critical. In our efforts to prove this theorem we have presented alternative equations that can be used for calculating the cyclic cutwidth of all complete graph. We also introduced an optimal method to labeling a complete graph with an even number of vertices for the advantage of obtaining the exact cyclic cutwidth.

ACKNOWLEDGEMENTS

I would like to thank Dr. Joseph Chavez and Dr. Rolland Trapp for their academic contribution and guidance to this research project. I would also like to thank the National Science Foundation (NSF) for funding the REU Program at California State University, San Bernardino (CSUSB) and making the research possible (CSUSB 2006 NSF-REU Grant DMS-0453605).

Bibliography

[1] Dolores Chavez. Investigation of 4-cutwidth critical graphs, (2006)

[2] Yixun Lin, Aifeng Yang. On 3-cutwidth critical graphs, Discrete Math.275(2004) 339-346

[3] Y.Lin, The cutwidth of trees with diameter at most 4, Appl. Math. J. Chinese Univ. Ser. B (2002), 17(3):361-369

[4] Francisco Ramon Rios, "Complete Graphs as a First Step towards Finding the Cyclic Cutwidth ot the n-Cube",Preprint, 1996.