On Spanning Tree Congestion of Product Graphs

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August 23, 2007

Abstract

In this paper we consider the spanning tree congestion for several families of graphs. We find the exact spanning tree congestion for toroidal meshes, $C_m \times C_n$, and cylindrical meshes, $P_m \times C_n$,. Also we find bounds for the spanning tree congestion of Q_n, and a construction that gives the upper bound.

1 Background

The linear cutwidth and cyclic, grid, and tree congestion of many graphs has been explored previously. The linear cutwidth and grid congestion of the Q_n was established by Bezrukov et al. [1]. The linear and cyclic cutwidth for toroidal meshes, $C_m \times C_n$, along with the linear cutwidth for cylindrical meshes, $P_m \times C_n$, was established by Rolim et al. [5]. The cyclic cutwidth of cylindrical meshes was later established by Schröder, et al. [6]. Tree congestion was first explored by Ostrovskii [4], who found bounds on the tree congestion for any graph. Hruska [3] then established the exact spanning tree congestion for several types of graphs including the grid, $P_m \times P_n$. This paper continues Ostrovskii and Hruska's work by finding the spanning tree congestion for several additional families of graphs. Furthermore it builds on Bezrukov's work by exploring the tree congestion of Q_n, and extends Schröder et al. [6] and Rolim et al.'s. [5] work by finding the spanning tree congestion of toroidal meshes, $C_m \times C_n$, and cyclindrical meshes, $P_m \times C_n$.

2 Introduction

A graph G is a set of vertices V_G and edges E_G connecting pairs of vertices.
A tree, T, is a connected acyclic graph.
A spanning tree of a graph G is a tree, T, with vertices $V_T = V_G$ and edges $E_T \subset E_G$.
A path P_n is a tree with n vertices where every vertex is degree 2 except for the two end vertices which are degree 1.
A cycle C_n is a graph with n vertices where every vertex is degree two.
Define $\Delta(G)$ to be the maximum degree of any vertex in G.

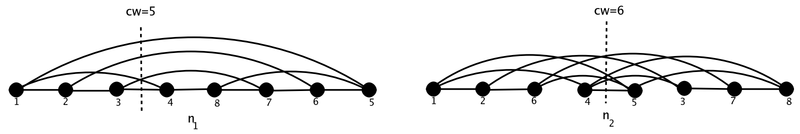

Figure 1: Linear cutwidths for different η of G, where G is the cube.

A vertex numbering, η of G, is a one to one function

$$\eta : V_G \to \{1, 2, ...|V_G|\}.$$

Given a graph with vertex numbering η, the linear cutwidth of η is

$$cw(\eta) = max_l |\{(v, w) \in E : \eta(v) \leq l \leq \eta(w)\}|.$$

and the linear cutwidth of a graph G is the minimum cutwidth over all numberings, η. That is

$$cw(G) = min_\eta \{cw(\eta)\}.$$

That is, given a linear arrangement of V_G, $cw(G)$ is the minimum over all η of the maximum number of edges cut by a line perpendicular to any set of two consecutive vertices. In Figure 1 we have two different numberings of the vertices of G, the cube, one with $cw(\eta_1) = 5$ and one with $cw(\eta_2) = 6$. In fact the numbering η_1 gives the minimum cutwidth for $\eta(G)$ when G is the cube, so $cw(G) = 5$.

The tree congestion, $t(G)$ is essentially the tree cutwidth of G. That is, instead of a linear arrangement of the vertices in G, the vertices are arranged into some tree T, and we evaluate the cutwidth on E_T.

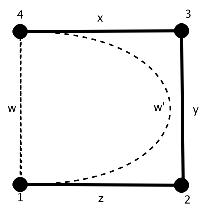

Figure 2: $cw(G) = t(G) = s(G) = 2$

The spanning tree congestion, $s(G)$ is the tree congestion of G where T is a spanning tree of G. (See Figure 2). Here we have a spanning tree with edges x, y, z. Vertices 1 and 4 must be connected; however we cannot use edge w as it does not lie on the spanning tree. Instead we create a detour, w', along the edges of the spanning tree, edges x, y, z. This gives $s(G) = 2$.

Given an edge e of a tree or spanning tree, define $c(e)$ to be the congestion along e in a given tree or spanning tree congestion. The congestion of an edge is equivalent to the number of detours in which the edge is used in. (In Figure 2 $c(x) = c(y) = c(z) = 2$.)

Given a set of vertices, M, of a graph G, we define $\theta_G(M)$ to be the number of edges with one vertex in M and one not in M. That is

$$\theta_G(M) = |\{(u, v) : u \in M, v \notin M\}|.$$

If we say that the set of vertices M is of size m, define $\theta_G(m)$ to be the minimum over all $\theta_G(M)$. That is

$$\theta_G(m) = min_{|M|=m}\{\theta_G(M)\}.$$

When it is clear what graph, G, we are evaluating θ_G on, we may omit the subscript and simply write θ.

