On 3-cyclic cutwidth critical graphs

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Abstract

In this paper we will investigate 3-cyclic cutwidth critical graphs. We conjecture that the list of 3-cyclic cutwidth critical graphs has 11 elements. We also quickly look at the cyclic cutwidth of stars.

1 Introduction

1.1 Graph Theory

Graphs are used in a variety of fields from computer science to sociology. They are applied in these fields in various ways; for example, graphs can be used to represent the different ways websites are linked to one another, the structure of molecules, or to model a social networking situation. Now let us define what a graph is.

Definition 1. *A graph G is an ordered triple (V_G, E_G, ψ_G) consisting of a nonempty set V_G of vertices, a set E_G, disjoint from V_G, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G [1]. See Figure 1.*

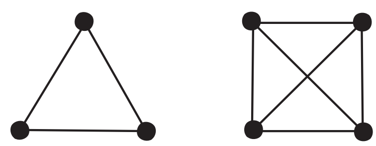


Figure 1: Two different graphs.

Definition 2. *A simple graph is a graph that does not contain any multiple edges or loops. See Figure 2.*

Definition 3. *A path, $v_0v_1v_2\dots v_n$, is a sequence of vertices such that v_i is adjacent to v_{i+1} for all $i = 0, \dots, n-1$. The number of edges in a path is called the length (here the length is n). See Figure 3.*

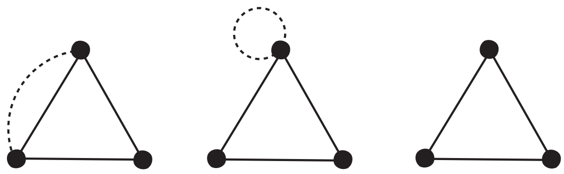


Figure 2: A graph with a multiple edge, a graph with a loop, and a simple graph.

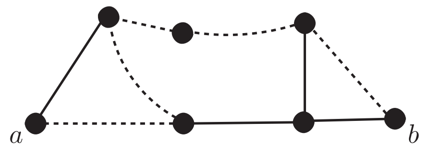


Figure 3: The dotted line illustrates a path from a to b of length 5.

Definition 4. *A graph is said to be connected if given any pair of vertices, say a, b, there exists a path from a to b. See Figure 4.*

All graphs throughout this paper will be simple and connected.

Figure 4: A connected graph, and a disconnected graph.

Definition 5. *A cycle is a closed path; that is, a path with $v_0 = v_n$. We denote a cycle of length n as C_n. C_3 is often called a triangle. In Figure 4 the graph on the left is C_3 (a triangle).*

Definition 6. *A tree is a simple connected graph with no cycles.*

Definition 7. *A vertex of degree 1 is called a pendant vertex. We may sometimes call an edge incident to a degree 1 vertex a pendant edge.*

Definition 8. *A tree is called a caterpillar if after removing all pendant vertices what remains is a single path (called the spine).*

See Figure 5.

1.2 Labeling Definitions and Examples

Definition 9. *A labeling (linear embedding) of a simple graph $G = (V, E, \psi)$ (with $|V| = n$) is a bijection $f : V \to \{1, 2, \ldots, n\}$, which can be regarded as an embedding of G into a path P_n. See Figure 6.*

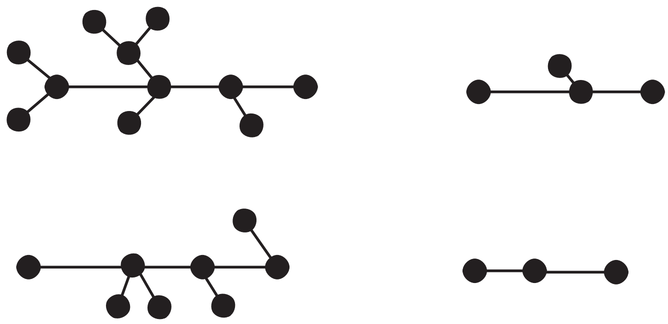


Figure 5: Two trees (top left, bottom left) and their resulting subgraphs after removing all pendant vertices (top right, bottom right). The top tree is not a caterpillar, but the bottom tree is a caterpillar.

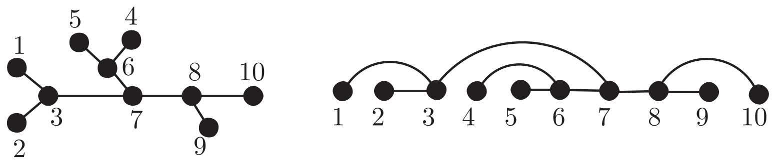


Figure 6: A tree and a (linear) labeling of it.

Definition 10. *For a given labeling f of G, the linear cutwidth of G with respect to f is*

$$lcw(G, f) = \max_{1 \leq i < n} |\{uv \in E : f(u) \leq i < f(v)\}.$$

Definition 11. *The linear cutwidth of a graph G is defined by*

$$lcw(G) = \min_{f} lcw(G, f).$$

See Figure 7 for an example of a graph labeling and its linear cutwidth.

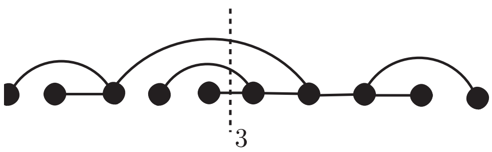


Figure 7: This labeling has a linear cutwidth of 3.

Definition 12. *A cyclic labeling, H, of a graph $G = (V, E, \psi)$, with $|V| = n$ is an ordered pair $H = (\pi, P_\pi)$ consisting of:*
(i) a bijection, $\pi : V \to C_n$, that maps the set of vertices onto the cycle with n vertices, and

(ii) a collection P_π of directed paths in H, one directed path joining $\pi(v)$ to $\pi(w)$ for each pair of adjacent vertices v and w in G. By directed path we mean a path that has a direction of either clockwise or counter-clockwise around the outside of the cycle. See Figure 8.

