Edge-Bandwidth

Taijiro Bernstein *

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Abstract

In this paper we discuss the edge-bandwidth of some families of graphs and characterize graphs by edge-bandwidth. In particular, bounds for $m \times n$ grids, triangular grids of size l, and the closure of the triangular grid T_l^*.

1 Introduction

Let $G = (V, E, \delta)$ be a simple graph with a set V of vertices, a set E of edges, and a function $\delta : E \to \binom{V}{2}$ which indentifies the two distinct vertices incident to each edge. Let f be a bijection from V to the set $\{1, 2, 3, \ldots, |V|\}$, called a labelling of the vertices of G. The *bandwidth* of G is defined to be

$$B(G) := \min_f \max\{|f(a) - f(b)| : \{a, b\} \in E\},$$

where the minimum is taken over all possible labellings of V.

Some of the motivations for investigating the bandwidth problem include: sparse matrix computations, representing data structures by linear arrays, VLSI layouts, and mutual simulations of interconnection networks [3, 4, 9]. The problem of computing bandwidth however, is known to be NP-Complete and bandwidths are only known for a few infinite families of graphs. Bandwidths were found for hypercubes [2], complete trees [5], and other various mesh-like graphs [3, 5, 6, 7, 8].

The analog of the bandwidth problem for edges is finding the *edge-bandwidth* of G. If g is a bijection from E to the set $\{1, 2, 3, \ldots, |E|\}$, an edge-labelling of G, we define the *edge-bandwidth* of the labelling g to be

$$B'(g) := \max\{|g(a) - g(b)| : a, b \in E, \text{ where } a, b \text{ are incident}\},$$

and the *edge-bandwidth* of G to be

$$B'(G) := \min_g B'(g)$$

where the minimum is taken over all possible labellings of E.

Authors such as Grünwald and Weber determined the edge-bandwidths for complete binary trees, complete, and complete bipartite graphs in [10, 11].

*Department of Mathematics, California State University, San Bernardino, CA 92407

Edge-bandwidths are also known for caterpillars and theta graphs [12, 13]. A useful graph for determining the edge-bandwidth of a graph is $L(G)$, the line graph of G, where $V(L(G)) = E(G)$ and x, y are adjacent in $L(G)$ if and only if x and y share a vertex in G. Then combining definitions, we obtain the equality

$$B'(G) = B(L(G)). \tag{1}$$

This equality allows us to establish bounds for edge-bandwidth using known results about bandwidth. Two useful inequalities stand out. If $G = (V, E)$ is a graph, for $S \subseteq V$, let

$$\delta(S) := \{v \in V - S : (u, v) \in E, u \in S\}$$

be the boundary of S. In [14], Harper proved:

Theorem 1. *For any k, $0 \leq k \leq |V|/2$*

$$B(G) \geq \min_{\substack{S \\ |S|=k}} \max\{|\delta(S)|, |\delta(V - S)|\}. \tag{2}$$

Another important estimation from [15]:

Theorem 2. *Let H be a graph on p vertices of $diam(H) > 0$. Then*

$$B(H) \geq \left\lceil \frac{p-1}{diam(H)} \right\rceil. \tag{3}$$

Where $diam(H) := \max_{u,v \in V} \min\{length(P) : P \text{ is a path from } u \text{ to } v\}$.
Now we can establish some bounds on the edge-bandwidth of an $m \times n$ grid, which was conjectured in [1] to be $2n - 1$, and later proved in [16]. Later, we will show how the ideas in (2) were extended to determine the edge-bandwidth of a rectangular grid by an error of one and to obtain the correct asymptotic growth of triangular grids.

2 Rectangular and Triangular Grids

G is an $m \times n$ grid where $m \geq n$, if V is the set of ordered pairs of positive integers (a, b) such that $a \leq n$ and $b \leq m$, and x is adjacent to y if and only if $\|x - y\| = 1$.

