Convexity and Minimum Distance Energy

Rosanna Speller *

Abstract

We look at the problem of reducing the minimum distance energy of polygonal knots as an isoperimetric problem. Building on techniques used to show that a regular n-gon maximizes area for a given perimeter, we have been able to prove that convex figures minimize the minimum distance energy for all polygons in $\mathbb{E}^3$. We hope that this will be a helpful step towards showing the regular n-gon has the least minimum distance energy for all polygonal knots, and give suggestions for isoperimetric problem solving tools that could be used to further explore this problem.

Introduction

Simon defined and gave preliminary explorations of the minimum distance energy of polygonal knots. He found that in experiments seeking to find the position for least minimum distance energy of various polygons, edges seem to try to push apart from each other, attempting to equalize edge lengths and open angles [12]. Similar experiments using the program Ming [15], find that knots open and equalize into convex shapes which have a greater area, for a certain perimeter.

In order the simplify the problem, we investigated all equilateral $2n$-gons placed on a lattices, for $n \leq 4$. As expected, the minimum distance energy of a regular $2n$-gon was less than or equal to that of corresponding lattice $2n$-gons. It was also noted that convex lattice $2n$-gons had a smaller minimum distance energy than non-convex polygons with the same number of sides. This gave us further evidence that convex n-gons might reduce the minimum distance energy and motivated us towards proving this conjecture.

Several works explore the relationship between the Möbius energy of smooth knots and the minimum distance energy of polygonal knots [10, 9, 13]. Möbius energy is minimized for a perfect circle [10, 9], and a regular n-gon is a polygonal approximation of a circle. It is well known that for a fixed perimeter, the perfect circle is the shape with maximum area. Similarly, the regular n-gon maximizes area for all n-gons [2]. So, because there is a relationship between Möbius energy and minimum distance energy [10, 9], the fact that both a perfect circle and a

*Adviser: Dr. Rolland Trapp, and Department of Mathematics, California State University, San Bernardino, CA 92407

regular n-gon maximize area for a given perimeter seems significant. Thus, we explored techniques used in isoperimetric problems in order to build evidence for our argument that the regular n-gon minimizes minimum distance energy.

Preliminary Definitions and Theorems

Here we give the definitions which should give an understanding of what we are referring to as the minimum distance energy. Throughout the paper, we refer to U'_{md}, a variant of Simon's Minimum Distance Energy, as the minimum distance energy. This variant, given in [9], is more analogous to a double integral, and therefore helpful in drawing connections between Minimum Distance Energy and Möbius Energy. Note also that $U'_{md}(P) = 2 \cdot U_{md}(P)$, so the difference, in relation to the material of this paper, is otherwise trivial.

Definition 1. *Simon's Minimum Distance Energy is defined for a pair of non-consecutive edges, X and Y, of an n-gon, as $U_{md}(X,Y) = \frac{\ell(X)\ell(Y)}{md(X,Y)^2}$. Here $\ell(X)$ gives the length of the segment X and $md(X,Y)$ gives the minimum distance between edges X and Y. Simon's original formula for the Minimum Distance Energy of a polygon, P is given by*

$$U_{md}(P) = \sum_{X \neq Y \, or \, adjacent} U_{md}(X,Y).$$

Definition 2. *The Möbius Energy (or O'Hara Energy) of a smooth knot, K is*

$$E_0(K) = \iint_{C \times C} \frac{1}{|x(t) - x(s)|^2} - \frac{1}{|s-t|^2} \, ds \, dt$$

where $t \to x(t)$ gives a unit-speed parameterization of K on a circle, C. The notation, E_0, reminds us that we are using a definition of Möbius Energy such that $E_0(C) = 0$, for a circle, C.