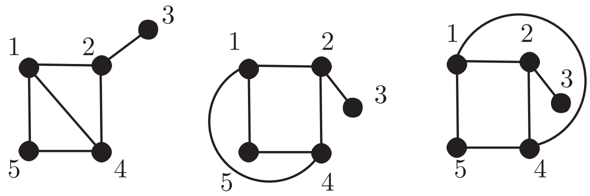


Figure 8: A graph and two different cyclic labelings of it.

Now we will give an intuitive way to find the cyclic cutwidth of a labeling, and then later define it rigorously.

If we take a line from the center of the cycle and draw it out and through a sector, the number of times the line intersects with the cyclic labeling will be the cutwidth of that sector. The cutwidth of the labeling is then the maximum of these sector cutwidths. See Figure 9.

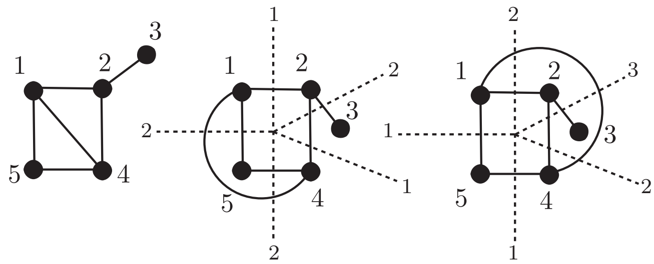


Figure 9: The same graph and two labelings as before, but now with cutwidths through each sector. Note that the cutwidth of two different labelings of the same graph can be different; in fact, here the first labeling has a cyclic cutwidth of 2, while the second labeling has a cyclic cutwidth of 3.

Definition 13. *For a given cyclic labeling H of G we define the cyclic cutwidth of G with respect to $H = (\pi, P_\pi)$, denoted by $ccw(G, H)$, as the maximum number of times an edge e of H appears in the set of paths P_π.*

Definition 14. *The cyclic cutwidth of a graph G is defined as:*

$$ccw(G) = \min\{ccw(G, H) : H \text{ is a cyclic labeling of } G\}.$$

Definition 15. *A cyclic labeling is said to be an optimal labeling if it minimizes the cyclic cutwidth.*

The graph in Figure 9 has cyclic cutwidth of 2 (this can be proven by looking at each labeling, or by noting that a degree three vertex forces a cyclic cutwidth of at least 2), and hence we would say that the first labeling is an optimal labeling.

1.3 More Definitions and Examples

Definition 16. *A graph $G' = (V', E', \psi')$ is said to be a proper subgraph of a simple connected graph $G = (V, E, \psi)$ if $V' \subseteq V$, $E' \subset E$, and if for each $e \in E'$ we have $\psi'(e) = \psi(e)$. We would also say that G is a proper supergraph of G'. See Figure 10.*

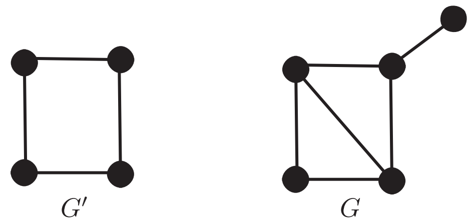


Figure 10: G' is a proper subgraph of G.

Definition 17. *By inserting new vertices of degree two into the edges of a graph G, we obtain a subdivision G'.*

Definition 18. *Two graphs are said to be homeomorphic if they are subdivisions of the same graph.*

Definition 19. *A graph is said to be homeomorphically minimal if it is not a subdivision of any simple graph.*

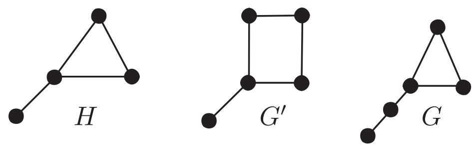


Figure 11: G and G' are subdivisions of H. Each graph is homeomorphic to the other graphs. H is homeomorphically minimal.

Definition 20. *Let G be a simple connected graph. We say that G is k-linear cutwidth critical if:*
(i) $lcw(G) = k$;
(ii) If G' is a proper subgraph of G, then $lcw(G') < lcw(G)$, and
(iii) G is homeomorphically minimal.

We also have the same idea for cyclic cutwidth.

Definition 21. *Let G be a simple connected graph. We say that G is k-cyclic cutwidth critical if:*
(i) $ccw(G) = k$;
(ii) If G' is a proper subgraph of G then $ccw(G') < ccs(G)$;
(iii) G is homeomorphically minimal.

Figure 12: K_2; and the only cyclic labeling of K_2.

1.4 1 and 2-Cyclic Cutwidth Critical Graphs

Proposition 1. *The only 1-cyclic cutwidth critical graph is K_2.*

Proof. First let us show that K_2 is 1-cyclic cutwidth critical. There is only one cyclic embedding of K_2 (using symmetry). See Figure 12.

(i): The cyclic cutwidth is 1.
(ii): Any proper subgraph will not have any edges, and hence the cyclic cutwidth will be 0.
(iii): There are no vertices with degree 2, so this property also holds.

Now let us show that K_2 is the only 1-cyclic cutwidth critical graph.

Let G be a simple connected graph, and suppose that G is 1-cyclic cutwidth critical, and that $G \neq K_2$.

So G has the following properties:
(i) $ccw(G) = 1$;
(ii) If G' is a proper subgraph of G, then $ccw(G') < 1$;
(iii) G is homeomorphically minimal.

Let E_G be the edge set of G.

Case 1: ($|E_G| > 1$) In this case K_2 will always be a subgraph of G, and hence property (ii) will not hold, a contradiction.
Case 2: ($|E_G| = 1$) Since G is simple and connected, we must have $G = K_2$, a contradiction.
Case 3: ($|E_G| = 0$) Here we have that $ccw(G) = 0$, a contradiction.

