The Minimum Distance Energy and Midpoint Insertion

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Abstract

For any crossing knot K, it appears that the minimum stick representation will not yield the lowest minimum distance energy. Experimentally it is obvious that this is the case. In fact, by just adding a midpoint to a minimum stick representation we find that a lower energy results. In an attempt to show this, we will show that the minimum distance energy of an edge from knot K will be greater than or equal to the composite sum of the same knot's edge divided by a midpoint. We call savings the amount our U_{MD} decreases for any given edge. We will show what our savings is after adding a midpoint to an edge. Last we will show our findings for a rough upper bound describing the contribution of newly non-adjacent edges.

1 Background

A **Stick Knot** also known as a Polygonal Knot, is a simple closed curve in Euclidean space $\mathbb{R}^3$ produced by joining finitely many points, called vertices, with straight line segments, called edges. See figure 1.

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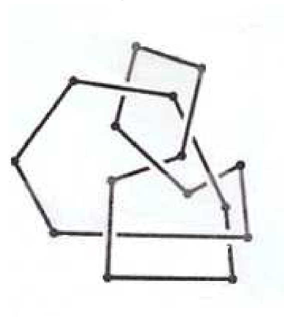
Figure 1:



Simon introduces the minimum distance energy for a stick knot and we review his definitions, [1]. The U_{MD} for the stick knot K with n sticks, sums all unordered pairs of nonconsecutive segments within the knot. We know that the total number of unordered pairs is $\frac{n(n-3)}{2}$. If we let W,Z be disjoint line segments in $\mathbb{R}^3$ then the minimum distance between W and Z is a positive number which we denote MD(W,Z). Letting l(W) denote the length of segment W, and l(Z) denote the length of segment Z. We define the minimum distance energy as:

$$U_{MD}(W,Z) = \frac{l(W)l(Z)}{MD(W,Z)^2}$$

For a stick knot K, we define:

$$U_{MD}(K) = \sum_{W,Z\ non-consecutive segments\ of\ K} U_{MD}(W,Z)$$

Note that the U_{MD} of a stick knot is scale invariant. This emphasizes the shape not the size of the knot. This can be seen through simple manipulations of the above equation.

1.1 Data and Examples

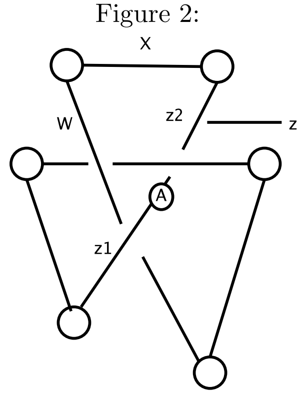
Ming is a program that allowed us to manipulate knots and determine the U_{MD}. Using Ming it was observed that adding a stick to any knot, that knot's energy is immediately reduced. This can be seen in Table 1. The Knot type represented by $K_{3,1e}$ explains the knot K. Here the 3 describes the crossing number, 1 explains where this knot appears in the table of knots with the same crossing number, and e means this is an equillateral knot. Notice that while the energy for each knot type goes down, for some knots the decrease in energy is more significant than others.

Knot Type	Number of Edges	U_{MD}
$K_{3,1}$	7	184.48508082
	8	180.72377345
$K_{3,1e}$	6	1290.89530575
	7	1071.74585483
$K_{4,1}$	8	466.79796198
	9	439.76469024
$K_{5,1}$	8	401.42316488
	9	377.01424621
$K_{6,1}$	8	1047.44375784
	9	1003.68536269

Table 1: Ming, experimental data [2]

Our observations from Ming led us to manually determine the minimum distance energy for ourselves. We began by taking knot coordiantes, found in Ming, and determining the total energy of that knot. Then we took the same knot and divided an edge at a midpoint. Using Ming, again we determined the energy for that knot. Then we went back to the original knot and divided another edge at it's midpoint. Throughout this process we realize that there is one edge, when divided, that will yield the largest decrease in U_{MD}. Once we knew what edge was best for subdividing, we wrote a program in Maple which allowed us to look at the contribution of each edge to the total energy. We compared the original knot with the knot plus its subdivided edge. By doing this we made several observations which led us to the Lemma's found in section 2.

Figure 2:



The above diagram shows us how adding vertex A, we change the edge Z into Z_1, and Z_2. We also can see that while Z had been adjacent to X, it division causes Z_1 to be non-adjacent to X. Z_2 also has a newly non-adjacent edge.

