Spanning tree congestion critical graphs

Daniel Tanner

August 24, 2007

Abstract

The linear or cyclic cutwidth of a graph G is the minimum congestion when G is embedded into either a path or a cycle respectively. A graph is cutwith critical if it is homeomorphically minimal and all of its subgraphs have lower cutwitdth. Our purpose is to extend the study of congestion critical graphs to embeddings on spanning trees.

1 Introduction

We let $G = (V_G, E_G)$ represent a graph with vertex set V_G and edge set E_G. Edges of a graph connect pairs of vertices. For the purposes of this paper we define a notational convention. For graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, we define $G \cup H$ as a graph such that

$$G \cup H = (V_G \cup V_H, E_G \cup E_H).$$

An edge and a vertex are *incident* if the vertex is an end of the edge. Two vertices $u, v \in V_G$ are *adjacent* if there is an edge in G connecting them. We define the *degree* of a vertex $u \in V_G$ as the number of vertices in V_G that are adjacent to u:

$$d_u = |\{v \in V_G : (u, v) \in E_G\}|$$

Let G be a graph. We define the *components* of G to be some number $w(G)$ of pairwise-disjoint connected subgraphs of G: $A_1, A_2, \ldots, A_{w(G)}$ such that $G = A_1 \cup A_2 \cup \ldots \cup A_{w(G)}$. Note that G is a connected graph if and only if $w(G) = 1$ [1].

A *path* in G is a finite sequence $P_{(v_0,v_k)} = v_0, e_0, v_1, e_1, v_2, e_2, \ldots, v_k$, whose terms are alternately distinct vertices and edges, such that for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i [1]. A *tree* is a graph in which there is exactly one path between any two vertices. Given graph G, we let T_G denote the set of all trees T such that $V_T = V_G$. A tree $T_i \in T_G$ is called a *spanning tree* of G if $E_{T_i} \subseteq E_G$. Let S_G denote the set of all spanning trees of G.

Let $T \in T_G$. Given edge $h = (u, v)$ in E_G, we let P_h be a path in T that connects vertices u and v. We call such a path on a tree T a *detour*. Since T is a tree we know that P_h is unique. Let L denote the set of all paths P_e in T where e is any edge in E_G. For $g \in E_T$, we define the *congestion of* g as the number of paths in L in which g is an element:

$$c(g, T) = |\{P_e \in L : g \in P_e\}|.$$

We then define the *congestion of* G embedded onto T as the maximum $c(g,T)$ over all edges g in E_T:

$$c(G:T) = \max\{c(g,T) : g \in E_T\}.$$

The *tree congestion* of G, denoted $t(G)$, is the minimum value of $c(G:T)$ over all trees T in T_G. Similarly, the *spanning tree congestion* of G, denoted $s(G)$, is the minimum value of $c(G:T)$ over all spanning trees S in S_G:

$$t(G) = \min\{c(G:T) : T \in T_G\},$$

$$s(G) = \min\{c(G:S) : S \in S_G\}.$$

It should be noted that in the above definitions tree congestion and spanning tree congestion are only defined on connected graphs. For the purposes of this paper it will be necessary to extend the domain of these functions to disconnected graphs. An obvious adjustment is to use the maximum value of the individual spanning tree congestions of each disconnected component. We define:

$$t(G) = \begin{cases} \min\{c(G,T) : T \in T_G\} & \text{if } G \text{ is connected,} \\ \max\{t(A_i) : i \in \{1,2,\ldots,w(G)\}\} & \text{if } G \text{ is disconnected,} \end{cases}$$

and

$$s(G) = \begin{cases} \min\{c(G,S) : S \in S_G\} & \text{if } G \text{ is connected,} \\ \max\{s(A_i) : i \in \{1,2,\ldots,w(G)\}\} & \text{if } G \text{ is disconnected.} \end{cases}$$

Graphs G and G' are said to be *homeomorphic* if they can both be obtained from some graph H by subdividing its edges (inserting new vertices of degree two into its edges). Both G and G' are said to be subdivisions of graph H.

Let $G = (V_G, E_G)$ be a simple connected graph with vertex set V_G and edge set E_G. G is said to be k*-spanning tree congestion critical* if:

1. $s(G) = k$
2. If G' is a proper subgraph of G then $s(G') < k$
3. G is homeomorphically minimal, that is, G is not a subdivision of any simple graph.