Thus K_2 is the only 1-cyclic cutwidth critical graph. □

Proposition 2. *The only 2-cyclic cutwidth critical graph is $K_{1,3}$.*

Proof. First let us show that $K_{1,3}$ is 2-cyclic cutwidth critical.
(i): If we look at the degree 3 vertex of $K_{1,3}$ and try to embed it on a cycle we will be unable to get a cyclic cutwidth of 1, but we can have $ccw(K_{1,3}) = 2$. See Figure 13.
(ii): Removing any edge of $K_{1,3}$ will give us a graph with cyclic cutwidth of 1.
(iii): There are no degree 2 vertices in $K_{1,3}$ so this property is satisfied.

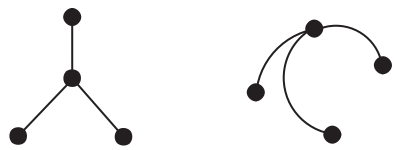


Figure 13: $K_{1,3}$; and an optimal cyclic labeling of $K_{1,3}$.

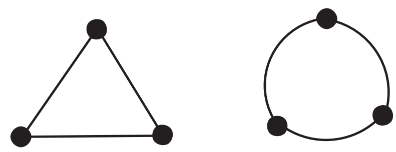


Figure 14: K_3; and a labeling of K_3 with cyclic cutwidth of 1.

Now let us prove that $K_{1,3}$ is the only 2-cyclic cutwidth critical graph.

Let G be a simple connected graph and suppose that $G \neq K_{1,3}$ and that G is 2-cyclic cutwidth critical. That is,
(i) $ccw(G) = 2$;
(ii) If G' is a proper subgraph of G, then $ccw(G') < 2$.
(iii) G is homeomorphically minimal.

Now we will prove some lemmas that will restrict the structure of our 3-cyclic cutwidth critical graph G which will help prove the proposition.

Lemma 1. *$deg(G) \leq 2$. That is, each vertex of G has degree 1 or degree 2.*

Proof. Suppose G has a vertex with degree 3 or greater. Then $K_{1,3}$ is a proper subgraph of G (remember $G \neq K_{1,3}$), but $K_{1,3}$ has cyclic cutwidth of 2, which contradicts property (ii). □

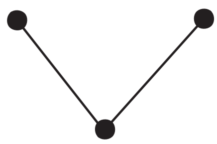
Lemma 2. *G has at least one vertex of degree 1.*

Proof. Suppose not, then each vertex in G must have degree 2. Since G is simple, G must have at least three vertices.

If G has four or more vertices then G will not be homeomorphically minimal as there will be at least one vertex of degree 2 that can be removed. So G must have exactly three vertices, and since G is connected we must have $G = K_3$. But K_3 can be labeled in a way that gives a cyclic cutwidth of less than 2. See Figure 14.

Thus G has at least one vertex of degree 1. □

Lemma 3. *G has at least one vertex of degree 2.*



G :

Figure 15: G is not homeomorphically minimal.

Proof. Suppose not; that is, each vertex of G has degree 1. Since G is simple and connected we must have that $G = K_2$, but $ccw(K_2) = 1$. Thus G has at least one vertex of degree 2. □

Lemma 4. *G has exactly one vertex of degree 2*

Proof. From the previous lemma we know that G has at least one vertex of degree 2.

Suppose G has more than one vertex of degree 2, and let a, b be any two of these degree 2 vertices.

If a, b are adjacent then G is not homeomorphically minimal, so a, b must not be adjacent. Thus we have that a is adjacent to two degree 1 vertices, and the same is true for b. However, this means we must have at least two connected components and hence G is not connected, a contradiction. Hence G has exactly one vertex of degree 2. □

So we know that G has exactly one vertex of degree 2, no vertex of degree greater than 3, and since G is simple and connected we must have two vertices of degree 1. However, this means that G is not homeomorphically minimal. See Figure 15.

Therefore $K_{1,3}$ is the only graph that is 2-cyclic cutwidth critical. □

1.5 More Cyclic Cutwidth Critical Graphs

In [4] we see that K_2 and K_n (where n is a multiple of 4) are the only complete cyclic cutwidth critical graphs. Specifically, K_2 is 1-cyclic cutwidth critical, and, for n a multiple of 4, K_n is $\frac{n^2+8}{8}$-cyclic cutwidth critical.

A generalization can also be made for stars. S_n, the star with n vertices, is a tree with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1. See Figure 16.

We now give the cyclic cutwidth of stars and then show that for n even S_n is $\frac{n}{2}$-cyclic cutwidth critical.

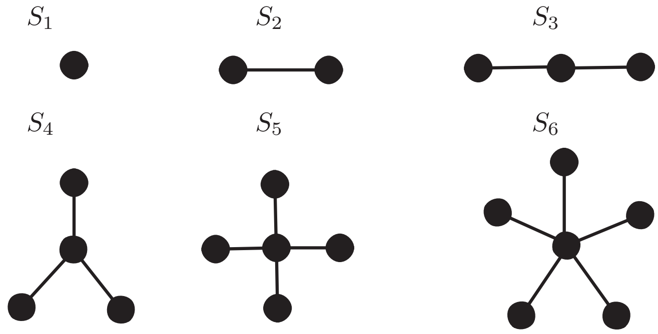


Figure 16: S_1 through S_6

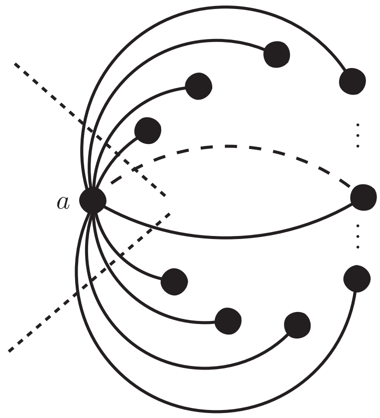


Figure 17: S_n for n even has cyclic cutwidth of $\frac{n}{2}$

Proposition 3.

$$ccw(S_n) = \begin{cases} \dfrac{n}{2} & \textit{for } n \textit{ even} \\ \dfrac{n-1}{2} & \textit{for } n \textit{ odd} \end{cases}$$

Proof. Suppose n is even. Let a be the vertex in S_n with $deg(a) = n - 1$. Now draw a in a cycle with the other $n - 1$ vertices. Suppose that $ccw(S_n) < \frac{n}{2}$. Since a is adjacent to each of the other $n - 1$ vertices, we must have each edge either contributing to the cyclic cutwidth on the left of a, or on the right of a. However, there are n edges, meaning that if one side has less than $\frac{n}{2}$ then the other side must have more than $\frac{n}{2}$, a contradiction. We can embed S_n such that $ccw(S_n) = \frac{n}{2}$ giving us the equality. See Figure 17.