Definition 3.

$$U'_{md}(P) = \sum_{all \ edges \ X} \sum_{Y \neq X \, or \, adjacent} U_{md}(X,Y)$$

Example: Finding U'_{md} for all n-gons

It is well known that the length of a chord, c, of a circle with radius R is $\ell(c) = 2R\sin(\frac{\theta}{2})$, where θ is the related central angle. We can use these basic rules of trigonometry to compute U'_{md} for all n-gons. A regular n-gon can be inscribed on a perfect circle so that all of its vertices rest along the curve. See Figure 1. Here we have an octagon. The grey paths give the minimum distances between the edge p and all edges (r, s, t, u, and v) to which p is not adjacent. You will note that these distances are all chords of a circle. We can create a general formula for the U'_{md} of an even n-gon and the U'_{md} of an odd n-gon.

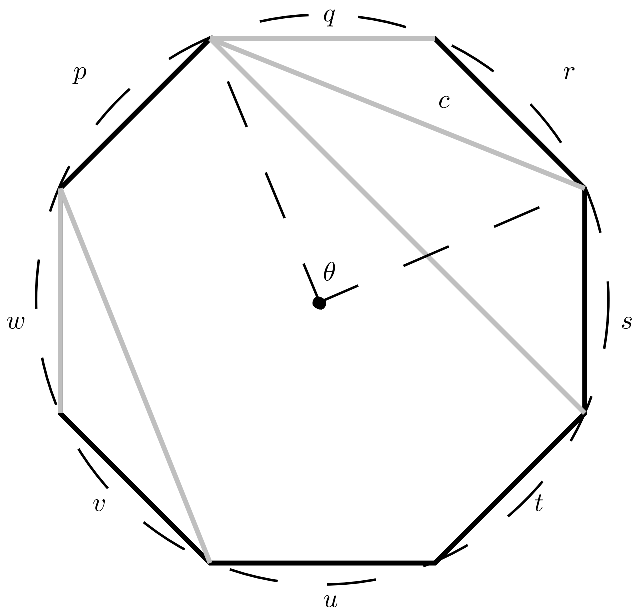


Figure 1: Inscribing an Octagon on a Circle

For an n-gon, where n is odd,

$$U'_{md}(R_n) = 2n \cdot \sin^2\left(\frac{\pi}{n}\right) \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor - 1} \csc^2\left(\frac{j\pi}{n}\right)$$

Similarly, for an even n,

$$U'_{md}(R_n) = n \cdot \sin^2\left(\frac{\pi}{n}\right) \left(\csc^2\left(\frac{\pi(n-2)}{2n}\right) + 2 \cdot \sum_{j=1}^{\frac{n}{2}-2} \csc^2\left(\frac{j\pi}{n}\right) \right)$$

Convexity and U'_{md}

Many proofs relating to the isoperimetric problem begin by proving that a region with maximum area must be convex [2, 5, 16, 6]. Generally, we can note that the perimeter of the convex hull, $H(P)$, will be less than or equal to that of P, whereas the area inside $H(P)$ will be greater than or equal to that of P [2, 5]. (See Figure 2.) Similarly, reflecting collections of edges can give a new polygon with greater area without changing edge length [16].