2.1 Cubes

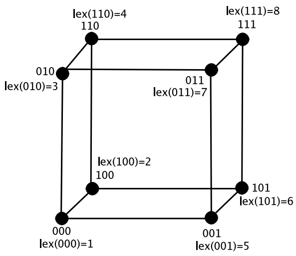

Figure 3: Q_3 with labeling and lexicographic numbering

A cube, Q_n, is a graph with vertices labeled by unique n-tuples of $0's$ and $1's$ and edges connecting every pair of vertices whose n-tuple differs in exactly one entry (See Figure 3).
Given an n-cube with vertices numbered as n-tuples $(x_1, x_2, ...x_n)$ where $x_i = 0$ or 1, the lexicographic numbering (see Figure 3) is

$$lex(x) = 1 + \sum_{i=1}^{d} x_i 2^{i-1}. \text{ [2]}$$

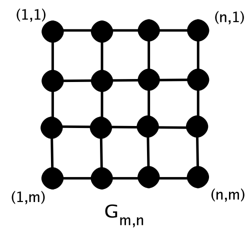

Figure 4: Grid with vertex numbering

2.2 Grids

Define an $m \times n$ grid, $G_{m,n}$ to be $P_m \times P_n$ where $m \leq n$.
Label each vertex with a pair of coordinates, (n_i, m_j), where the horizontal coordinate range from 1 to n from left to right and the vertical coordinate range from 1 to m from top to bottom (See Figure 4).

2.3 Toroidal Meshes, $C_m \times C_n$

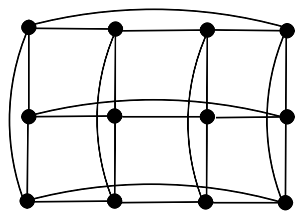

Figure 5: $(C_3 \times C_4)$

Let $C_m \times C_n$, where $m \leq n$, be the product of C_m and C_n. That is, take m copies of C_n and connect each vertex in the kth copy of C_n to the corresponding vertex in the $(k-1)$ (mod m) th and $(k+1)$(mod m)th copy of C_n (see Figure 5). We consider only non-degenerate cycles, that is $n, m \geq 3$.

2.4 Cylindrical Meshes, $P_m \times C_n$

Let $P_m \times C_n$ be the product of P_m and C_n. That is, take m copies of C_n and connect each vertex in the kth copy of C_n to the corresponding vertex in the $k+1$st copy of C_n (see Figure 6). Again we consider only non-degenerate paths and cycles, so $m \geq 2$ and $n \geq 3$.
Without loss of generality we will consider $P_m \times C_n$ to be drawn such that the vertices of P_m form vertical columns and the vertices of C_n form horizontal rows.

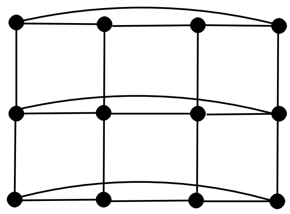

Figure 6: $(P_3 \times C_4)$

3 Tree Congestion of Q_n

Let u and v be any two be vertices of G. Define $m(u,v)$ to be the number of edge disjoint paths between u and v. Define m_G, to be the maximum of $m(u,v)$ over all pairs of vertices in G. That is

$$m_G = max|\{m(u,v) : u,v \in V_G\}|. \text{ [4]}$$

Theorem 1. (Ostrovskii) [4] $m_G = t(G)$**.**

Theorem 2. $t(Q_n) = n$.

Proof. By Theorem 1 this is equivalent to saying $m_{Q_n} = n$. Clearly $m_G \leq \Delta(G)$ and since $\Delta(Q_n) = n$, it follows that $m_{Q_n} \leq n$.
To show $m_{Q_n} = n$, choose any two adjacent vertices. Without loss of generality consider (00...0) and (10...0). We have n paths between (00...0) and (10...0) as follows. The first path is simply the edge connecting vertices (00...0) and (10...0). Let $1 < m \leq n$. Then the mth path is of the form (00...0), $(0i_2i_3...i_n)$, $(1i_2i_3...i_n)$, (10...0) where $i_k = 1 \iff k = m$. This gives n paths, which are clearly edge disjoint. Therefore $m_{Q_n} = t(Q_n) = n$. $\square$

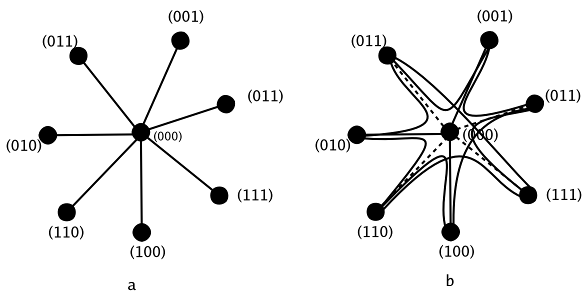

Figure 7: Tree for which $t(Q_3) = 3$

The tree that gives $t(G) = n$ is $K_{1,n-1}$, where, due to symmetry, any vertex can be the vertex of degree $n-1$ (see Figure 7).