Now suppose that n is odd. Again let a be the vertex in S_n with $deg(a) = n-1$, and draw a in a cycle with the other $n - 1$ vertices. Again we use the same idea. Suppose $ccw(S_n) < \frac{n-1}{2}$. Now if we have fewer than $\frac{n-1}{2}$ contributing to

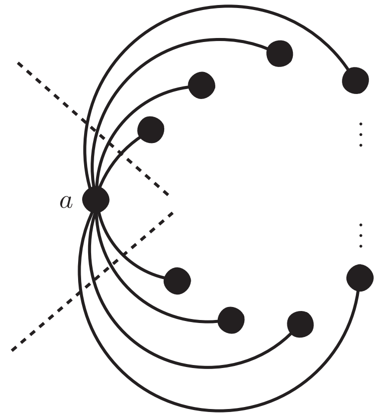


Figure 18: S_n for n odd has cyclic cutwidth of $\frac{n-1}{2}$

the cyclic cutwidth on the left of a, we get that the cyclic cutwidth on the right of a is more than $\frac{n-1}{2}$ (and vice versa). We can, however, embed S_n such that $ccw(S_n) = \frac{n-1}{2}$ giving us the other equality. See Figure 18. □

Corollary 1. *For n even, S_n is $\frac{n}{2}$-cyclic cutwidth critical.*

Proof. From the previous proposition we get property (i).

For property (ii) consider the following. Removing an edge, or a degree 1 vertex, from a star gives us a smaller star, and from the previous proposition we know that for n even, $ccw(S_n) = \frac{n}{2}$ and $ccw(S_{n-1}) = \frac{(n-1)-1}{2} < \frac{n}{2}$. If we remove the degree $n-1$ vertex we will have no edges and the remaining cyclic cutwidth will be 0. Thus we have property (ii).

Property (iii) is clear because the only star with a degree 2 vertex is S_3 (which we are not considering anyway).

Therefore whenever n is even, S_n is $\frac{n}{2}$-cyclic cutwidth critical. □

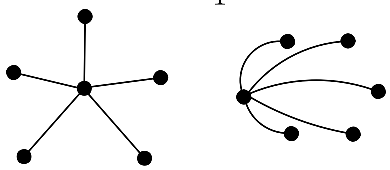
Corollary 2. *For n odd, S_n is not $\frac{n-1}{2}$-cyclic cutwidth critical.*

Proof. Property (ii) fails here. For n odd, $ccw(S_n) = \frac{n-1}{2}$, but $ccw(S_{n-1}) = \frac{n-1}{2}$. So removing an edge from S_n to get a smaller star will yield a subgraph that does not have a smaller cyclic cutwidth. □

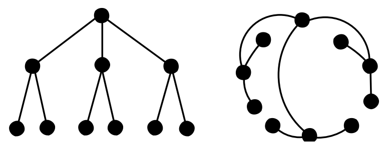
2 3-Cyclic Cutwidth Critical Graphs

First let us list all the 3-cyclic cutwidth critical graphs and an optimal embedding of each.

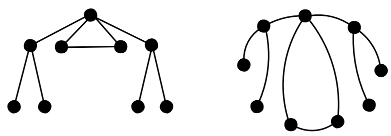
J_1



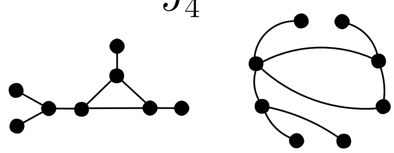
J_2



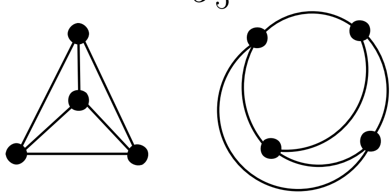
J_3



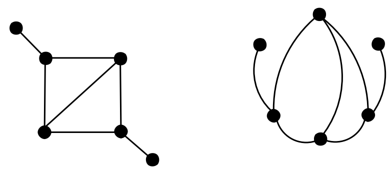
J_4



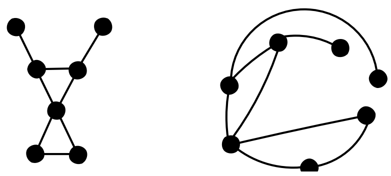
J_5



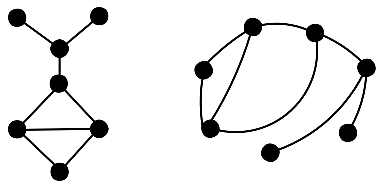
J_6



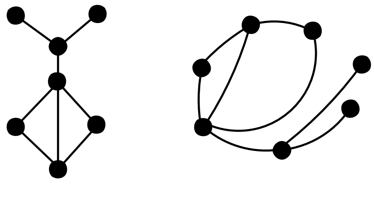
J_7



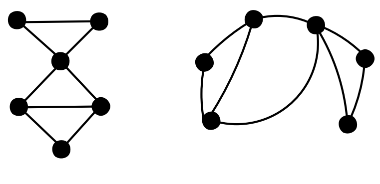
J_8



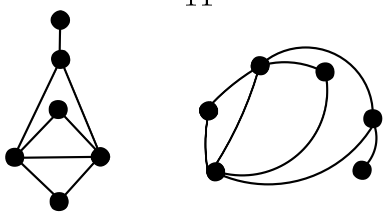
J_9



J_{10}



J_{11}



The following proposition will be used frequently throughout the paper.