In an attempt to understand why adding a stick will lower the energy despite the above knowledge, we will also compare the knot's energy before and after the addition of a midpoint. This helped us understand which component of the knot energy is independent of the equation. Let us begin by viewing table 2, see below. This table compares the minimum distances and U_{MD}'s from the knots $K_{3,1e}:6$ and $K_{3,1e}:7$. $K_{3,1e}:6$ is the original knot and $K_{3,1e}:7$ is $K_{3,1e}:6$ with a midpoint added. We see in this table that U_{MD} is the energy for the combination of edges listed. For example knot $K_{3,1e}:6$'s edge $(1,3)$ has the U_{MD} of 2.5811 and the corresponding edges in $K_{3,1e}:7$ include $(1,3)$ with U_{MD} of 1.2905 and $(1,4)$ with the U_{MD} of 1.2863. Here we see that by dividing edge 3 the edge $(1,3)$ in $K_{3,1e}:6$ turns into two edges in $K_{3,1e}:7$, $(1,3)$ and $(1,4)$. Thus we can see the comparisons of U_{MD} from the original knot and the knot with the midpoint added.

$K_{3,1e}:6$	U_{MD}	$K_{3,1e}:7$	U_{MD}
$(1,3)$	2.5811	$(1,3)$ $(1,4)$	1.2905 1.2863
$(1,4)$	372.7404	$(1,5)$	372.7404
$(1,5)$	240.3246	$(1,6)$	240.3246
$------$	——	$(2,4)$	3.5891
$(2,4)$	11.2088	$(2,5)$	11.2088
$(2,5)$	35.6330	$(2,6)$	35.6330
$(2,6)$	11.2105	$(2,7)$	11.2105
$------$	——	$(3,5)$	3.0216
$(3,5)$	240.3164	$(3,6)$ $(4,6)$	120.1582 49.0561
$(3,6)$	372.7117	$(3,7)$ $(4,7)$	186.3558 31.7020
$(4,6)$	4.1688	$(5,7)$	4.1688
TOTAL ENERGY	1290.8953		1071.7459

Table 2: Corresponding Edges

As stated earlier, we divided edge 3. We chose to do this because the lowest U_{MD} occurs under this division. However, the reasoning behind this will be explained in later. Notice that following the division of edge 3 the first half will continue to be called edge 3, while the second half will be renamed 4. Hence the reason why edge $(1,3)$ in $K_{3,1e}:6$ became the edges $(1,3)$ and $(1,4)$ in $K_{3,1e}:7$. The labeling counts up from there, thus we will have one extra edge in the knot $K_{3,1e}:7$.

Over all, we see that the U_{MD} of $K_{3,1e}$ is lower when there are seven sticks. We also notice that the energies involving the edges 3 and 4 in $K_{3,1e}:7$, as a sum are less than the corresponding edges from $K_{3,1e}:6$. For example if we look at $(3,5)$ from $K_{3,1e}:6$ we see that the energy for that edge is 240.3164. Now if we look at the corresponding edges in $K_{3,1e}:7$ we see that the edge $(3,6)$ with energy 120.1582 and $(4,6)$ with energy 49.0561 totals 169.2143. Clearly 169.2143 is less than the 240.3164 from the original knot.

2 Structural Properties of U_{MD}

In this section we will describe how inserting a midpoint in knot K, effects the U_{MD} for the knot. We will denote the knot with the midpoint added, K'. We will show how a knot edge's energy is greater as a whole edge versus adding the energies of the subdivided edge together. We also find an equation to describe the savings when a midpoint is added, and also find that the savings is independent of the edges length. Through this we determine an upper bound for the contribution to U_{MD} from the newly non-adjacent edge. Although Lemma 1 doesn't explain the edges that became newly non-adjacent, it remains useful as it helps us in furthuring our understanding on how adding a stick lowers the total energy.

Lemma 1 Let W and Z be disjoint edges in $\mathbb{R}^3$, and let Z_1 and Z_2 denote edges obtained by subdividing Z at its midpoint. Then we have:

$$U_{MD}(W,Z) \geq U_{MD}(W,Z_1) + U_{MD}(W,Z_2)$$

Proof. As we know how to find the energy of any knot we apply the equation in the following way: $U_{MD} = \frac{l(W)l(Z)}{MD^2(W,Z)}$. Since U_{MD} is scale invarient we can assume $l(W) = 1$. Thus we will compare:

$$\frac{1 \cdot l(Z)}{MD(W,Z)^2}$$

and

$$\frac{1(1)l(Z)}{2MD^2(W,Z_1)} + \frac{1(1)l(Z)}{2MD^2(W,Z_2)}$$

Now we can factor out $l(Z)$. Thus our equation becomes:

$$\frac{1}{MD^2(W,Z)}$$

and

$$\frac{1}{2MD^2(W,Z_1)} + \frac{1}{2MD^2(W,Z_2)}$$

We call point on Z that is closest to W, z_x. Since Z_1 and Z_2 are the subdivisions of Z, without the loss of generality z_x will exist on Z_1. Thus $MD(W,Z) = MD(W,Z_1)$. We will denote this minimum distance by m.