2 Preliminary theorems

Theorem 1. *Let A_1 and A_2 be graphs where $s(A_1) = n$, $s(A_2) = m$ and $m \leq n$. Let A be the graph formed when A_1 and A_2 are pendantly attached at vertex x; that is, A is a graph formed when A_1 and A_2 are made to share exactly one vertex, x. Then $s(A) = n$.*

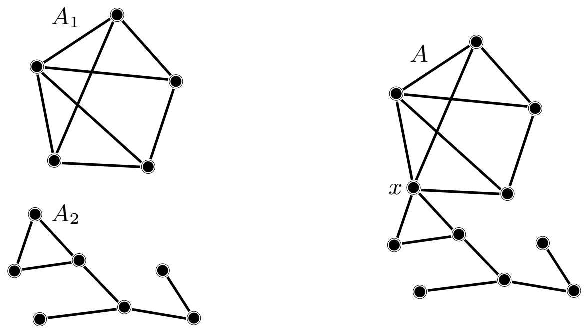


Figure 1: *An example of A_1 and A_2 being attached pendantly at vertex x.*

Proof. Let A_1 and A_2 be graphs where $s(A_1) = n$, $s(A_2) = m$ and $m \leq n$. Let A be the graph formed when A_1 and A_2 are pendantly attached at vertex x. Let $T_1 \in S_{A_1}$ and $T_2 \in S_{A_2}$ such that $c(A_1 : T_1) = n$ and $c(A_2 : T_2) = m$. Let $T^* = T_1 \cup T_2$.

In A there are no edges from a vertex in $V_{A_1} \setminus \{x\}$ to a vertex in $V_{A_2} \setminus \{x\}$ so in any $T \in S_A$ there are no detours from a vertex in $V_{A_1} \setminus \{x\}$ to a vertex in $V_{A_2} \setminus \{x\}$. Then for any edge $e \in E_{A_i}$, $i = 1, 2$, we know that $c(e : T)$ depends only on the edges and vertices in A_i. Then $c(A : T^*) = \max\{s(A_1), s(A_2)\} = n$. There also can not be a spanning tree of A that yields a congestion lower than n because then there would be a subtree spanning A_1 that would yield congestion less than n. Then $s(A) = n$. $\square$

Corollary 1. *Let G be a k-spanning tree congestion critical graph. Then G cannot be formed by pendantly attaching two non-trivial subgraphs of G. More technically, for all $x \in V_G$, $G' = (V_G \setminus \{x\}, E_G \setminus \{(x, v) : v \in V_G\})$ is a connected graph.*

Proof. By contrapositive suppose that G is a graph that can be formed by pendantly attaching A_1 and A_2, two non-trivial subgraphs of G. Without loss of generality let $s(A_1) \geq s(A_2)$. Then by Theorem 1 $s(G) = s(A_1)$. Since A_1 and A_2 are both non-trivial they most also both be proper subgraphs of G. Then G is not a spanning tree congestion critical graph. $\square$

Theorem 2. *Let G and G' be homeomorphic graphs such that subdividing a single edge of G yields G'. Then $s(G) = s(G')$.*

Proof. Let G be a graph and let $(a_1, a_2) \in E_G$.
Let $G' = (V_G \cup \{x\}, E_G \cup \{(a_1, x), (x, a_2)\} \setminus \{(a_1, a_2)\})$.

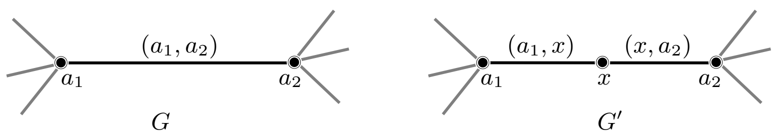


Figure 2: *The edge $(a_1, a_2) \in E_G$ is subdivided to create G'.*

Claim 1: Let $B \in S_G$ where $(a_1, a_2) \in E_B$. Let $B' \in S_{G'}$ where
$B' = (V_{G'}, E_B \cup \{(a_1, x), (x, a_2)\} \setminus \{(a_1, a_2)\})$. Then $c(G', B') = c(G, B)$.