Lemma 1. *Let G and H be simple graphs such that G is a subgraph of H, then*

$$B'(G) \leq B'(H) \tag{4}$$

Proof. Let $g : E(H) \to \{1, 2, 3, \dots, |E(H)|\}$ be an optimal edge-bandwidth labelling of H, i.e. $B'(g) = B'(H)$. Consider $K := \{g(e) : e \in E(G)\}$, and define an equivalence relation $\sim$ on K where

$$i \sim j \iff \forall k \in \mathbb{Z}^+ \text{ and } \min\{i, j\} \leq k \leq \max\{i, j\}, \quad k \in K.$$

Now suppose K consists of l equivalence classes, and we say that a class C is less than a class C' if every element of C is less than every element of C'. This induces a labelling of the classes with the integers $1, 2, \dots, l$. Then we define

$x_1 := 0$ and $x_{i+1} := \min C_{i+1} - \max C_i + 1$ for $i \leq l-1$.

Next, we construct a labelling g^* of the edges of G:

$$\forall e \in \{e : g(e) \in C_i\}, \quad g^*(e) = g(e) - \sum_{k \leq i} x_i \qquad i \leq l$$

We observe now that given two edges of G, their edge difference in g^* is no more than their edge difference in g so $B'(G) \leq B'(g^*) \leq B'(g) = B'(H)$. □

Lemma 2. *Let G be an $n \times n$ grid, then*

$$B'(G) \geq n + 1. \tag{5}$$

Proof. We consider the line graph of G, $H := L(G)$, and by (1), we need only show that $B(H) \geq n + 1$.

First, we see that $|V(H)| = 2n(n-1)$ since G has $n(n-1)$ horizontal edges and $n(n-1)$ vertical edges on the cartesian plane.

Second, an induction yields that $diam(H) = 2n - 3$. If $n = 3$, $diam(H) = 3$. Suppose now that for an $n \times n$ grid G, $diam(L(G)) = 2n - 3$, and consider the line graph H of an $(n+1) \times (n+1)$ grid. We observe that the two edges furthest apart are at the corners. A path P, of length $2n - 1 = 2(n + 1) - 3$, can be constructed by extending a path of length $2n - 3$, between the corners of an $n \times n$ grid which is a subraph of H. We see that P is the shortest such path since the choices of moving horizontally or vertically from one edge to another does not affect the length of the entire path.

Combining these observations with (3),

$$\begin{aligned} B(H) &\geq \left\lceil \frac{|V(H)| - 1}{diam(H)} \right\rceil \\ &= \left\lceil \frac{2n(n-1) - 1}{2n - 3} \right\rceil \\ &= \left\lceil n + \frac{n-1}{2n-3} \right\rceil \\ &= n + \left\lceil \frac{n-1}{2n-3} \right\rceil \\ &= n + 1. \qquad\qquad n \geq 3 \end{aligned}$$

□

Lemma 3. *Let G be an $m \times n$ grid, then*

$$B'(G) \leq 2n - 1. \tag{6}$$

Proof. We construct an edge-labelling g of G such that $B'(g) = 2n - 1$.

First, we label the edges adjacent to row 1:

$$\begin{array}{cc} g((1,1),(2,1)) := 1, & g((1,1),(1,2)) := n \\ g((2,1),(3,1)) := 2, & g((2,1),(2,2)) := n+1 \\ \vdots & \vdots \\ g((n-1,1),(n,1)) := n-1, & g((n,1),(n,2)) := 2n-1. \end{array}$$

Then we label the unlabelled edges adjacent to row $i > 1$ similarly:

$$\begin{array}{rl} g((1,i),(2,i)) := x+1, & g((1,i),(1,i+1)) := x+n \\ g((2,i),(3,i)) := x+2, & g((2,i),(2,i+1)) := x+n+1 \\ \vdots & \vdots \\ g((n-1,i),(n,i)) := x+n-1, & g((n,i),(n,i+1)) := x+2n-1, \end{array}$$

where x is the last label of row $i-1$. Note that there are no vertical edges to label for row $i = n$.