4 Spanning Tree Congestion of Q_n

Theorem 3. $s(Q_n) \leq 2^{n-1}$.

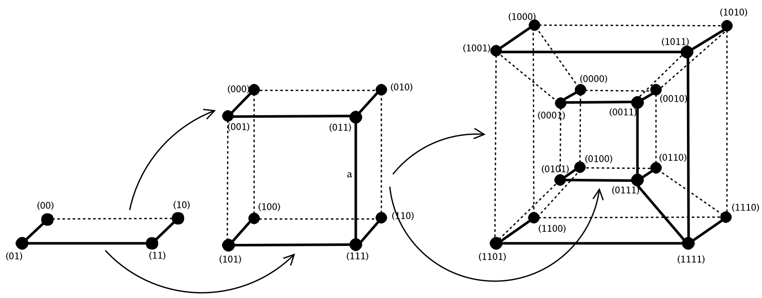

Figure 8: Q_2, Q_3 and Q_4 with spanning trees

Proof. By construction.
Begin with a spanning tree on Q_2, which is simply a path of length 3. This implies that $s(Q_2) = cw(Q_2) = 2$ [1]. Take two copies of Q_2, call them Q_2 and Q_2', numbered as in Figure 8, with spanning trees S_{Q_2} and $S_{Q_2'}$. Join S_{Q_2} and $S'_{Q_2'}$ at the vertices labeled (11) and $(1'1')$ with an edge, a. We now have Q_3 with a new spanning tree S_{Q_3}. Relabel the vertices by adding a 1 to the beginning of ever vertex in Q_2 and a 0 to the beginning of every vertex in Q_2' (see Figure 8). $s(S_{Q_3}) = 2^2 = 4$ since every vertex in Q_2 is connected to a vertex in Q_2' along edge a, meaning that $c(a) = 4$.
In the same manner, take Q_{n-1} and Q'_{n-1} and connect them at vertex (11...1) and $(1'1'...1')$ with an edge b and relabel Q_n by adding a 1 (respectively 0) to the beginning of each vertex in Q_{n-1} (respectively Q'_{n-1}). This gives a spanning tree S_{Q_n}. Then $s(S_{Q_n}) = 2^{n-1}$ since every vertex $v \in Q_{n-1}$ must connect to a vertex $v' \in Q_{n-1}$, and the detours for these edges run along edge b. No other vertices are connected using b and for any edge $d \in Q_{n-1}$ (respectively $d' \in Q'_{n-1}$) $c(d) \leq 2^{n-1}$ since there are 2^{n-1} vertices in Q_{n-1} (respectively Q'_{n-1}). Therefore $s(Q_n) \leq s(b) = 2^{n-1}$.
Note that there are other spanning trees with a congestion of 2^{n-1}, however we must only produce one. $\square$

Recall that $\theta_G(M) = |\{(u, v) : u \in M, v \notin M\}|$ and that $\theta_G(m) = min_{|M|=m}\{\theta_G(M)\}$. It is known that the lexicographic ordering minimizes the linear congestion of Q_n [1].
Therefore consider a lexicographic ordering of the vertices of G, and let M be the set of the first m vertices. Then define l to be the minimum $|M|$ such that $\theta_{Q_n}(l) \geq 2^{n-1}$. Consider a specific spanning tree, S_{Q_n}. If, when we remove an edge to form a disconnected graph, both of the resulting disconnected components are of size $\geq l$, it follows that $s(S_{Q_n}) \geq 2^{n-1}$.

We calculate l by ordering the vertices of the cube according to their lexicographic numbering, and computing θ and finding where $\theta \geq 2^{n-1}$. For Q_2 we have

$\lvert M\rvert$	0	1	2	3	4
θ	0	2	2	2	0.

For Q_3 we take two copies of Q_2 and add edges connecting them. This gives

$\lvert M\rvert$	0	1	2	3	4	5	6	7	8
θ_{Q_2}	0	2	2	2	0	2	2	2	0
$+$	0	1	2	3	4	3	2	1	0
θ_{Q_3}	0	3	4	5	4	5	4	3	0.

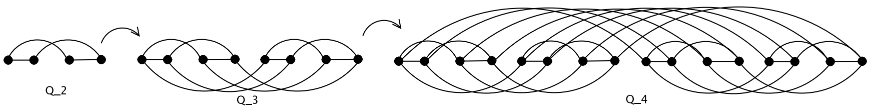

Figure 9: construction of linear cutwidth for Q_n

We continue calculating θ in this manner, taking two sets of $\theta_{Q_{n-1}}$ (using only one of $|M| = 0$ and $|M| = 2^{n-1}$) and adding $1, 2, 3, ...2^{n-1}, 2^n, 2^{n-1}, ...3, 2, 1$ to each (see Figure 9).
From this we have:

n	l
1	1
2	1
3	2
4	3
5	11
6	22.