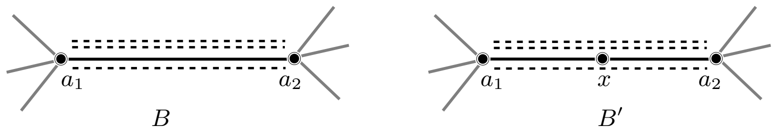


Figure 3: *Spanning trees B of G and B' of G'. Detours containing both vertices a_1 and a_2 are represented by dotted lines.*

As we see in Figure 3 above, by subdividing (a_1, a_2) we are not changing the number of detours containing both vertices a_1 and a_2, nor are we adding any detours through other edges. Thus $c((a_1, x) : B') = c((x, a_2) : B') = c((a_1, a_2) : B)$ and for all other edges $e \in E_B \setminus \{(a_1, a_2)\}$, $c(e, B) = c(e, B')$. Then $c(G', B') = c(G, B)$ so Claim 1 holds.

Claim 2: Let $B \in S_G$ where $(a_1, a_2) \notin E_B$. Let $B' \in S_{G'}$ where $B' = (V_{G'}, E_B \cup \{(a_1, x)\}$. Then $c(G', B') = c(G, B)$.

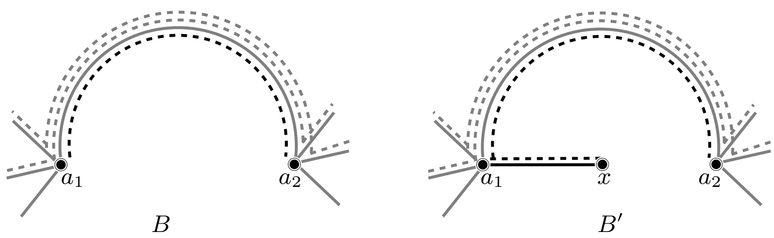


Figure 4: *Spanning trees B of G and B' of G'. The solid arc represents the uniqe path in each tree from a_1 to a_2. Detours containing both vertices a_1 and a_2 are represented by dotted lines.*

In this case G must have a cycle so $s(G) \geq 2$. We can see that $c((a_1, x) : B') = 2 \leq s(G) \leq c(B, G)$. As we see in Figure 4, the only detour that is different is the black dotted one, which in B' just contains additional edge (a_1, x) and vertex x. Then for all edges $e \in E_B$, $c(e, B) = c(e, B')$. Then $c(G', B') = c(G, B)$ so Claim 2 holds.
We now return to the main proof of Theorem 2.

Want to show: $s(G') \leq s(G)$
Let $T \in S_G$ such that $c(T : G) = s(G)$.

Case 1: $(a_1, a_2) \in E_T$
Consider $T' \in S_{T'}$ where $T' = (V_{G'}, E_T \cup \{(a_1, x), (x, a_2)\} \setminus \{(a_1, a_2)\})$. By Claim 1 $c(G' : T') = c(G : T)$. Then $s(G') \leq c(G' : T') = c(G : T) = s(G)$ so $s(G') \leq s(G)$.

Case 2: $(a_1, a_2) \notin E_T$
Consider $T' \in S_{T'}$ where $T' = (V_{G'}, E_T \cup \{(a_1, x)\})$. By Claim 2 $c(G' : T') = c(G : T)$. Then $s(G') \leq c(G' : T') = c(G : T) = s(G)$ so $s(G') \leq s(G)$.

Want to show: $s(G') \geq s(G)$
By condtradiction suppose that there exists some $T' \in S_{G'}$ such that $c(G' : T') < s(G)$.

Case 1: $(a_1, x), (x, a_2) \in E_{T'}$

Consider $T \in S_G$ where $T = (V_G, E_{T'} \cup \{(a_1, a_2)\} \setminus (a_1, x), (x, a_2))$. Then by Claim 1 $c(G, T) = c(G', T') < s(G)$. $\Rightarrow\Leftarrow$

Case 2: Without loss of generality $(a_1, x) \in E_{T'}$ and $(x, a_2) \notin E_{T'}$

consider $T \in S_G$ where $T = (V_G, E_{T'} \setminus \{(a_1, x)\})$. Then by Claim 2 $c(G, T) = c(G', T') < s(G)$. $\Rightarrow\Leftarrow$