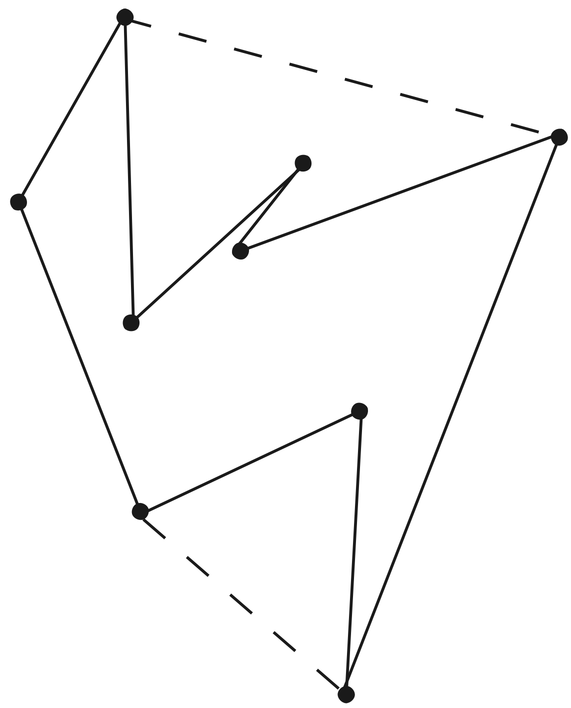


Figure 2: Non-Convex Polygon with Convex Hull of Greater Area

Convexifying Planar n-gons

It was relatively simple to prove that we can decrease the U'_{md} with a reflection. The next problem to address was whether or not a finite number of these reflections could produce a convex polygon. Since, as is clear from Figure 3, some polygons require many reflections before they are made convex. Fortunately, we found that this problem was first proposed by the famous mathematician Paul Erdös in 1934, and then solved by Bèla Nagy in 1939 [3, 14]. Toussaint gives a history and summary of related contemporary problems in [14]. We borrow his terms "flip," "pocket," and "pocket lid" in order to better express our proof.

Definition 4. *The convex hull of an n-gon, P, is the smallest convex set containing all vertices of P, it is denoted $H(P)$. $\partial H(P)$ gives the boundary.*

Definition 5. *A pocket is a set of edges of a polygon not in $\partial H(P)$ between the vertices i and j on $\partial H(P)$. Its pocket lid is the line $\overline{ij}$.*

Definition 6. *A flip is the reflection of a pocket across a pocket lid.*

Erdös-Nagy Theorem. *Every simple planar polygon can be made convex with a finite number of flips.*

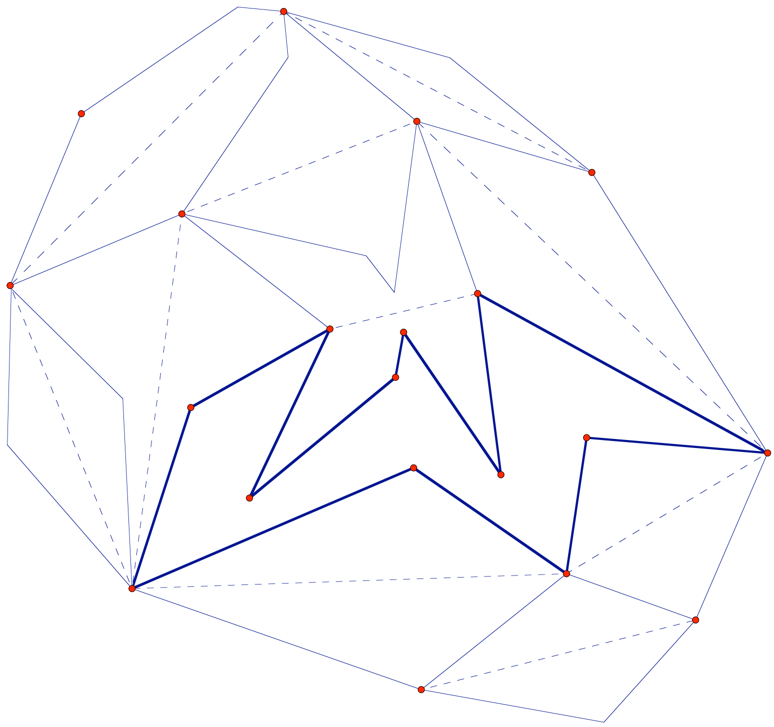


Figure 3: Reflecting Pockets over Pocket Lids

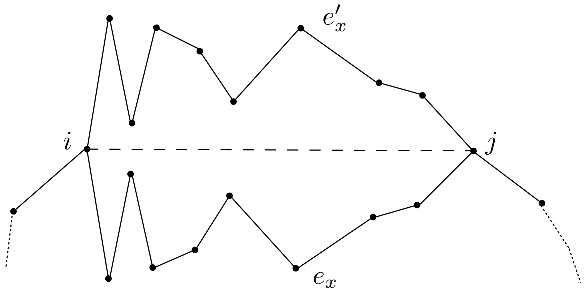


Figure 4: A Pocket, a Pocket Lid and a Flipped Pocket

Theorem 1. *If P is a planar n-gon with minimized U'_{md}, then P is convex.*

Proof.