Furthermore we claim that $l_n = \lceil \frac{2^n}{6} \rceil \ \forall n$.
Note also that $\theta_{Q_n}(2^{n-1}) = 2^{n-1}$ and that $\theta_{Q_n}(k) = \theta_{Q_n}(2^n - k)$.

Write $m = 2^{i_1} + 2^{i_2} + 2^{i_3} + ... + +2^{i_k}$ where $i_1 > i_2 > ... > i_k$.

Proposition 1. *Given a lexicographic ordering of vertices,*

$$\theta_{Q_n}(m) = nm - \sum_{j=2}^{k} (2^{i_j})[i_j + 2(j-1)].$$

Proof. If we begin with a set M with m vertices and assume there are no edges between any of them, then there are nm edges adjacent to these m vertices. We must then subtract off all edges that connect two vertices in M. In the largest Q_{i_1} that is contained completely within M, there are 2^{i_1} vertices of degree i_1, so we have counted an extra $(2^{i_1})(i_1)$ edges. For Q_{i_2} we have 2^{i_2} vertices, which are of degree i_2. Within this n-cube we have counted an extra $(2^{i_2})(i_2)$ edges. Also these i_2 vertices connect to i_2 vertices in Q_{i_1}, giving a total of $(2^{i_2})(2)$ extra edges. In general in Q_{i_m} there are 2^{i_m} degree i_m vertices, giving an extra $(2^{i_m})(i_m)$ edges. Also each such vertex is connected to i_{m-1} vertices in each of the larger Q_n with an edge that has been counted twice. This gives an extra $(2^{i_m})[i_m + 2(m-1)]$ edges. Subtracting off all these extra edges gives

$$\theta_{Q_n}(m) = nm - \sum_{j=2}^{k} (2^{i_j})[i_j + 2(j-1)].$$

□

Theorem 4. *$s(Q_n) = 2^{n-1}$ for $n \leq 6$.*

Proof. For $n = 1$ and $n = 2$ this is trivial.
For $3 \leq n \leq 6$, a spanning tree, T will have $\Delta(T) \leq n$ and $|V_T| = 2^n$. Since $l = \lceil \frac{2^n}{6} \rceil$ then $l \leq \lceil \frac{2^n}{n} \rceil \leq 2^{n-1}$. Then $s(Q_n) \geq \theta_n(l) = 2^{n-1}$ if for every spanning tree, by removing an edge we can always disconnect a set of vertices, W where $|W| \geq \lceil \frac{2^n}{n} \rceil$. To show this we prove the following lemma.

Lemma 1. *Let G be a graph with spanning tree, T. We can always remove an edge of T to give two disconnected sets of size at least $\lceil \frac{|V|}{\Delta(G)} \rceil$.*

Proof. Given a set of vertices W and a vertex, v, with $v \notin W$, we say W is adjacent to v if there is an edge, (v, v_w) where $v_w \in W$.
Let v_1 be a vertex with degree $\Delta(G)$. We want to show that either removing an edge incident with v_1 gives us two sets of size at least $\lceil \frac{|V|}{\Delta(G)} \rceil$ or by considering some vertex v_2 adjacent to v_1, v_3 adjacent to v_2, ..., v_{m+1} adjacent to v_m, we eventually reach a point where by removing edge (v_m, v_{m+1}) we have two disconnected sets each of size at least $\lceil \frac{|V|}{\Delta(G)} \rceil$.

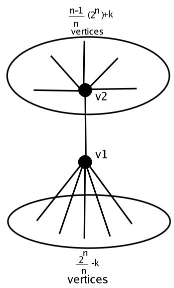

Figure 10: Division of vertices for Lemma 1, Case 1

Case 1. *The vertices in T are not evenly distributed around v_1. In particular disconnecting T along the edge with endpoints v_1 and v_2 leaves total of $\lceil \frac{|V|}{\Delta(G)} \rceil - k_1$, where $k_1 > 0$, vertices in the set containing v_1.*