Then $s(G') = s(G)$. $\square$

Corollary 2. *Let G and G' be finite connected graphs. Let G be homeomorphic to G'. Then $s(G) = s(G')$.*

Proof. Let G and G' be finite connected graphs. Let G be homeomorphic to G'. Then G and G' are both homeomorphic to some homeomorphically minimal graph H. G can be turned into H by undoing a finite number of edge subdivisions, each of which yields a new graph that by Theorem 2 has the same spanning tree congestion as G. Then we know that $s(H) = s(G)$. Similarly $s(H) = s(G')$. Then $s(G) = s(G')$. $\square$

Theorem 3. *Let G be an acyclic graph. Then $s(G) \leq 1$.*

Proof. Let G be an acyclic graph. Then each component of G is a tree. If $|E_G| = 0$ then the components of G are single vertices and $s(G) = 0$. If $|E_G| > 0$ then there exists some non-trivial tree A_i that is a component of G. Then $s(A_i) = 1$ so $s(G) = 1$. $\square$

Theorem 4. *Let G be a unicyclic graph. Then $s(G) = 2$.*

Proof. Let G be a unicyclic graph. Let T be a spanning tree of G. Then T has exactly one detour of length greater than one. Then there is at least one edge in E_T that is contained in more than one detour and there are no edges in E_T that are contained in more than two detours. Then for all $T \in S_G$, $c(G, T) = 2$. Then $s(G) = 2$. $\square$

From [2] we have complete results for spanning tree congestion of complete n-partite graphs. The main results of that work follow:

Theorem 5. *For $G = K_{a_1, a_2, \ldots, a_n}$, with $0 < a_1 \leq a_2 \leq \ldots \leq a_n$, $|A_1| = a_1$, $|A_2| = a_2, \ldots$, $|A_n| = a_n$, and $n \geq 2$,*

$$s(G) = \begin{cases} 1 + a_3 + a_4 + \ldots + a_n & \text{if } a_1 = 1; \\ 2(a_1 + a_2 + \ldots + a_{n-2}) + a_{n-1} + a_n - 2 & \text{if } a_1 \geq 1. \end{cases}$$

3 Identification of spanning tree congestion critical graphs

In this section we will begin identifying all k-spanning tree congestion critical graphs for small k.

3.1 1-spanning tree congestion critical graphs

Proposition. *The following graph, $K_{1,1}$, is the only 1-spanning tree congestion critical graph.*

Figure 5: $K_{1,1}$.

$K_{1,1}$ obviously meets the conditions of being 1-spanning tree congestion critical. Any graph with an empty edge set has a spanning tree congestion of zero. Then we can assume that if there were any other 1-spanning tree congestion critical graph it would have at least one edge. Suppose G is a graph where $|E_G| \geq 1$ and $G \neq K_{1,1}$. Then $K_{1,1}$ is a proper subgraph of G and since $s(K_{1,1}) = 1$, G does not meet condition 2) of being 1-spanning tree congestion critical.

3.2 2-spanning tree congestion critical graphs

Proposition. *The following graph, $K_{1,1,1}$, is the only 2-spanning tree congestion critical graph.*

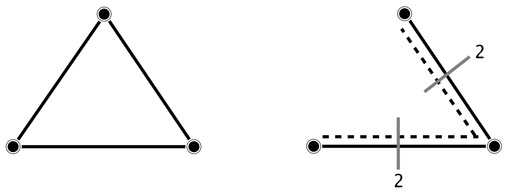


Figure 6: $K_{1,1,1}$ *and its only possible spanning tree. Dotted lines represent detours of length greater than one and grey dashes identify edge congestion.*

$K_{1,1,1}$ is homeomorphically minimal and $s(K_{1,1,1}) = 2$. If G' is a proper subgraph of $K_{1,1,1}$ then G' is acyclic so by Theorem 3 $s(G') \leq 1 < s(G)$. Then $K_{1,1,1}$ is 2-spanning tree congestion critical.

Suppose G is a 2-spanning tree congestion critical graph. By Theoerem 3 we know that G is cyclic. Then G has a cycle as a subgraph, call it C. Any cycle is homeomorphic to $K_{1,1,1}$ so by Corollary 2 $s(C) = 2$. Then we know that C is not a proper subgraph of G. Then $G = C$ and since $K_{1,1,1}$ is the only homeomorphically minimal cycle, $G = K_{1,1,1}$.