Lemma 1. *Let P be a non-convex simple planar polygon and P' the result of a flip on P, then $U'_{md}(P) \geq U'_{md}(P')$.*

Proof. Take a pocket, p of P. We shall refer to $P - p$ as p'. Perform a flip on p. We will call the reflected collection of edges r and the new polygon made, up of p' and r, as P'.

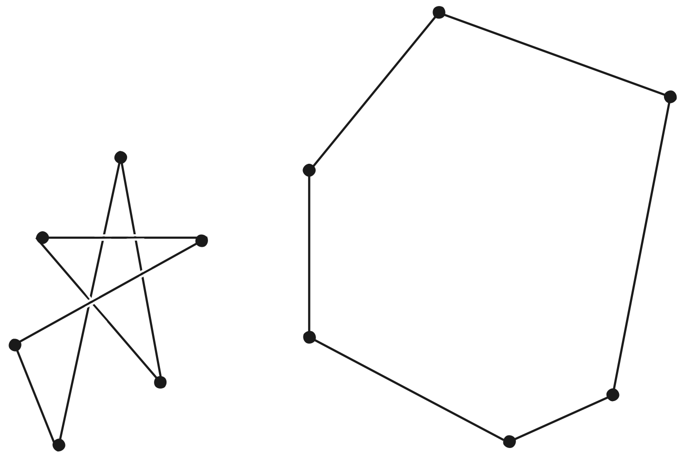
Edge length is not changed by reflections, so $\ell(e_x) = \ell(e'_x)$, where e'_x is a corresponding edge in r. Therefore $\ell(e_x)\ell(\Delta) = \ell(e'_x)\ell(\Delta)$, where Δ is a non-adjacent edge in p'. Thus, we need only show that $md(e'_x, \Delta) \geq md(e_x, \Delta)$ in order to show $\frac{\ell(e'_x)\ell(\Delta)}{md(e'_x,\Delta)^2} \leq \frac{\ell(e_x)\ell(\Delta)}{md(e_x,\Delta)^2}$.

Two nonadjacent edges in P' will either be on opposite sides of $\overline{ij}$ or on the same side.

Case 1 For the pairs (e_x, e_y) on the same side of $\overline{ij}$, the edges e_x and e_y are reflected together so $\frac{\ell(e'_x)\ell(e'_y)}{md(e'_x,e'_y)^2} = \frac{\ell(e_x)\ell(e_y)}{md(e_x,e_y)^2}$. The position of edges in p' is not affected by the flip, so distances between these edges will also remain the same after the flip. Thus, the U'_{md} of pairs of nonadjacent edges on the same side of $\overline{ij}$ is unchanged.

Case 2 Now lets us investigate the distance between nonadjacent edges which are on opposite sides of $\overline{ij}$ after the flip. Let us say that the minimum distance, $md(e_x, \Delta)$, occurs between some point δ in Δ and some point $\alpha \in e_x$. Similarly, $md(e'_x, \Delta)$ occurs between some δ' in Δ and some $\beta \in e'_x$. If we divide the plane along the pocket lid, $\overline{ij}$, we see that P is all on oneside of $\overline{ij}$, by definition of convex hull. All of r, however, is on the opposite side of $\overline{ij}$ (although i and j are on the line). Thus, $\forall\delta,\ \beta,\ |\delta - \beta| \geq |\delta - \alpha|$ which implies $\frac{\ell(e'_x)\ell(\Delta)}{md(e'_x,\Delta)^2} \leq \frac{\ell(e_x)\ell(\Delta)}{md(e_x,\Delta)^2}$.