Since v_2 has degree at most $\Delta(G)$, there are at most $\Delta(G) - 1$ sets adjacent to v_2 not including the set containing v_1. As there are $\lceil \frac{|V|}{\Delta(G)} \rceil - k_1$ vertices in the set containing v_1 there are $|V| - \lceil \frac{|V|}{\Delta(G)} \rceil + k_1 \geq \lceil \frac{\Delta(G)-1}{\Delta(G)}|V| \rceil$ total vertices in the other $\Delta(G) - 1$ sets. Therefore at least one set, W_2, which is adjacent to v_2, has $|W_2| \geq \lceil \frac{|V|}{\Delta(G)} \rceil$. If upon disconnecting the set W_2, the set containing v_1 has at least $\lceil \frac{|V|}{\Delta(G)} \rceil$ vertices, we are done. If however it still contains less that $\lceil \frac{|V|}{\Delta(G)} \rceil$ vertices, we continue by considering a vertex, $v_3 \in W_2$ where v_3 is adjacent to v_2. We continue iterating in this manner until we reach the first vertex, v_m, where disconnecting a set, W_{m+1}, by removing the edge between v_m and v_{m+1} leaves at least $\lceil \frac{|V|}{\Delta(G)} \rceil$ vertices in the set containing v_1 through v_m and their adjacent vertices. Since W_{m+1} is the first such set, the set containing $v_1$through v_{m-1} along their adjacent vertices (excluding v_m) now has $\frac{|V|}{\Delta(G)} - k_2$ vertices in it. This leaves $\frac{\Delta(G)-1}{\Delta(G)}|V| + k_2$, where $k_2 \geq 0$, vertices in the $\Delta(G) - 1$ sets adjacent to v_m excluding the one set containing v_1 through v_{m-1}. Therefore removing one edge, (v_m, v_{m+1}), gives a set W_{m+1} with $|W_{m+1}| \geq \lceil \frac{|V|}{\Delta(G)} \rceil$ vertices in it, and we have disconnected T into two components, both with at least $\lceil \frac{|V|}{\Delta(G)} \rceil$ vertices.

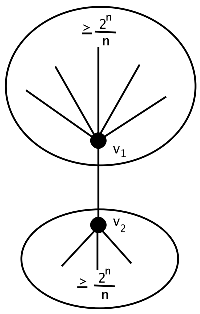

Figure 11: Division of vertices for Lemma 1, Case 2

Case 2. *The vertices in T are more evenly distributed around v_1. In particular removing any edge adjacent to v_1 leaves at least $\lceil \frac{|V|}{\Delta(G)} \rceil$ vertices in the set containing v_1. (See Figure 11)*

As there are $|V|$ vertices and at most $\Delta(G)$ sets adjacent to v_1, then the average size of these sets is $|W_{ave}| \geq \frac{|V|}{\Delta(G)}$. Therefore we can always find some set W_k which contains a vertex, v_k, adjacent to v_1, where $|W_k| \geq \frac{|V|}{\Delta(G)}$. We disconnect the set W_k by removing the edge (v_1, v_k). This gives two sets with at least $\frac{|V|}{\Delta(G)}$ vertices in them since by the initial set up the set containing v_1 has at least $\frac{|V|}{\Delta(G)}$ vertices in it.

Case 3. *T has no vertices with degree $\Delta(G)$.*

Consider v_1 to be the vertex of highest degree. Either the distribution of the vertices around v_1 in T falls into Case 1 or Case 2 above. Since we have fewer sets in which to distribute the vertices, there is no way to decrease $|W|$, where W is the largest set you must always be able to disconnect and leave the remaining set with at least $\frac{|V|}{\Delta(G)}$ vertices.

Every spanning tree falls into one of the three above cases. In each we can remove an edge to give two sets of size at least $\frac{|V|}{\Delta(G)}$, so for any spanning tree T of a graph G we can disconnected a set of size at least $\frac{|V|}{\Delta(G)}$.

□

Now that we have proved the above Lemma, we can continue with the proof of Theorem 4. Since for Q_n we have $|V| = 2^n$ and $\Delta(T) = n$ we can always remove an edge to disconnect two sets of size at least $\lceil \frac{2^n}{n} \rceil$. If $n \leq 6$ then $\lceil \frac{2^n}{n} \rceil \geq \lceil \frac{2^n}{6} \rceil = l$. Therefore we can always disconnect a set of size at least l which gives $s(Q_n) = 2^{n-1}\ \forall n \leq 6$.

□

Theorem 5. *If $n > 6$ then $\frac{2^n}{n} < l$ and $s(G) \geq \theta_{Q_n}(\frac{2^n}{n})$.*

Proof. This follows immediately from the definition of θ and the fact that $\Delta(Q_n) = n$. □

Conjecture 1. *$s(Q_n) = 2^{n-1}\ \forall n$.*

Although for $n > 6$ we have $l \geq \frac{2^n}{n}$ and one can produce a spanning tree that has either $\lceil \frac{2^n}{n} \rceil$ or $\lfloor \frac{2^n}{n} \rfloor$ vertices on every path, it appears that one cannot do so in a manner that minimizes congestion.

5 Spanning Tree Congestion of $C_m \times C_n$

Theorem 6. *$s(C_m \times C_n) = 2m$ where by definition $m \leq n$.*

Proof. We show this by constructing a spanning tree on $C_m \times C_n$ that has congestion $2m$, to establish an upper bound. We then prove that no spanning tree can have congestion less than $2m$ which proves that $s(C_m \times C_n) = 2m$.