3.3 3-spanning tree congestion critical graphs

Proposition. *The following graph, $K_{1,1,2}$, is the only 3-spanning tree congestion critical graph.*

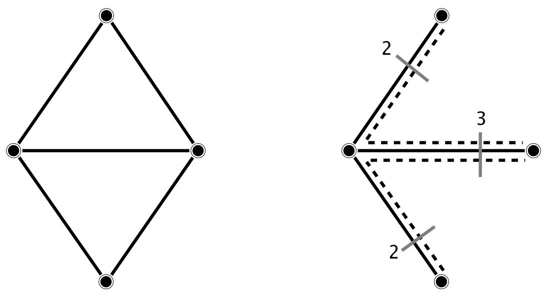


Figure 7: $K_{1,1,2}$ *and its optimal spanning tree. Dotted lines represent detours of length greater than one and grey dashes identify edge congestion.*

$K_{1,1,2}$ is clearly homeomorphically minimal and by Theorem 5 we know that $s(K_{1,1,2}) = 3$. Let G be a proper subgraph of $K_{1,1,2}$. Then either G is acyclic or G is unicyclic. If G is acyclic then by Theorem 3 $s(G) \leq 1 < s(K_{1,1,2})$. If G is unicyclic then by Theorem 4 $s(G) = 2 < s(K_{1,1,2})$. Then $K_{1,1,2}$ is 3-spanning tree congestion critical.

Theorem 6. *Let G be a k-spanning tree congestion critical graph where $3 \leq k$. Then there exists some subgraph G' of G such that G' is homeomorphic to $K_{1,1,2}$.*

Proof. Let G be a k-spanning tree congestion critical graph where $k \geq 3$. By Theorems 3 and 4 we know that G must be polycyclic. By Corollary 1 we know that there must exist two cycles in G that share at least one edge. Call these cycles C_1 and C_2. If C_1 and C_2 share exactly one edge then $C_1 \cup C_2$ is homeomorphic to $K_{1,1,2}$. Suppose C_1 and C_2 share more than one edge.

Case 1: C_1 and C_2 share only consecutive edges.

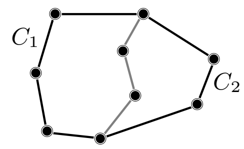


Figure 8: *An example of a 7-cycle and a 6-cycle sharing 3 consecutive edges. Shared edges are colored grey.*

Then $C_1 \cup C_2$ is homeomorphic to $K_{1,1,2}$.

Case 2: C_1 and C_2 share non-consecutive edges.

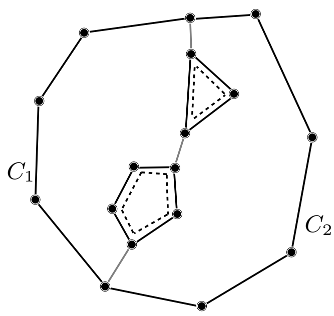


Figure 9: *An example of C_1 and C_2 sharing non-consecutive edges. Shared edges are colored grey. The dotted lines denote new cycles.*

Then there must be some cycle C_3 (one of the dotted cycles in Figure 9) such that C_1 and C_3 share only consecutive edges. Then $C_1 \cup C_3$ is homeomorphic to $K_{1,1,2}$. $\square$

Let G be a 3-spanning tree congestion critical graph. Then by Theorem 6 there exists some subgraph of G, call it G', such that G' is homeomorphic to $K_{1,1,2}$. Then by Corollary 2 $s(G') = 3$. Then $G = K_{1,1,2}$. Then $K_{1,1,2}$ is the only 3-spanning tree congestion critical graph.