Proposition 4. *(a) If G' is a subgraph of G then $ccw(G') \leq ccw(G)$. (b) If G' is homeomorphic to G, then $ccw(G') = ccw(G)$.*

Proof. Part (a) is clear; removing a vertex or an edge will certainly not increase the cyclic cutwidth.

For part (b) first note that it is enough to show that this holds if G and G' differ by only one degree 2 vertex, without loss of generality assume that G' has the extra degree 2 vertex.

Let us show that $ccw(G') \leq ccw(G)$.

Suppose we have an optimal labeling of G, inserting a degree 2 vertex will just cut a sector in half, which will not increase the cyclic cutwidth, giving us that $ccw(G') \leq ccw(G)$.

Now let us show that $ccw(G') \geq ccw(G)$.

Suppose not, that is we have $ccw(G') < ccw(G)$. Recall that the only difference between G and G' is that G' has an extra vertex of degree 2. What we will now show is that removing this vertex of degree 2 does not increase the cyclic cutwidth. That is, $ccw(G) \leq ccw(G')$, a contradiction.

There are only two ways to embed a degree 2 vertex onto a cycle. See Figure 19.

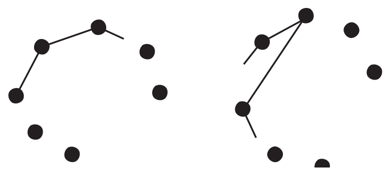


Figure 19: The two ways to embed a degree 2 vertex onto a cycle.

In both of these cases, removing the degree 2 vertex does not increase the cyclic cutwidth giving us that $ccw(G) \leq ccw(G')$.

Therefore if G' is homeomorphic to G, then $ccw(G') = ccw(G)$. $\square$

2.1 3-Cyclic Cutwidth Critical Trees

Lemma 5. *A tree T is k-cyclic cutwidth critical if and only if T is k-linear cutwidth critical.*

Proof. Suppose a tree T is k-cyclic cutwidth critical. That is, $ccw(T) = k$, if T' is a subgraph of T then $ccw(T') < ccw(T)$, and T is homeomorphically minimal. We now wish to show that T is k-linear cutwidth critical.

From [3] we have for any tree T, $ccw(T) = lcw(T)$, so property (i) is clear. For property (ii) use that $lcw(T') = ccw(T') \leq ccw(T) = lcw(T)$ and we also have that T is homeomorphically minimal. Thus T is also k-linear cutwidth critical.

Suppose a tree T is k-linear cutwidth critical. That is, $lcw(T) = k$, if T' is a subgraph of T then $lcw(T') < lcw(T)$, and T is homeomorphically minimal. Again using that $ccw(T) = lcw(T)$ we get that T is k-cyclic cutwidth critical. $\square$

Proposition 5. *A tree T is 3-cyclic cutwidth critical if and only if T is either J_1 or J_2.*

Proof. From the previous lemma we know that a tree is 3-cyclic cutwidth critical if and only if it is 3-linear cutwidth critical, and from [5] we know that J_1 and J_2 are the only 3-linear cutwidth critical trees. $\square$

2.2 3-Cyclic Cutwidth Critical Unicyclic Graphs

We will now prove the following lemma which will help us find all unicyclic 3-cyclic cutwidth critical graphs.

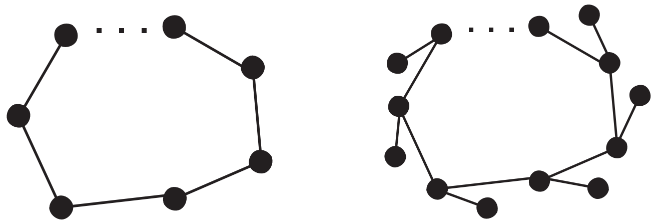
Lemma 6. *Any unicyclic 3-cyclic cutwidth critical graph must contain C_3 as its unique cycle.*

Proof. Suppose we start with C_n for $n \geq 4$. See Figure 20(a). This cycle by itself is not homeomorphically minimal, and since we are considering only unicyclic graphs at the moment, we must add an edge to each vertex in the cycle. However, this can be embedded to give cyclic cutwidth of 2. See Figure 20(b).

So now we have two options:
(i) We can add edges incident to one of the pendant vertices, or
(ii) we can add edges incident to any of the non-pendant vertices (giving that vertex degree 4).

Note that the cyclic cutwidth of $K_{1,5} (= J_1)$ is 3, and so the only 3-cyclic cutwidth critical graph with a degree 5 vertex is J_1, and any graph with a vertex of degree 5 or greater will be a supergraph of J_1.

In case (i) we must add two edges to the end of a pendant (we can't add just one because of the homeomorphic property). But now we have a subgraph homeomorphic to J_4. See Figure 21.



(a) (b)

Figure 20: (a) An n-gon. (b) An n-gon with pendant vertices off each vertex in the cycle.

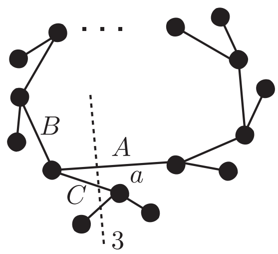


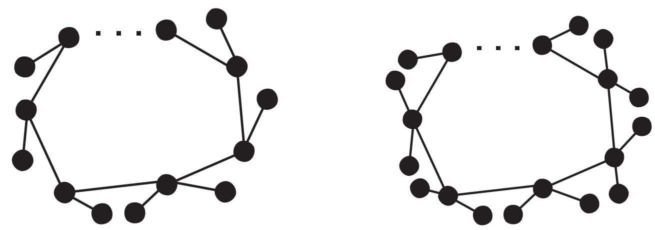
Figure 21: In case (i) we have a cyclic cutwidth of 3, but we now have a subgraph homeomorphic to J_4.