Thus, we have looked at all minimum distances energies for edge pairs in P and P', and in all cases those for P' are less than or equal to corresponding energies for P. So, the sum of the energies for P' will be less than or equal to those of P. Thus, we can say $U_{md}(P') \leq U_{md}(P)$. $\square$



(a) (b)

Figure 5: (a) Non-Convex, Non-Planar Polygon and (b) A Convex Polygon Made by Stretching

By applying Erdös-Nagy Theorem, we know that this flipping process will eventually result in a convex polygon. Therefore, the convex polygon minimizes the minimum distance energy for planar polygons. □

Convexifying n-gons in $\mathbb{E}^3$

Sallee "stretches" polygonal curves in $\mathbb{E}^n$, a process which can increase the distance between points on different edges by changing angles, not edge lengths, and thus can be thought of as the 3-dimensional version of a flip. He finds that any polygonal curve in $\mathbb{E}^n$ can be made planar and convex with a finite number of stretches, we concern ourselves, here, with only 3 dimensions [11]. Supported by his work, we are able to prove that the planar convex n-gon minimizes U'_{md} for higher dimensions.

Definition 7. *A stretch is made by a change in angles. For P and P', polygons with corresponding lengths, P' is a stretched version of P, if $\forall\ x,\ y\ \in P$ and corresponding $x',\ y'\ \in P'$, $|x-y| \leq |x'-y'|$ [11].*

Lemma 2. *If P is a non-convex polygon in $\mathbb{E}^n$, $\exists$ a stretched polygon, P' which is planar and convex, such that $\forall$, points $x,\ y \in P$, with x and y not on the same edge of P, and corresponding $x',\ y' \in P'$, $|x-y| < |x'-y'|$.*

Theorem 2. *If P is a polygon in $\mathbb{E}^3$ there exists a convex planar polygon, P', created by stretching such that $U'_{md}(P) \geq U'_{md}(P')$.*

Proof. Let P be any polygon in $\mathbb{E}^3$ and P' the stretched convex planar polygon guaranteed by Lemma 2. Let e_x be an edge in P and e'_x be the corresponding edge in P'. Stretching does not change edge lengths. Therefore, again, we need only examine the minimum distances between two edges in each polygon. Applying [11]'s lemma, we know that for all points $x \in e_x$ and $y \in e_y$, and corresponding x' and $y' \in P'$, $|x-y| < |x'-y'|$. Thus, $md(e_x, e_y) \leq md(e'_x, e'_y)$, and $\frac{\ell(e'_x)\ell(e'_y)}{md(e'_x,e'_y)^2} \leq \frac{\ell(e_x)\ell(e_y)}{md(e_x,e_y)^2}$. □

Suggestions for Further Research

This paper has contributed towards the goal of proving that the regular n-gon minimizes U'_{md} by proving that an n-gon with minimum U'_{md} must be planar and convex. It is still necessary to prove that the ideal n-gon is equilateral and equiangular. The following are tools gained from researching isoperimetric problems which may apply to this larger problem.

The Ellipse

In isoperimetric problems, proving that an equilateral polygon will maximize area for convex polygons is often done through a study of two consecutive edges [6, 1]. Let $\overline{ab}$ and $\overline{bc}$, be consecutive edges with $\ell(\overline{ab}) \neq \ell(\overline{bc})$. Benson draws b on an ellipse with foci at a and c. Using properties of ellipses, it is easy to find a point on the ellipse b', such that $\ell(\overline{ab'}) = \ell(\overline{b'c})$ and $2\ell(\overline{ab'}) = 2\ell(\overline{b'c}) = \ell(\overline{ab}) + \ell(\overline{bc})$. The area of $\triangle abc$ is less than that of the new $\triangle ab'c$ and thus equalizing the length produces a polygon with greater area [1].