3.4 4-spanning tree congestion critical graphs

Proposition. *The graphs in figures 10, 11, and 12 are* 4*-spanning tree congestion critical.*

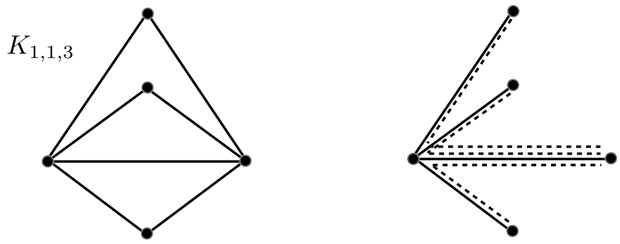


Figure 10: *Graph $K_{1,1,3}$ with a congestion minimizing spanning tree.*

$K_{1,1,3}$ is clearly homeomorphically minimal and by Theorem 5 we know that $s(K_{1,1,3}) = 4$. It is then left to show that all proper subgraphs of $K_{1,1,3}$ have lower spanning tree congestion. We can think of $K_{1,1,3}$ as two degree four vertices with four edge disjoint paths between them. The reason $s(K_{1,1,3}) = 4$ is that in any spanning tree three of these paths have to have detours through the fourth. Any proper subgraph of $K_{1,1,3}$ will have fewer paths between these two vertices, or one or both of the vertices will be nonexistent. Then any proper subgraph of $K_{1,1,3}$ has a lower spanning tree congestion. Then $K_{1,1,3}$ is 4-spanning tree congestion critical.

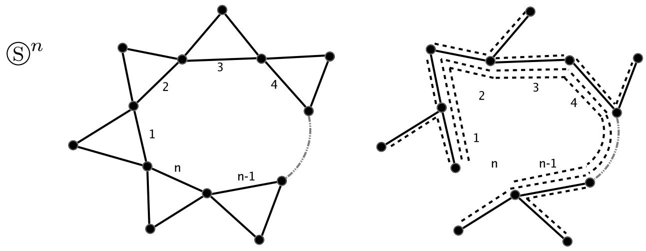


Figure 11: *A star* Ⓢ^{n} *with a congestion minimizing spanning tree.*

We construct a star Ⓢ^{n} with n three cycles, each of which share one edge of an n-cycle. In any spanning tree of Ⓢ^{n} where $n \geq 3$, one of the added outside 3-cycles must be missing both an outside edge and the edge shared with the n-cycle. Otherwise we would have a cycle, not a tree. As we can see in Figure 11, this sends two detours all the way around the inside of the rest of the spanning tree. Since $n \geq 3$ there must be some other 3-cycle that also must be missing an edge in the spanning tree and thus has a detour along its two remaining edges. Then one of these two edges has to have a congestion of four, so $s($Ⓢ$^n) = 4$.

The symmetry of Ⓢ^{n} is the reason it is critical. If, to make a subgraph of Ⓢ^{n}, any one edge were removed, we could create a spanning tree where only one detour is sent all the way around the inside. This would give us a congestion of three. Removing any more edges or vertices will either allow for a spanning tree with even fewer detours around the inside, or fewer three cycles adding to detours around the outside. Then all proper subgraphs of Ⓢ^{n} are going to have congestion of three or less. Then Ⓢ^{n} where $n \geq 3$ is 4-spanning tree congestion critical.

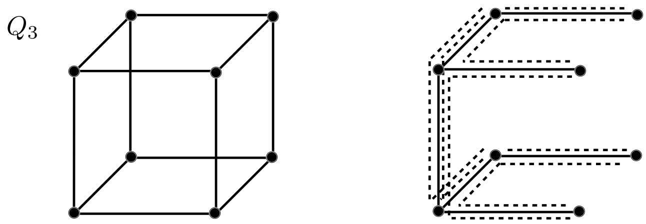


Figure 12: *Q_3 with a congestion minimizing spanning tree.*

By [6] we know that $s(Q_3) = 4$. Figure 12 shows an example of a minimizing spanning tree. In it there are three edges with congestion four. There are two edges in E_{Q_3} whose representative detours in this spanning tree both contain all three of these edges. If any one edge were removed to form a subgraph of Q_3, the symmetries of the cube would allow us to draw this same spanning tree in such a way that one of the two above mentioned edges was the removed edge. This would reduce the congestion on the three edges where it is highest, and reduce the overall spanning tree congestion of the subgraph to three. This may give the reader an intuitive feeling that *any* subgraph of Q_3 has spanning tree congestion less than four. The method used for proving this was to exhaustively examine the subgraphs of Q_3. The symmetries of the cube combined with the theorems in this paper allow us to reduce this to a manageable number of cases but the proof is omitted here.

4 Further exploration

The main focus of this paper has been to identify all of the k-spanning tree congestion critical graphs for small k. In this section we will examine other interesting results that could be potentially useful in further exploration of spanning tree congestion critical graphs.