In case (ii) we can still embed our graph to have cyclic cutwidth of 2 (See Figure 22(a)), so we must add more edges. If we add to the non-pendant vertices (without increasing the degree of any vertex past 4) we can again embed our graph in a way to have cyclic cutwidth of 2. See Figure 22(b). Thus eventually we must add edges to a pendant vertex, and this is covered in case (i). Hence this case also leads to a contradiction. In each case we reached a contradiction, thus any unicyclic 3-cyclic cutwidth critical graph must contain C_3 as its unique cycle. $\square$

Proposition 6. *A unicyclic graph G is 3-cyclic cutwidth critical if and only if G is either J_3 or J_4.*

Proof. First let us show that J_3 and J_4 are 3-cyclic cutwidth critical graphs.

From the previous lemma we know that any unicyclic 3-cyclic cutwidth critical graph has C_3 as its unique cycle, and this yields nine cases. These cases cover every possible way to begin a unicyclic graph with our desired properties. We break the nine cases up depending on the degrees of the three vertices in the



(a)

(b)

Figure 22: (a) An n-gon with one vertex of degree 4, still has cyclic cutwidth of 2. (b) An n-gon with each vertex having degree 4 still with cyclic cutwidth of 2.

cycle. Each of these cases will have an initial cyclic cutwidth of 2, forcing us to add more edges in a specific way. From adding edges the graph will either not be 3-cyclic cutwidth critical, or it will be either J_3 or J_4. See Figure 23.

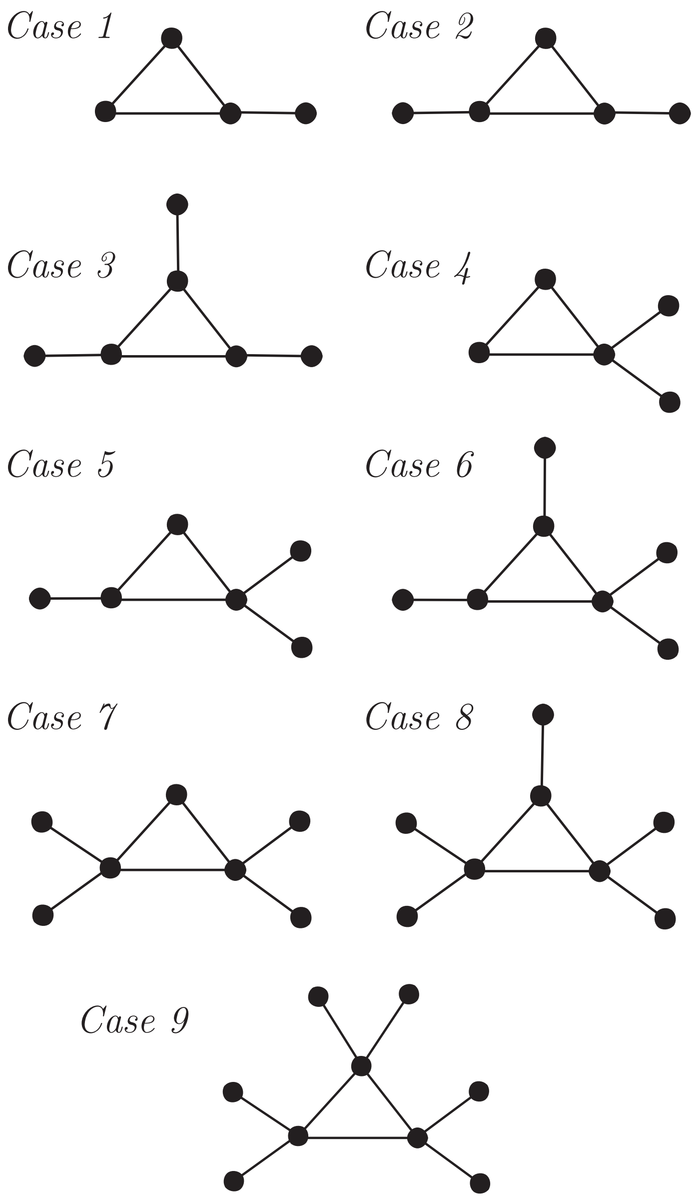


Figure 23: The nine cases for unicyclic 3-cyclic cutwidth critical graphs.

Case 1
The initial graph has cyclic cutwidth of 2, so we must add more edges. See Figure 24. The only vertex we can increase the degree of is vertex a (since increasing the degree of another vertex would fall under one of the other nine cases). Note

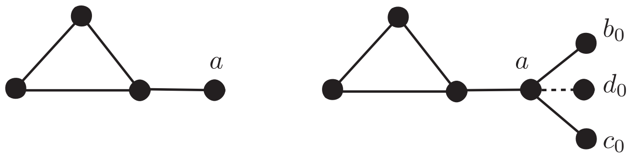


Figure 24: Case 1

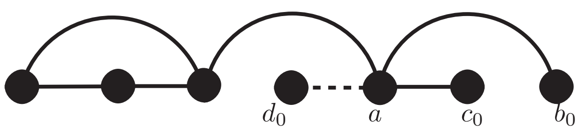


Figure 25: Has a linear cutwidth of 2.

that if we add only one edge our graph would no longer be homeomorphically minimal, hence we have to add two edges incident to vertex a. See Figure 24.

We have two cases right here. Either the degree of a is three, or the degree of a is four. We will consider the case where the degree of a is four, as we will be giving a construction of how to linearly embed the graph such that its linear cutwidth is 2, or such that we have a subgraph homeomorphic to J_2. If the degree of a is three, the new graph formed will be a subgraph of our graph with $deg(a) = 4$ which has a linear cutwidth of 2. This will finish case 1.

The following idea will be used in many of the nine cases (1, 2, 4, 5, and 7). Our graph currently has a linear cutwidth of 2 so we must add more edges in order to create a graph with a cyclic cutwidth of 3. See Figure 25. If we add edges incident to any two of the vertices b_0, c_0, d_0 we will get a subgraph of J_2, so without loss of generality we will add edges incident to b_0 only. See Figure 26.