It seems that this technique of using an ellipse could be applied in attempt to minimize U'_{md} of an n-gon. Let us assume that $\ell(\overline{ab}) < \ell(\overline{bc})$. (See Figure 6.) Let e refer to an arbitrary edge between the vertices c and a which is not b or b'. The polygon P and the edge $\overline{b'c}$ are on opposite sides of the line $\overline{bc}$, so it is clear that $md(\overline{bc}, e) \leq md(\overline{b'c}, e)$. The opposite, however, is true for lines $\overline{ab}$ and $\overline{ab'}$, following a similar argument. Thus, the problem is to find a way to determine a small enough error bound on the overall contribution of the length change on the U'_{md} of the polygon, which will tell us if length equalizing decreases U'_{md}.

Another issue with applying the ellipse is that there can be situations where changing the lengths of edges can force a convex polygon to become nonconvex. Note that this has occured in the example given by Figure 6. If we allow for a combination of edge length changes and flips, however, we will also have $U'_{md}(ce, X) \geq U'_{md}(c'e, X)$, where c' is the reflection of c. So, in these cases, it seems more likely that the overall contribution of length change will be a decrease in U'_{md}.

The Circle

It is known that a polygon inscribed in a circle has a greater area that any other polygon with the same sides [8, 1]. Polygons which can be inscribed on a circle

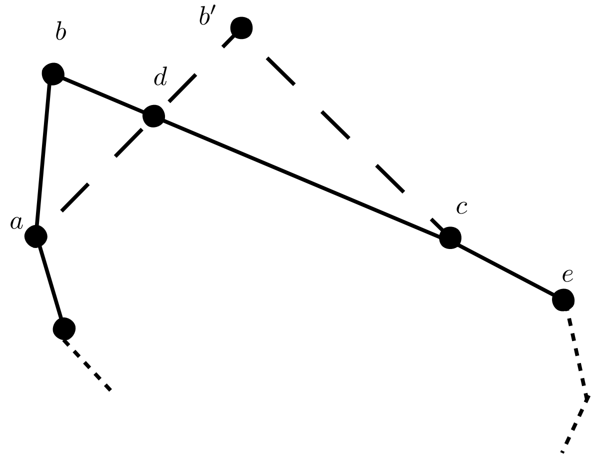


Figure 6: Changing Edge Length Can Change Convexity

are known as “chordal” or “cyclic” polygons [7]. A regular n-gon is cyclic, so perhaps investigating if cyclic polygons have minimized U'_{md} can help us to get closer to our goal. Pinelis proves that for a set of edge lengths, their exists a unique cyclic polygon with those edge lengths [7]. Thus, the task is to create an algorithm for inscribing edges on a circle that also allows us to track changes in the minimum distance between edges.

Also, the circle is valuable in determining minimum distance, since chord length can often be used to relate distances on a polygon inscribed on a circle. This may simplify the problem since equilateral polygons inscribed on a circle must also be equiangular. We can no longer apply our ellipse trick to two edges if we wish to keep them on a circle, however, which means that we must take into account new relations because of a change in the perimeter.

Fortunately, there exists a great deal of published research on cyclic polygons. This material gives us several possible “moves” which we can apply to equilize edge length. Hitt and Zhang study sequences of “midpoint-stretching polygons.” They take a chordal polygon, and create another chordal polygon with vertices at the midpoints of all edges on the first polygon. They find that if this process is repeated it creates a sequence of polygons that will converge to a regular n-gon [4]. The problem that occurs, however, when one is trying to apply this move to study U'_{md}, is that it is hard to generalize where the point of minimum distance will occur on an edge.