4.1 Infinite families of spanning tree congestion critical graphs

We have shown that for $1 \leq l \leq 4$ $K_{1,1,l-1}$ is l-spanning tree congestion critical. This is in fact true for all $l \geq 1$.

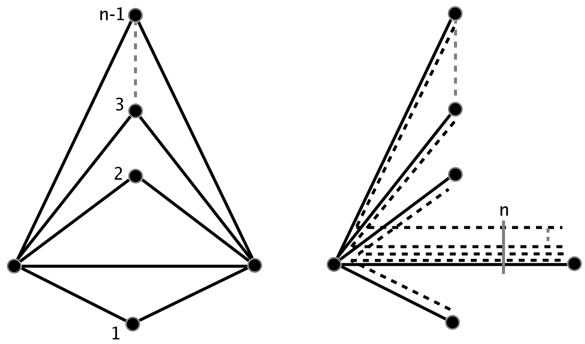


Figure 14:$K_{1,1,n-1}$.

To show this we think of $K_{1,1,l-1}$ as two degree l vertices with l edge disjoint paths between them. We then use the same argument we used to show that $K_{1,1,3}$ is 4-spanning tree congestion critical.

A more general infinite family of stars that are k-spanning tree congestion critical is known.

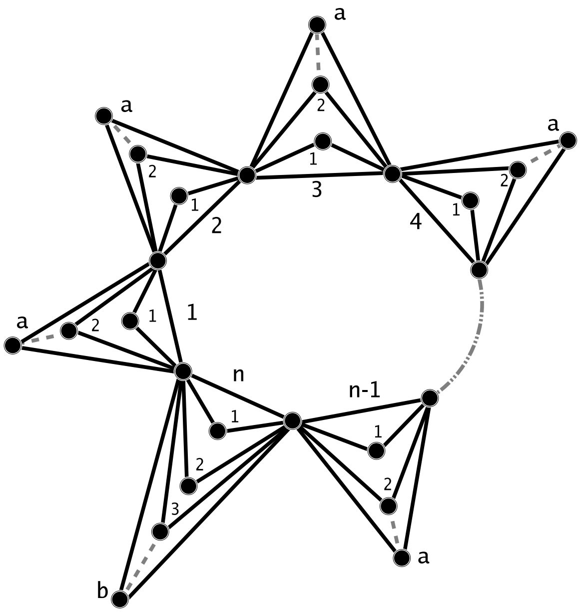


Figure 15: *A general star $\circledS^n_{a,b}$, $a \leq b$. Notice that when $n = 3$ and $a = 0$ this becomes our $K_{1,1,l-1}$ case.*

A general star $\circledS^n_{a,b}$ is constructed around an n-cycle. $n-1$ of the edges are shared with a 3-cycles and one of the edges is shared with b 3-cycles where $a \leq b$.

Proposition. *$\circledS^n_{a,b}$ where $a \leq b$ is $(a+b+2)$-spanning tree congestion critical.*

We give a non-rigorous explanation of why this is true. In any spanning tree of $\circledS^n_{a,b}$ there must be two consecutive vertices of the n-cycle that are not adjacent and do not connect through one of their shared 3-cycles (otherwise there would be a cycle so we wouldnt be looking at a spanning tree). We chose to draw a spanning tree such that the two vertices that satisfy the situation above are not the two shared in b 3-cycles. Then this will route $a+1$ detours around the inside of the spanning tree. On each of the sides with a shared 3-cycles there is an edge with another $a+1$ detours routed through it. Then the congestion of this edge is $2a+2$. On the side with b shared 3-cycles there is an edge with another $b+1$ detours through it. Then the congestion of this edge is $a+b+2$ and $s(\circledS^n_{a,b}) = a+b+2$.

We now examine subgraphs created by removing a single edge from $\circledS^n_{a,b}$. If we were to remove an edge from one of the sides with a shared 3-cycles then we could draw a spanning tree that routes only a detours around the inside, reducing the overall congestion to $(a)+(b+1)$. If we were to remove an edge from the side with b shared 3-cycles then we could draw a spanning tree that still routes $a+1$ detours around the inside, but our overall congestion would still be reduced to $(a+1)+(b)$. To finish the argument we would need to extend this to all possible subgraphs of $\circledS^n_{a,b}$. This argument would be done in a similar fashion to the one showing that $\circledS^n_{1,1}$ is 4-spanning tree congestion critical in section 3.4.