Now if we add edges incident to any two of b_1, c_1, d_1 then our graph will have a subgraph homeomorphic to J_2. So again we have to add edges incident to just b_1, just c_1, or just d_1. We can keep doing this and our graph will not reach a linear cutwidth of 3 until we add edges to at least two of b_i, c_i, d_i, but doing this would give a subgraph homeomorphic to J_2. See Figure 27.

Case 2

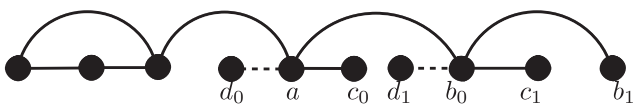


Figure 26: Has a linear cutwidth of 2.

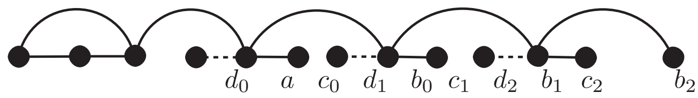


Figure 27: Again the graph has a linear cutwidth of 2.

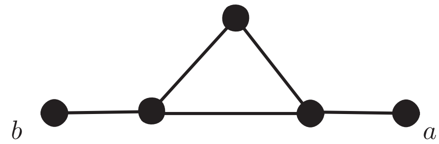


Figure 28: Case 2

We can use the same construction to embed the graph with a linear cutwidth of 2 as we did in Case 1. Except now the construction is done on both sides. This finishes Case 2.

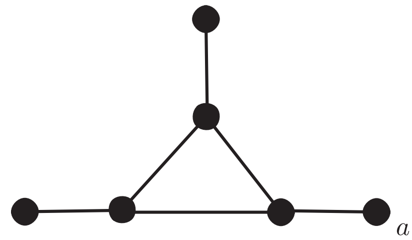


Figure 29: Case 3

Case 3
Again $ccw(G) = 2$, so adding more edges is necessary. Without loss of generality we will add two edges (if we added only one edge our graph would not be homeormohpically minimal) incident to vertex a. However, this gives us J_4.

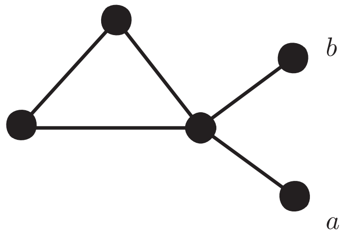


Figure 30: Case 4

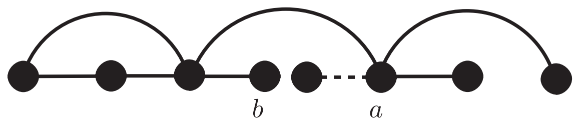


Figure 31: Case 4

Case 4
Again we use the same construction as in Cases 1 and 2. Except now we have an extra pendant edge. See Figure 31.

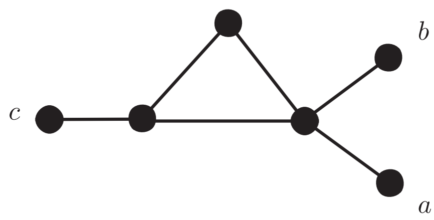


Figure 32: Case 5

Case 5
The same idea as in Case 4 works here as well.

Case 6
The initial graph has cyclic cutwidth of 2, but adding any more edges will give us a supergraph of J_4.

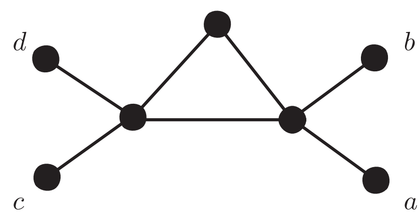


Figure 33: Case 7

Case 7
Adding edges incident to just a will force us to get the same problem of needing a linear cutwidth of 3, but only attaining the cutwidth after having lost criticality.

Adding edges incident to a, and edges incident to b (or to both c and d) will give us a supergraph of J_3.

And adding to both a and c will give us the same problem of needing a linear cutwidth of 3, but only attaining the cutwidth after having lost criticality.

Case 8
The initial graph has cyclic cutwidth of 2, but adding any more edges will give us a supergraph of J_4.

Case 9
The initial graph has cyclic cutwidth of 2, but adding any more edges will give us a supergraph of J_4.

Therefore J_3 and J_4 are the only unicyclic 3-cyclic cutwidth critical graphs. □

2.3 3-Cyclic Cutwidth Critical Polycyclic Graphs

Now we will begin our proof that $J_5 - J_{11}$ are all the 3-cyclic cutwidth critical polycyclic graphs. Note that we are not done with the proof yet, but we give partial proof, and conjecture that we have all the graphs.

We will break up the polycyclic case as follows:

First we will show that there are no 3-cyclic cutwidth critical polycyclic graphs that contain at least two cycles that are edge and vertex disjoint from every other cycle.

Then we will show that J_7 is the only 3-cyclic cutwidth critical polycyclic graph that contains only cycles that share exactly one vertex with another cycle. That is, each pair of cycles is edge disjoint.

Finally we conjecture that $J_5, J_6, J_8, J_9, J_{10}$, and J_{11} are the only 3-cyclic cutwidth critical polycyclic graphs that contain at least one pair of cycles that share at least one edge.

Proposition 7. *If a polycyclic graph G contains a pair of cycles that are edge and vertex disjoint from every other cycle in G, then G is not 3-cyclic cutwidth critical.*

Proof. First let us show that for $n > 3$, C_n cannot be one of the cycles.
The graph in Figure 34 has a cyclic cutwidth of 3 (in a linear arrangement the square and its pendants give a cutwidth of 3, and in a cyclic embedding with the square around the cycle, we have a cyclic cutwidth of 3 because of the square, its pendants, and vertex a). Anything larger will have a subgraph homeomorphic to this graph.

Thus somewhere in our graph we have two triangles seperated by a tree.