In order to verify that $B'(g) = 2n-1$, we observe that there are nine types of vertices to check: the four corners, the four degree three vertices (top,bottom,left,right), and the degree four internal vertices. The corners have edge difference at most n. The top and bottom degree three vertices contribute a difference of n. The remaining three types contribute $2n-1$. Thus, $B'(g) = 2n-1 \geq B'(G)$. □

Since every $n \times n$ grid is a subgraph of an $m \times n$ grid, combining (4), (5), and (6), we immediately obtain:

Theorem 3. *If G is an $m \times n$ grid with $n \geq 3$,*

$$n + 1 \leq B'(G) \leq 2n - 1.$$

In order to determine the edge-bandwidth exactly, Pikhurko in [16] improved the lower bound to $2n-2$. Then he gathered enough structural information about G, so that if the lower bound held with equality, it would yield a contradiction.

We will say that the *support* $V(S)$, of a set of edges $S \subseteq E(G)$, is the set of vertices which are an end to an edge in S. Two subsets of $E(G)$ *touch* if their supports intersect.

The *complement* of a set of edges S will be denoted by $\overline{S} = E(G) \setminus S$. For an edge $D \in \overline{S}$ the *distance* from D to S is the order of the shortest path in G joining a vertex of D to a vertex of $V(S)$. The *i-th neighborhood* $\sigma^i(S)$, of S is the set of edges in $\overline{S}$ which are at most distance i from S. The following extension of (2) will be useful:

Theorem 4. *For any edge labelling η of G, any $j \in [|E|]$, and any $i \geq 1$, if we take $S = \eta^{-1}([j])$,*

$$B'(\eta) \geq \frac{\max\{|\sigma^i(S)|, |\sigma^i(\overline{S})|\}}{i}. \tag{7}$$

Proof. We fix η, j, and i. We see that the largest label in S is j. Choose e_1, which is the edge with the largest label in $\sigma^i(S)$, for which $\eta(e) = j + |\sigma^i(S)|$. Choose P to be the shortest path from a vertex of e_1 to a vertex v in $V(S)$, and suppose it has $p \leq i$ vertices. Choose an edge e_2 in S with vertex v as an end. Now extend P to P' by linking on e_2 and e_1 to P. At each vertex of P, an edge difference between edges of P' results. Let A_1 denote the edge difference at the first vertex, A_2 at the second, and so on. We see that

$$B'(\eta) \geq \frac{\sum_{[p]} A_i}{p} \geq \frac{|\sigma^i(S)|}{p} \geq \frac{|\sigma^i(S)|}{i}.$$

The argument for the $\overline{S}$ case is similar. □

We now show that the lower bound of $B'(G)$ is greater than or equal to $2n-2$. We denote a row of edges with R_i and a column with C_i for $i \in [n]$. We call $R'_i := \{e \in E : e \text{ has ends } (j,i),(j,i+1) \text{ for some } j \in [n]\}$ a quasi-row and $C'_i := \{e \in E : e \text{ has ends } (i,j),(i+1,j) \text{ for some } j \in [n]\}$ a quasi-column for $i \in [n-1]$.

Proof. Consider a labelling η of G which optimizes edge-bandwidth. We let s be the smallest integer such that $\eta^{-1}([s+1])$ contains two lines (where a line is a row or column). We see that $\eta^{-1}([s])$ contains precisely one line. We may assume without loss of generality that the line is a row, say R_p for $p \in [n]$. Now we let,

$$K := \{i \in [n] : V(R_i) \cap V(S) \neq \emptyset\},$$

be the set of indices (of rows) which touch S. Either $|K| = k = n$, or $k < n$.