Paths of Minimum Distance

For the regular n-gon, we know that all paths of minimum distance occur between endpoints of edges. Since we have already found that planar convex n-gons minimize the minimum distance energy, we need only deal with planar

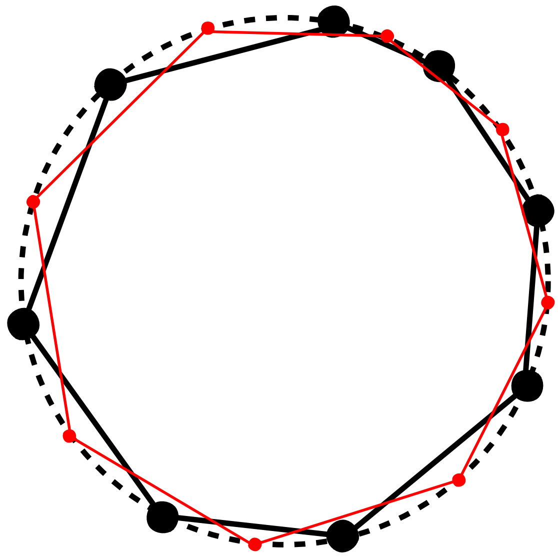


Figure 7: Midpoint-Stretching

cases. In the plane there can only be situations where the minimum distance between two edges occurs between two endpoints, or between an endpoint and an interior point. Perhaps polygons with minimum distances exclusively between endpoints have a reduce U'_{md}. If this was found to be true, it could help with the problems described above.

Acknowledgments

I feel incredibly fortunate to have been able to work on this project amongst such enthusiastic and intelligent people. I would like to thank all of the faculty and staff members who have made this program possible, as well as my professors at Smith College. I especially thank Dr. J.D. Chavez and Dr. R. Trapp for their support and insight. This work was completed during the 2007 REU program in Mathematics at California State University, San Bernardino, and was jointly sponsored by CSUSB and NSF-REU Grant DMS-0453605.

References

[1] R. V. Benson, Euclidean Geometry and Convexity, McGraw-Hill, 1966.

[2] R. F. Demar. A simple approach to isoperimetric problems in the plane, Mathematics Magazine **48**:1 (1975) 1-12.

[3] P. Erdös, Problem number 3763, Amer. Math. Monthly **42** (1935) 627.

[4] L. R. Hitt and X. Zhang, Dynamic Geometry of Polygons, Elem. Math. **56** (2001) 21-37.

[5] L.A. Lyusternik, Convex Figures and Polyhedra, Boston: D.C. Heath & Company, 1966.

[6] I. Niven, Maxima and Minima Without Calculus, Mathematical Association of America, 1981.

[7] I. Pinelis, Cyclic polygons with given edge lenths: Existence and uniqueness, J. Geom. **82** (2005) 156-171.

[8] G. Polya, Induction and Analogy in Mathematics, Princeton University Press, 1973.

[9] E. J. Rawdon and J. K. Simon, Polygonal approximation and energy of smooth knots, J. Knot Theory Ramifications **15**:4 (2006), 429-451.

[10] E. J. Rawdon and J. Worthington. Error Analysis of the Minimum Distance Energy of a Polygonal Knot and the Möbius Energy of an Approximating Curve. 2005, to appear.

[11] G. T. Sallee, Stretching chords of space curves, Geometriae Dedicata **2** (1973) 311-315.

[12] J. Simon, Energy functions for polygonal knots, J. Knot Theory Ramifications **3**:3 (1994), 299-320.

[13] J. Tam, The minimum distance energy for polygonal unknots, REU Project, California State University San Bernardino (CSUSB), 2006.

[14] G. Toussaint, The Erdös-Nagy theorem and its ramifications, Computational Geometry **31** (2005) 219-236.

[15] Y. Wu, MING, a computer program used to minimize the minimum distance energy of and visualize polygonal knots, University of Iowa http://www.math.uiowa.edu/ wu/.

[16] I. M. Yaglom and V. G. Boltyanskiĭ, Convex Figures, Holt, Rinehart and Winston, 1961.