Lemma 2. *$s(C_m \times C_n) \leq 2m$ where $m \leq n$.*

Proof. Create a spanning tree, T by taking all the vertical edges and the row of horizontal edges $[(x_i), \lceil \frac{m}{2} \rceil), (x_{i+1}, \lceil \frac{m}{2} \rceil)]$, where i ranges from 1 to $n-1$, as shown in Figure 12. The horizontal edges are used in m detours from the edges of the grid plus an additional m detours from the edges added to complete the horizontal cycles, giving them a total congestion of $2m$. If m is even, $m < n$, and $m \geq 4$, then some vertical edges have congestion $m+1$ on the underlying grid. However the vertical edges only have a detour for 1 vertical cycle, giving a congestion of $m + 2 \leq 2m$ (if m is odd the vertical edges have congestion $m + 1 \leq 2m$). Therefore $s(C_m \times C_n) \leq 2m$. □

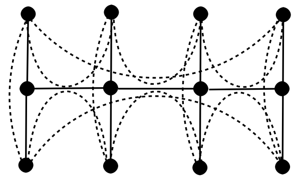

Figure 12: $s(C_3 \times C_4) \leq 6$

Lemma 3. *$s(C_m \times C_n) \geq 2m$ where $m \leq n$.*

Proof. Note that $\Delta(C_m \times C_n) = 4$ and that all vertices are degree 4. Also, $|V| = mn$. Therefore we can always disconnect a set of vertices of size l where $\frac{mn}{4} \leq l \leq mn - \lceil \frac{mn}{4} \rceil$ by Lemma 1. Now we must show that $\theta(l) \geq 2m$ whenever $\lceil \frac{mn}{4} \rceil \leq l \leq mn - \lceil \frac{mn}{4} \rceil$. To do so we need the following result.

Proposition 2. *Given l vertices where $k^2 \leq l \leq (k+1)^2$ and $jm \leq l \leq (j+1)m$, where j, k are positive integers, $\theta(l)$ is minimized if $l = k^2$ and the vertices form a complete $k \times k$ square giving $\theta = 4k$ or $l = jm$ and the vertices fill j complete rows of m vertices, which gives $\theta = 2m$.*

Proof. To minimize $\theta(l)$ we must maximize the degree of the l vertices being considered. In a rectangular layout, a square minimizes perimeter for a given area, so it maximizes the number of interior vertices, which in turn maximizes degree. Therefore for any rectangle with s vertices, where $t^2 \leq s \leq (t+1)^2$, θ is minimized when $s = t^2$ and the vertices are arranged into a $t \times t$ square giving $\theta = 4t$.
A non-rectangular layout of vertices clearly has fewer interior vertices than a rectangular layout, and hence has a larger θ.
Also taking complete rows of m vertices could minimize θ as this arrangement uses all the edges from a cycle, and thereby increasing the degree of the vertices. Clearly taking the smaller cycles, C_m minimizes θ, and gives $\theta = 2m$. $\square$

Using the above Proposition we complete our prove of Lemma 2. Observe that we could have the minimum θ when we disconnect a set of size $s = k^2$. Since $m \leq n$, s is minimized when $m = n$ and $\frac{mn}{4} = \frac{m^2}{4}$. Then we can arrange the vertices into a $k \times k$ square where $k = \frac{m}{2}$, assuming k is an integer. This gives $\theta = 2m$.
Likewise when $l = \frac{mn}{4}$, where $l = jm$ for some positive integer j arranging the vertices into j rows of m give $\theta = 2m$.
Therefore for all possible cases of $\lceil \frac{mn}{4} \rceil \leq l \leq mn - \lceil \frac{mn}{4} \rceil$ vertices, $\theta(l) \geq 2m$. $\square$

Now by Lemma 2 we have $s(C_m \times C_n) \leq 2m$ and by Lemma 3 that $s(C_m \times C_n) \geq 2m$. This gives equality, which proves the theorem. $\square$

6 Spanning Tree Congestion of $P_m \times C_n$

By definition $m \geq 2$ and $n \geq 3$. When $m = 2$, $n = 3$ then $s(P_m \times C_n)$ does not fit into the formula's outlined later in this section so we consider it separately.

Theorem 7. *$s(P_2 \times C_3) = 3$.*

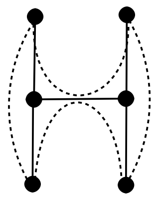

Figure 13: A spanning tree of $P_2 \times C_3$ with congestion 3

Proof. $\Delta(P_2 \times C_3) = 3$ so $m_G \leq 3$. We can find 3 edge disjoint paths between the vertices $(1,2)$ and $(2,2)$ by taking the edge between them and the paths $(1,2),(1,1),(2,1),(2,2)$ and $(1,2),(1,3),(2,3),(2,2)$ so $m_G = 3 \leq s(P_2 \times C_3)$. We can construct a spanning tree, T, with $s(T) = 3$ by taking the vertical edges from both C_3 and taking the middle P_2, as shown in Figure 13. $\square$

Theorem 8. *If $n > 2$ or $m > 3$*

$$\begin{aligned} s(P_m \times C_n) &= min\{2m, n+1\} \quad n \text{ is odd} \\ &= min\{2m, n+2\} \quad n \text{ is even.} \end{aligned}$$

Proof. To prove this we first show that we can construct a spanning tree with congestion equal to $min\{2m, n+1\}$ for n odd and congestion equal to $min\{2m, n+2\}$ for n even. Then we prove that if $n \geq 2m$ the congestion is at least $2m$, which is less than $n+1$ ($n+2$ if n is even). Then we show if $n < 2m$ then the congestion is at least $n+1 \leq 2m$ if n is odd and $n+2 \leq 2m$ if n is even.