If $k = n$, then S has n vertical edges which belong to $\sigma(S)$. For every row R_i, $i \neq p$, $R_i \cap \sigma(S) \neq \emptyset$ since $k = n$. Hence,

$$B'(G) = B'(\eta) \geq |\sigma(S)| \geq n + n - 1 = 2n - 1.$$

Suppose now that $k < n$. Let $Y := \sigma^{n-k}(S)$ and $Y' := Y \setminus \sigma(S)$. We wish to find a lower bound for Y, so we partition Y into three disjoint sets and estimate the size of each separately.

$$Y = \left(\bigcup_{j \in [n]} (Y \cap C_j) \right) \cup \left(\bigcup_{j \in [n-1]} (Y' \cap C'_j) \right) \cup \left(\bigcup_{j \in [n]} (\sigma(S) \cap R_j) \right).$$

First, we see that there are at least $n-k$ vertices of $V(C_j)$ which do not belong to $V(S)$. We observe that the vertices of $V(C_j) - V(S)$ form paths. If a path P has m vertices, it has $m-1$ internal edges. In addition, it has at least one edge with end in $V(P)$ and one end in $V(S)$. Hence each path contributes m edges. Thus, $|C_j - S| \geq n - k$ and it then follows that

$$|Y \cap C_j| \geq n - k. \tag{8}$$

Consequently,

$$\left| \bigcup_{j \in [n]} (Y \cap C_j) \right| \geq (n-k)n.$$

Second, we wish to show that $|Y' \cap C'_j| \geq n - k - 1$ for $j \in [n-1]$. We observe that C'_j has at least $n-k$ elements which do not touch S. We also see that none of these edges belong to $\sigma(S)$, otherwise they would touch S. We also see that $|\sigma^m(S) \cap C'_j| \geq m - 1$. But since none of the $n-k$ elements of C'_j belong to $\sigma(S)$, $|Y' \cap C'_j| \geq n - k - 1$. As a result,

$$\left| \bigcup_{j \in [n-1]} (Y' \cap C'_j) \right| \geq (n-k-1)(n-1) \tag{9}$$

Finally, we consider $\sigma(S) \cap R_j$. In each of the k rows which touch S, $R_j - S \neq \emptyset$

for $i \neq p$. Thus,

$$\left| \bigcup_{j \in [n]} (\sigma(S) \cap R_j) \right| \geq k - 1. \tag{10}$$

Combining these estimations (8), (9), and (10), we get,

$$\begin{aligned} |Y| &\geq (n-k)n + (n-k-1)(n-1) + (k-1) \\ &= (n-k)n + (n-k)(n-1) - (n-1) + k - 1 \\ &= (n-k)n + (n-k)n - (n-k) - n + k \\ &= (n-k)(2n-2). \end{aligned}$$

Which yields $B'(G) \geq B'(\eta) \geq |Y|/(n-k) = 2n-2$. □

The similar idea of considering the l-neighborhood was used to establish the proper growth of triangular grids. The vertices of a triangular grid T_l consist of $V := \{(a,b,c) \in \mathbb{Z}^3 : a,b,c \geq 0 \text{ and } a+b+c = l\}$ and we say two vertices $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ are adjacent if and only if they agree in one coordinate but differ by one in every other coordinate. Akhtar, Jiang, and Pritikin determined the following bounds for T_l in [17]:

Theorem 5.

$$3n - o(n) \leq B'(T_l) \leq 3n - 1.$$

We also add that the bandwidth of triangular grids were determined exactly in [6], where Hochberg, McDiarmid, and Saks showed that:

Theorem 6.

$$B(T_l) = l + 1.$$

In this paper, we relaxed the adjaceny condition of T_l and say that the *closure* of a T_l graph, T_l^* has the same vertex set, but two vertices are adjacent if they agree in one coordinate but differ in every other coordinate (not just one).

3 T_l^* Graphs

We now go on to establish lower bounds for $B'(T_l^*)$, but first we will need some small observations.