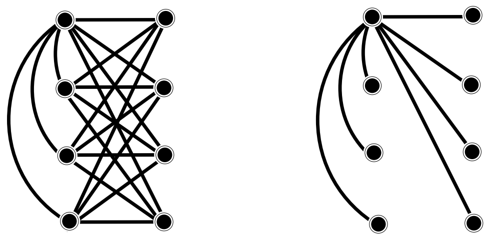
This star construction family of congestion critical graphs can be further extended. Consider m identical stars $\circledS^n_{a,a}$. Each star has vertices of degree two and vertices of degree $2a+2$.

These m stars are each made to share a vertex of degree $2a+2$ with two other stars, forming a shape like an m-gon with a star on each edge. We conjecture that this is also a spanning tree conjestion critical graph. This process could then be repeated with this new graph, always connecting some number of copies of itself at vertices of highest degree.

4.2 Subgraphs with higher spanning tree congestion

In studying critical cutwidth and critical congestion graph problems we are searching for graphs in which every subgraph has a strictly smaller congestion. It seems intuitively obvious that since removing edges and vertices from a graph removes detours from an embedding it is also going to either reduce the congestion or leave it unchanged. This is the case for both linear cutwidth [3] and cyclic cutwidth [5] but turns out not to be true for spanning tree congestion. When looking at spanning tree congestion, removing edges and vertices from a graph reduces the number of detours through a given spanning tree but it also reduces the number of possible spanning trees. This could possibly raise the spanning tree congestion rather than lower it. The following is an example of when this happens.

$K_{1,3,4}$



$K_{4,4}$

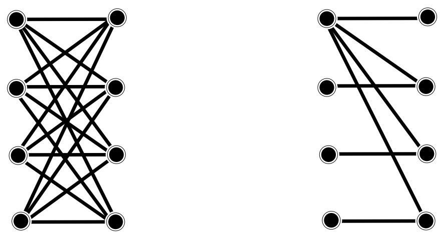


Figure 16: *$K_{4,4}$ and $K_{1,3,4}$ with congestion minimizing spanning trees* (see [2]).

In the above example $K_{4,4}$ is a subgraph of $K_{1,3,4}$ but $s(K_{4,4}) = 6$ while $s(K_{1,3,4}) = 5$ [2]. This demonstrates how complicated the process of checking criticality of a graph can get. In linear and cyclic cutwidth critical problems it is sufficient to check that removing any single edge from a graph will reduce cutwidth. In the spanning tree congestion critical problem it is necessary to check that all possible subgraphs of a graph have lower congestion.

5 Suggestions for further research

In this paper we compiled a complete list of all k-spanning tree congestion critical graphs for $k < 4$. We demonstrated three different 4-spanning tree conjestion critical graphs and conjecture that this is in fact all of them. One possible extension to this work would be to develop methods of proving that this list is complete.

Another possible area for further research is in product graphs of the form $C_n \times C_n \times \ldots \times C_n$, that is the cartesian product of the cycle C_n with itself m times. We conjecture that this graph is $2n^{m-1}$-spanning tree congestion critical.

Acknowledgements

This work was completed during the 2007 REU program in Mathematics at California State University, San Bernardino, and was jointly sponsored by CSUSB and NSF-REU Grant DMS-0453605

References

[1] J.A. Bondy, U.S.R. Murty. Graph Theory With Applications. New York: Elsevier Science Publishing Co., (1976)

[2] A. Cox. Tree Congestion for Complete n-Partite Graphs, REU Project, Cal State Univ., San Bernardino, 2007.

[3] Y. Lin, A. Yang. On 3-cutwidth critical graphs, *Discrete Mathematics* 275 (2004) 339-346.

[4] M.I. Ostrovskii. Minimal Congestion Trees, *Discrete Mathematics*. 285 (2004) 219-226.

[5] M. N. Straughn. On 3-cyclic cutwidth critical graphs, REU Project, Cal State Univ., San Bernardino, 2007.

[6] D. Tanner. Tree congestion of the n-cube, REU Project, Cal State Univ., San Bernardino, 2007.