$$\frac{1}{m^2}$$

and

$$\frac{1}{2m^2} + \frac{1}{2MD^2(W,Z_2)}$$

As we know $m \leq MD(W, Z_2)$ thus $\frac{1}{m} \geq \frac{1}{MD(W,Z_2)}$. Therefore we can conclude that:

$$\frac{1}{m^2} \geq \frac{1}{2m^2} + \frac{1}{2MD^2(W, Z_2)}$$

□

Now, that we see that the U_{MD} for an edge is lower after adding a midpoint, let us determine what the savings is from that midpoint. We will use the notation p in our savings equation, where $p = \frac{MD(W,Z_2)}{MD(W,Z)}$. As in the previous Lemma, z_x will exist on Z_1. Again we will call the point on Z that is closest to W, z_x. The point where Z is subdivided will be called p_o.

Lemma 2

Given the edges W and Z with unit length, the savings for any edge W is:

$$U_{MD}(W, Z) - (U_{MD}(W, Z_1) + U_{MD}(W, Z_2)) = \frac{1}{MD^2(W, Z)} \left(\frac{p^2 - 1}{p^2} \right)$$

Proof. Since W and Z have unit length

$$U_{MD}(W, Z) - (U_{MD}(W, Z_1) + U_{MD}(W, Z_2))$$

can be rewritten

$$\frac{1}{MD^2(W, Z)} - (\frac{(1)}{2MD^2(W, Z_1)} + \frac{(1)}{2MD^2(W, Z_2)})$$

because $MD(X, Z_2) = pMD(X, Z)$ where $p \geq 1$ we can substitute to get

$$\frac{1}{MD^2(W, Z)} - \left(\frac{1}{2MD^2(W, Z)} + \frac{1}{2p^2 MD^2(W, Z)} \right)$$

Since we know that $MD^2(W, Z) = MD(W, Z_1)$, we can cancel out these minimum distances and factor out the half to get

$$\frac{1}{2} \left(\frac{1}{MD^2(W, Z)} - \frac{1}{p^2 MD^2(W, Z_2)} \right)$$

when simplified we obtain

$$\frac{1}{2} \frac{1}{MD^2(W, Z)} \left(1 - \frac{1}{p^2} \right)$$

by finding a common denominator we can conclude the savings to be:

$$\frac{1}{MD(W, Z)^2} \left(\frac{p^2 - 1}{p^2} \right)$$

□

Now let us consider what specific pieces of our equation cause the knot energy to decrease when a midpoint is added. Again we will use the same notation as above to show that Z_1 is independent of the savings equation found in Lemma 2.

Lemma 3
Let us show if p is not necessarily the midpoint of Z and if z_x is on Z_1, then the savings is independent of $l(Z_1)$.

Proof. To show that Z_1 is independent we must begin with the equation from the last Lemma.

$$U_{MD}(W,Z) - (U_{MD}(W,Z_1) + U_{MD}(W,Z_2)) = \frac{1}{MD^2(W,Z)}\left(\frac{p^2-1}{p^2}\right)$$

However, for this case we make substitutions so that $U_{MD}(W,Z) - (U_{MD}(W,Z_1) + U_{MD}(W,Z_2))$ is rewritten:

$$\frac{l(W)(l(Z_1)+l(Z_2))}{MD^2(W,Z)} - \left(\frac{l(W)l(Z_1)}{MD^2(W,Z_1)} + \frac{l(W)l(Z_2)}{MD^2(W,Z_2)}\right)$$

Here we will factor out $l(W)$. As we already know that z_x will exist on Z_1 and $MD^2(W,Z) = MD^2(W,Z_1)$. Consequently we can cancel out these minimum distances to obtain:

$$l(W)\left(\frac{l(Z_2)}{MD^2(W,Z)} - \frac{l(Z_2)}{MD^2(W,Z_2)}\right)$$

□

Because $l(Z_1)$ is independent of the minimum distance energy, we realize that $l(Z_1)$ has minimal significance for the final energy. This said, lets use the previous Lemmas to help us find an upper bound for the contribution of newly non-adjacent edges. As we discussed earlier, the addition of a midpoint creates newly non-adjacent edges. However, through observation we know that there is a way to add a midpoint that will lower the U_{MD} the most.

Originally we assumed that any added midpoint would lower the total energy. Unfortunately, this is not the case. In fact, dividing an exterior edge wont necessarily lower the energy. On certain examples it has in fact increased the energy. While the reasoning behind this is still unknown, it indicates that dividing an interior edge is more preferable to lower the U_{MD}. This known, let us discuss how best to add a midpoint to decrease the U_{MD}.

We make the assumption that our knot is equilateral, consequently we assume that each knot edge is of length 1. Now we know that we must consider two types of angles, acute and obtuse.