Lemma 4. *For every vertex v,*

$$d(v) = 2l \tag{11}$$

Proof. We observe that given a vertex $x = (x_1, x_2, x_3)$, the number of vertices adjacent to x by agreement in the first coordinate, second coordinate, and third coordinate is $l - x_1$, $l - x_2$, and $l - x_3$ respectively. Thus, $d(x) = 3l - (x_1 + x_2 + x_3) = 2l = \Delta$. □

Given a set $S \subseteq V$ of vertices, we let $E(S) := \{e \in E : V(e) \subseteq S\}$ denote the internal edges of S, call $\Theta(S) := \{e \in E : V(e) \cap S \neq \emptyset, V(e) \cap V - S \neq \emptyset\}$ the edge-boundary of S, and say $g(S) := |E(S)| + |\Theta(S)|$ is the number of edges with ends in S.

Lemma 5. *Given a T_l^* graph, let $S_k := \{v \in V : v_1 = 0$ and $v_2 \leq k-1\}$ for $1 \leq k \leq l+1$ and $T \subseteq V$. Then we have,*

$$g(S_k) = \min_{\substack{T \\ |T|=k}} g(T). \tag{12}$$

Thus, no other arrangement T of k vertices has less edges with ends in T than S_k.

Proof. We proceed by induction on k. If $k = 1$, then S_k is a single vertex. Suppose the lemma holds for $k = n$, $1 \leq n \leq l$. Now consider S_k, for $k = n+1$. Let $T \subseteq V$ be an arbitrary set of $n+1$ vertices. Choose a vertex v in T. By the inductive hypothesis,

$$g(S_{n+1} \backslash \{(0, n, l-n)\}) = g(S_n) \leq g(T \backslash \{v\}).$$

Then by definition and the fact that every vertex has degree $2l$,

$$\begin{aligned}
g(S_{n+1}) &= |E(S_{n+1})| + |\Theta(S_{n+1})| \\
&= \Big[|E(S_{n+1})| + \big(|E(S_n)| - |E(S_n)|\big)\Big] + \\
&\quad \Big[|\Theta(S_n)| + 2l - 2\big(|E(S_{n+1})| - |E(S_n)|\big)\Big] \\
&= |E(S_n)| + |\Theta(S_n)| + 2l - \big(|E(S_{n+1})| - |E(S_n)|\big) \\
&= g(S_n) + 2l - \big(|E(S_{n+1})| - |E(S_n)|\big) \\
&\leq g(T\backslash\{v\}) + 2l - \big(|E(S_{n+1})| - |E(S_n)|\big) \\
&= g(T\backslash\{v\}) + 2l - n \\
&\leq g(T\backslash\{v\}) + 2l - \big(|E(T)| - |E(T\backslash\{v\})|\big) \\
&= |E(T\backslash\{v\})| + |\Theta(T\backslash\{v\})| + 2l - \big(|E(T)| - |E(T\backslash\{v\})|\big) \\
&= \Big[|E(T)| + \big(|E(T\backslash\{v\})| - |E(T\backslash\{v\})|\big)\Big] + \\
&\quad \Big[|\Theta(T\backslash\{v\})| + 2l - 2\big(|E(T)| - |E(T\backslash\{v\})|\big)\Big] \\
&= |E(T)| + |\Theta(T)| \\
&= g(T).
\end{aligned}$$

□

We then get a corollary which asserts that a "triangle", $V(T_{l-1}^*)$ is the best way to arrange $l(l+1)/2$ vertices to maximize the number of its internal edges.