This tree is actually a caterpillar with the spine being the unique path from one cycle to the other. If the tree were not a caterpillar, then there would have to be a vertex with degree three incident to one of the vertices on the spine. If this happens at the vertex of the cycle then we get a subgraph homeomorphic

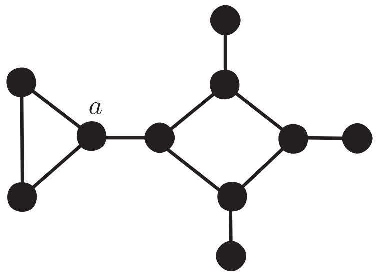


Figure 34: This graph has cyclic cutwidth of 3.

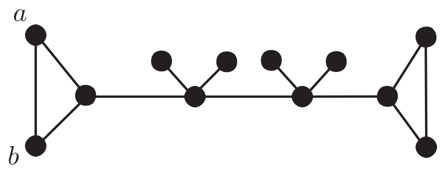


Figure 35: This graph has linear cutwidth of 2, so we must add more edges to it.

to J_3. If this happens at another vertex in the spine then we get a subgraph homeomorphic to J_2.

Now we give a way to construct our graph to have a linear cutwidth of 2, or have a subgraph homeomorphic to J_2 or J_3.

Since we know the two cycles are triangles and that they are seperated by a caterpillar, we can embed this graph such that the linear cutwidth is 2. See Figure 35. So we must add some more edges. We cannot add edges incident to a and edges incident to b as we will have a subgraph homeomorphic to J_4. So without loss of generality we can add edges incident only to a, and so we have either that $deg(a) = 3$ or $deg(a) = 4$. We will consider the case where $deg(a) = 4$ since the same construction will work for the case where $deg(a) = 3$. And we have the exact same linear construction as we saw previously in the Case 1 of Proposition 6. $\square$

Proposition 8. *A polycyclic graph G with no two cycles sharing an edge is 3-cyclic cutwidth critical if and only if G is J_7.*

Proof. Let G be a polycyclic graph with no two cycles sharing an edge. Using this assumption and the previous proposition we know that there must be two cycles that share a vertex.

If one of these cycles has more than four edges then we get a subgraph homeomorphic to J_7 (note that the vertices of the larger cycle must have degree greater than 2 (homeomorphic), and the edges cannot create cycles that share edges).

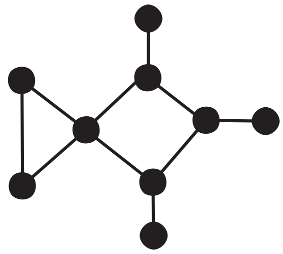


Figure 36: This is with C_4 sharing a vertex with C_3 and we get a subgraph homeomorphic to J_7 (the same idea works for $C_n (n > 3)$ sharing a vertex with C_3).

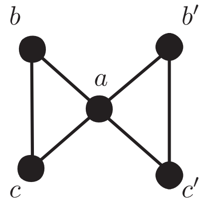


Figure 37: This graph has cyclic cutwidth of 2, so we must add more edges to it.

See Figure 36. So we need only consider the case where we have two triangles sharing a vertex. See Figure 37.

We cannot add any edges incident to a as we would have a vertex of degree 5 (or greater). If we add edges incident to b, and edges incident to c (or to b' and c') we get J_7 or a supergraph of J_7.

Thus we need only consider the case where we add edges or vertex-sharing cycles to just one side (i.e., to b), and the case where we add to both sides (i.e., to b and b').

Now we give a similar linear embedding to what we saw in the unicyclic case and in the previous proposition. Again we will have that our graph has a linear cutwidth of 2 (recall that we needa linear cutwidth of at least 3 in order to have a cyclic cutwidth of 3), or a subgraph homeomorphic to J_2.

First note that adding a triangle at the start does nothing and we still have the exact same problem. Also recall that we can't add two triangles off vertices of the same triangle as we would have a subgraph homeomorphic to J_2. See Figure 38.

So if we ever want a linear cutwidth of 3, which we need, we must eventually add 1 or 2 edges incident to one of the vertices on an outter triangle. We will show the construction with 2 edges, as the 1 edge case uses the same construction.

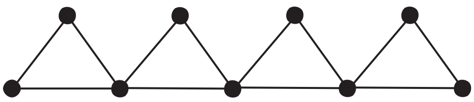


Figure 38: The graph still has a linear cutwidth of 2.

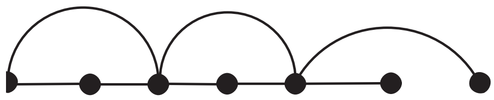


Figure 39: Still our graph has a linear cutwidth of 2.

The new graph still has a linear cutwidth of 2. See Figure 39. If we add two edges incident to b_0 and two edges incident to c_0 we will have a subgraph homeomorphic to J_2, so without loss of generality let us add two edges incident to just b_0. If we connect these edges to form a triangle we still have a linear cutwidth of 2. So in both cases we must add more edges, and we have the exact same problem. We either have a linear cutwidth of 2 forever, or we are forced to have a subgraph homeomorphic to J_2. See Figure 40. □

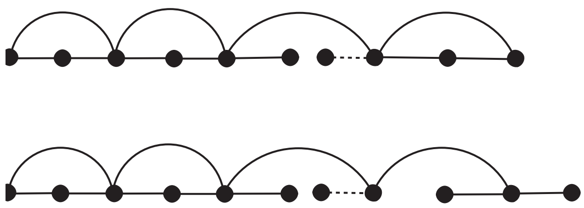


Figure 40: We can repeat this forever without getting a linear cutwidth of 3.

3 Further Research

There are some possibilities for further research in this area. First, the proof that we have all 3-cyclic cutwidth critical graphs is not yet complete. Also, one could try to discover a more elegant way to prove that this list is complete (if it is). Work could also be done in 4-cyclic cutwidth critical graphs, but this may prove to be difficult. And of course there are many things that can be looked at with cyclic cutwidth in general.

4 Acknowledgements

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