Figure 3: diagram not to scale

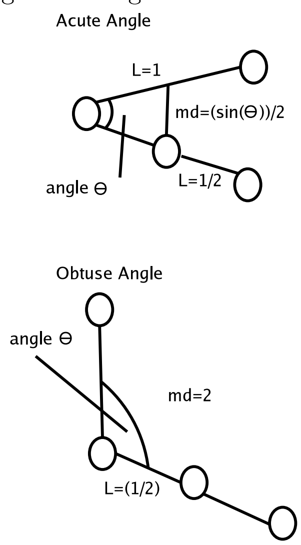


Figure 3 illistrates the two angles to be considered. It also helps us to see that the obtuse angle will always have a larger minimum distance. Table 3 show the contribution to U_{MD} from the newly non-adjacent edges.

Angle Type	θ	U_{MD}
Acute	≤ 90	$\frac{2}{\sin(\theta)^2}$
Obtuse	≥ 90	2

Table 3:

Lemma 4 The contribution to $U_{MD}(K')$ coming from newly non-adjacent edges is at most

$$\frac{(4)}{\sin(\theta_{min}^2)}$$

Proof. To make sure that the above properties are met we will denote θ_{min} as the smallest angle in any knot, where α and β are angles larger than θ_{min}. A simple illustration can be seen in figure 4. Consequently we know:

$$\frac{1}{\sin(\alpha)} \leq \frac{1}{\sin(\theta_{min})} \text{ and } \frac{1}{\sin(\beta)} \leq \frac{1}{\sin(\theta_{min})}$$

.

Therefore we can rewrite this as:

$$\frac{2}{\sin^2(\alpha)} + \frac{2}{\sin^2(\beta)} \leq \frac{4}{sin^2(\theta_{min})}$$

Since we added a midpoint to Z, the new subdivisions become Z_1 and Z_2. Here we have $\frac{2}{\sin^2(\alpha)}$. This is the energy from the new edge Z_2. We also have $\frac{2}{\sin^2(\beta)}$ to describe the energy from the new edge Z_1. $\frac{1}{\sin(\theta_{min})}$ is the energy from the original edge. This inequality makes sense because we want the energies, created from dividing our edge Z, to be less than the original edge.

Thus we have a rough upper bound for the contribution of newly non-adjacent edges. □

3 Conclusion

In this paper I have attempted to show that the minimum stick representation of a knot K does not yield the minimum distance energy. While this was not accomplished for the general case, We have found localized proofs. We found that the U_{MD} of an edge is lower after a midpoint is added. We were also able to find the savings caused after adding a midpoint. Last, we found an upper bound for the contribution from newly non-adjacent edges. Still there is room left to expand upon, such as tightening the upper bound for the contribution of newly non-adjacent edges.

This research was intended to prove that the minimum stick number of a knot does not yield the minimum U_{MD}. As this was not proven this too can be expanded upon. In fact, if we recall Lemma 2, and the savings caused by adding a midpoint to a knot:

$$\frac{1}{2MD^2(W,Z)}\left(\frac{p^2-1}{p^2}\right)$$

and we know from Simon's Lemma 8, [1]

$$\frac{1}{MD^2(W,Z)} \geq \frac{4}{sin^2(\theta)}$$

We can use these two known equations to prove that the minimum stick representation does not produce the minimum U_{MD}.
Since the savings for W is:

$$\frac{1}{2MD^2(W,Z)}\left(\frac{p^2-1}{p^2}\right) \geq \frac{4}{2sin^2(\theta)}\left(\frac{p^2-1}{p^2}\right) = \frac{2}{sin^2(\theta)}\left(\frac{p^2-1}{p^2}\right)$$

Since the savings for V is:

$$\frac{1}{2MD^2(V,Z)}\left(\frac{p^2-1}{p^2}\right) \geq \frac{4}{2sin^2(\theta)}\left(\frac{p^2-1}{p^2}\right) = \frac{2}{sin^2(\theta)}\left(\frac{p^2-1}{p^2}\right)$$

Lemma 2 told us the savings and Simon's Lemma 8 [1] told us the inequality between the savings and the contributions. Knowing these we then know that the:

$$\text{savings}_W + \text{savings}_V \geq \frac{2}{sin^2(\theta)} \left(\frac{p_V^2 - 1}{p_V^2} + \frac{p_W^2 - 1}{p_W^2} \right)$$

if we could prove for any knot K:

$$\left(\frac{p_V^2 - 1}{p_V^2} + \frac{p_W^2 - 1}{p_W^2} \right) \geq 2$$

then we would know

$$\left(\frac{p_V^2 - 1}{p_V^2} + \frac{p_W^2 - 1}{p_W^2} \right) \geq \frac{4}{sin^2(\theta)} \geq \text{ contributions from the newly non-adjacent edges}$$

Consequently we would know that the minimum stick does not yeild the minimum U_{MD}.

4 Acknowledgements

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References

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