Corollary 1. *Given a T_l^* graph, let $S := \{v \in V : v_1 \geq 1\}$ and $T \subseteq V$, then*

$$|E(S)| = \max_{\substack{T \\ |T|=|S|}} |E(T)|. \tag{13}$$

Proof. Consider an arbitrary set $T \subseteq V$ of $|S|$ vertices. We see that

$$|E(S)| = |E| - (|E(V-S)| + |\Theta(V-S)|) = |E| - g(V-S).$$

Since $V - S = \{v \in V : v_1 = 0\}$, $V - S$ consists of $l+1$ vertices and so does $V - T$. By (12),

$$|E(S)| = |E| - g(V-S) \geq |E| - g(V-T) = |E(T)|.$$

□

Finally, we obtain lower bounds of quadratic growth on the edge-bandwidth of T_l^* graphs:

Theorem 7.

$$l^2 + 2l - 1 \leq B'(T_l^*).$$

Proof. First, we consider an arbitrary labelling g, of the edges of T_l^*. We observe that the edges labelled $1, 2, 3, \ldots, \binom{l}{2} + 1$ must cover $l + 1$ vertices, since a complete graph maximizes the number of internal edges and a complete graph of l vertices has $\binom{l}{2}$ edges. We see that since every vertex has degree $2l$,

$$|E| = \frac{1}{2}\sum_{v \in V} d(v) = \left(\frac{1}{2}\right)\frac{(l+1)(l+2)}{2}2l = \frac{l(l+1)(l+2)}{2}$$

Also by (13), any arrangement of $l(l+1)/2$ vertices has no more than $(l-1)(l)(l+1)/2 = |E(T_{l-1}^*)|$ internal edges. Hence, the edges labelled $|E|$, $|E| - 1, \ldots, |E| - [(l-1)(l)(l+1)/2 + 1] + 1$ must cover $l(l+1)/2 + 1$ vertices. Thus at least one of the edges labelled $1, \ldots, \binom{l}{2} + 1$ must share a vertex with one of the edges labelled $|E|, \ldots, |E| - [(l-1)(l)(l+1)/2 + 1] + 1$, and we have,

$$\begin{aligned}
B'(T_l^*) &\geq B'(g) \\
&\geq |E| - \frac{(l-1)(l)(l+1)}{2} - 1 + 1 - \left(\binom{l}{2} + 1\right) \\
&= \frac{l(l+1)(l+2)}{2} - \frac{(l-1)(l)(l+1)}{2} - \frac{l(l-1)}{2} - 1 \\
&= l\frac{(l+1)(l+2) - (l-1)(l+1) - (l-1)}{2} - 1 \\
&= l\frac{l^2 + 3l + 2 - (l^2 - 1) - l + 1}{2} - 1 \\
&= l\frac{2l + 4}{2} - 1 \\
&= l^2 + 2l - 1
\end{aligned}$$

□

4 Classification of Graphs by Edge-Bandwidth

A natural question arises when considering edge-bandwidth: given a positive integer k, what do graphs of edge-bandwidth k look like? We first examine edge-bandwidth 1 graphs.

Theorem 8. *If G is a simple graph, $B'(G) = 1$ if and only if G consists of a union of vertex disjoint paths.*

Proof. (⇐) Label each path P individually so that $B'(P) = 1$.

(⇒) We say that two vertices u and v are connected if and only if there is a path in G withs ends u and v. This defines an equivalence relation on V. We call an equivalence class under this relation a component of G.

Consider C, an arbitrary component of G. The degree of every vertex in C

must be less than three, since otherwise $B'(G) > 1$. If C has any vertices of degree zero, then C consists of a single vertex. We also see that C cannot have all vertices degree two, then C is a cycle and again $B'(G) > 1$. Thus if C is not a single vertex, then C has some vertices degree one. Let P be a path of maximum length. The internal vertices of P are degree two, and are not adjacent to any other vertices. The ends of P are not adjacent to any other vertices since otherwise P can be extended, contradicting the maximum length of P. Thus $P = C$ and every component is a path. □

We see that as the edge-bandwidth increases, the graphs quickly become more complicated and difficult to describe. We should however, at least be able to determine all edge-bandwidth 2 graphs. For larger k, it would be interesting to find a non-trivial property which must exist in common among all edge-bandwidth k graphs. We already know some simple aspects of the edge-bandwidth k graphs which are guaranteed, such as bounds on the maximum degree.

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