Lemma 4.

$$\begin{aligned} s(P_m \times C_n) &\leq min\{2m, n+1\} \quad n \text{ is odd} \\ &\leq min\{2m, n+2\} \quad n \text{ is even.} \end{aligned}$$

Proof. We prove this by construction, considering the case where $2m \leq n+1$ for n odd ($2m \leq n+2$ for n even) and the case where $n+1 < 2m$ for n odd ($n+2 < 2m$ for n even).

Case 1 *$2m < n+1$ for n odd ($2m < n+2$ for n even).*

If $2m < n+1$, where n is odd ($2m < n+2$, where n is even) then $n \geq 2m$.

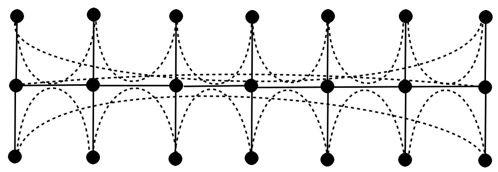

Figure 14: $s(P_3 \times C_8) \leq 6$

Create a spanning tree by taking all the vertical edges, that is, all the edges from the n copies of P_m, and take the horizontal row of edges $[(x_i), \lceil \frac{m}{2} \rceil), (x_{i+1}, \lceil \frac{m}{2} \rceil)]$ where i ranges from 1 to $n-1$, as shown in Figure 14. This gives a congestion of $2m$ on the horizontal edges since they have detours for 2 edges from each of m cycles. The vertical edges have congestion at most $m+1 \leq 2m$.

Case 2 *$n+1 \leq 2m$ for n odd ($n+2 \leq 2m$ for n even).*

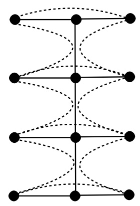

Figure 15: $s(P_4 \times C_3) \leq 4$

Create a spanning tree by taking all the horizontal edges and by taking the vertical row of edges $[(\lceil \frac{n}{2} \rceil, x_i), (\lceil \frac{n}{2} \rceil, x_{i+1})]$ where i ranges from 1 to $m-1$, as shown in Figure 15. If n is odd, then the horizontal edges adjacent to the vertical row have congestion $n+1$ since there are n detours from the underlying grid and 1 detour from the one edge needed to complete the cycle on the horizontal row of vertices. If n is even the congestion is $n+2$ since the horizontal row $(x_i, \frac{n}{2}), (x_i, \frac{n}{2}+1)$ where i ranges from 1 to n has congestion $n+1$ on the underlying grid, plus 1 from the additional edge needed to complete the cycle on its horizontal row of vertices.

□

Lemma 5. *If $n \geq 2m$ then $s(P_m \times C_n) \geq 2m$.*

Proof. Note that $n \geq 2m$ implies $2m \leq n+1$ if n is odd and $2m \leq n+2$.
By definition $m \geq 2$ and $n \geq 3$. If $m = 2$ then $\Delta(G) = 3$, otherwise $\Delta(G) = 4$. So for $m = 2$, we can always disconnect a set of size at least $l \geq \lceil \frac{mn}{3} \rceil$ and for $m \geq 3$ we can disconnect a set of size at least $l \geq \lceil \frac{mn}{4} \rceil$ by Lemma 1. By Proposition 2, θ is minimized when we take either complete rows or columns of vertices, or a square.

For $m = 2$ a complete square must be a 2×2 square, which is the same as taking two rows of $m = 2$ and gives congestion $2m = 4$. Therefore we only need to consider $m \geq 3$, which gives $\Delta = 4$ and $l \geq \lceil \frac{mn}{4} \rceil$.

Taking j complete columns gives $\theta = 2m$.
Taking j complete rows gives $\theta = n \geq 2m$.

A square of side length k gives $\theta = 3k$ since one side can be on the first or mth cycle which gives no contribution to θ along that side. We could have $\theta \leq 2m$ if $3k \leq 2m$, or equivalently $9k^2 \leq 4m^2$.
We know that $k^2 \geq \frac{mn}{4}$ and that $n \geq 2m$ which implies $\frac{mn}{4} \geq \frac{2m^2}{4}$. It then follows that $k^2 \geq \frac{m^2}{2}$ or equivalently $9k^2 \geq \frac{9}{2}m^2$. Therefore $9k^2 \geq 4m^2$ so $3k \geq 2m$ whenever $k^2 \geq \frac{mn}{4}$ and $n \geq 2m$. So $\theta(l) \geq 2m$ and $s(P_m \times C_n) \geq 2m$ if $l \geq \frac{mn}{4}$ and $n \geq 2m$. $\square$

Lemma 6. *If $n < 2m$*

$$s(P_m \times C_n) \quad \begin{array}{ll} \geq n+1 & n \textit{ is odd} \\ \geq n+2 & n \textit{ is even.} \end{array}$$

Proof. If $n < 2m$ then $n + 1 \leq 2m$ if n is odd and $n + 2 \leq 2m$ if n is even. $\Delta(G) = 4$ unless $m = 2$ which has $n < 2m$ only when $n = 3$ However this falls into the special case addressed by Theorem 7. Therefore we have $m \geq 3$ and $\Delta(G) = 4$ and hence can always disconnect a set of vertices of size $\lceil \frac{mn}{4} \rceil$. By Proposition 2, θ is minimized when we take complete rows or columns, or a complete square.
If we take complete columns, $\theta = 2m \geq n + 1$ if n is odd and $\theta \geq n + 2$ if n is even.
If we take a complete $k \times k$ square it has at least $\frac{mn}{4} = k^2$ vertices. The following cases show that $\theta(k^2) \geq n + 1$ if n is odd and $\theta(k^2) \geq n + 2$ if n is even.

Case 1 *n is odd*
If $3k \geq n + 1$ then $\theta \geq n + 1$. Since k and n are integers, if $3k \geq n + 1$, then $3k > n$ and equivalently $9k^2 > n^2$.
Since $n + 1 \geq 2m$ we have that $\frac{nm}{4} \geq \frac{n(n+1)}{8}$. Since $k^2 \geq \frac{nm}{4}$ then $k^2 \geq \frac{n(n+1)}{8}$ and equivalently $9k^2 \geq \frac{9}{8}n(n + 1)$. Then $3k \geq n + 1$ whenever $\frac{9}{8}n(n + 1) > n^2$. This holds for all n, so $3k \geq n + 1$ and $\theta \geq n + 1$.

Case 2 *n is even*
If $3k \geq n + 2$ then $\theta \geq n + 2$. Since k and n are integers, if $3k \geq n + 2$, then $3k > n + 1$ and equivalently $9k^2 > (n + 1)^2$.
Since $n + 2 \geq 2m$ we have that $\frac{nm}{4} \geq \frac{n(n+2)}{8}$. Since $k^2 \geq \frac{nm}{4}$ then $k^2 \geq \frac{n(n+1)}{8}$ and equivalently $9k^2 \geq \frac{9}{8}n(n + 2)$. Then $3k \geq n + 2$ whenever $\frac{9}{8}n(n + 2) > (n + 1)^2$. This holds for all $n > 2$, which is true by definition. Then $3k \geq n + 2$ and $\theta \geq n + 2$.

Therefore θ is minimized when every set we can disconnect of size $\geq \lceil \frac{mn}{4} \rceil$ contains jn vertices, where j is any positive integer, arranged into complete rows, that is, only complete C_n's. This gives $\theta = n$, but the only way to do this is the tree constructed in Lemma 4 Case 2. However, this spanning tree has congestion $n+1$ if n is odd and $n+2$ if n is even. Therefore if $n < 2m$, $s(P_m \times C_n) \geq n+1$ if n is odd, and $s(P_m \times C_n) \geq n+2$ if n is even. $\square$

By Lemma 4 we have that

$$\begin{aligned} s(P_m \times C_n) \quad & \leq min\{2m, n+1\} \quad n \text{ is odd} \\ & \leq min\{2m, n+2\} \quad n \text{ is even.} \end{aligned}$$

and by Lemmas 5 and 6 we have

$$\begin{aligned} s(P_m \times C_n) \quad & \geq min\{2m, n+1\} \quad n \text{ is odd} \\ & \geq min\{2m, n+2\} \quad n \text{ is even.} \end{aligned}$$

This gives equality, which proves the theorem.

$\square$

7 Conclusion

In this paper we have found the exact spanning tree congestion for $C_m \times C_n$ and $P_m \times C_n$, and have constructions for spanning trees with that congestion. For Q_n we have the exact spanning tree congstion for $n \leq 6$ and a construction that gives that congestion. For $n > 6$ we have a constructive upper bound and a separate lower bound for the spanning tree congestion. We conjecture that the exact congestion is in fact the upper bound, but do not currently have a method of proving this.

8 Acknowledgments

I would like to thank Dr. Chavez and Dr. Trapp for their help and guidance throughout my research and writing of this paper. I would also like to acknowledge Daniel Tanner, with whom I worked with to develop several of the ideas regarding Q_n. I would also like to acknowledge the other REU 2007 participants. This work was completed during the 2007 REU program in Mathematics at California State University, San Bernardino, and was jointly sponsored by CSUSB and NSF-REU Grant DMS-0453605.

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