On Ropelength of Alternating Links

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Abstract

For any given link conformation L which admits an alternating projection, we establish that $Rop(L) \geq 4Cr(L)$. We arrive at this conclusion by considering the sum of the change in heights of successive overcrossings and undercrossings as the link is traversed in a fixed direction.

1 Introduction

Several different pictures can represent the same link, such a picture is called a **projection** of the link. Figures 1 and 2 illustrate two distinct projections of the trefoil knot viewed from different angles. The **crossing number** of a link L, denoted by $Cr(L)$, is the least number of crossings that occur over all projections of the link. Figure 1 offers a projection that is non-alternating since the crossings do not abide by the over and under pattern, while Fig. 2 provides us with a projection in which the crossings alternate. It has been proven that the proportion of alternating links exponentially approaches zero with increasing crossing number [2]. Therefore as the crossing number increases, we find exponentially fewer alternating projections of any given link.

The **ropelength** of a link conformation L, denoted by $Rop(L)$, is the ratio of the arclength, $l(L)$, to the radius, $r(L)$. In other words, $Rop(L) = \frac{l(L)}{r(L)}$. Note that for a link type $\mathscr{L}$, we define the ropelength of $\mathscr{L}$ to be the minimum ropelength over all conformations L of $\mathscr{L}$; i.e., $Rop(\mathscr{L}) = min_L Rop(L)$. Also note that $Rop(L)$ is scale-invariant since $l(L)$ and $r(L)$ increase by the same factor. Without loss of generality, we assume $r(L) = 1$ which implies that $Rop(L)$=$l(L)$.

Definition 1: An **alternating conformation** of a link L is any link conformation which admits an alternating projection of L in the z direction and has $r(L)$=1. We observe that specifying the projection direction is unnecessary, but will simplify the rest of the work.

The relationship between the ropelength and crossing number has been studied by many; it has been established that for any knot K, $Rop(K)$ is bounded below by $O(Cr(K))^{\frac{3}{4}}$ (see [3] and [4]). We consider the possibility of a linear lower bound in terms of crossing number. In [1], Diao, Ernst, and Thistlethwaite pose the question, "Is the ropelength at least of the order $O(Cr(K))$ for any prime alternating knot K?" We provide a partial answer to this question.

Namely, we show that if a ropelength minimizer L of $\mathscr{L}$ admits an alternating projection, then $Rop(\mathscr{L}) \geq O(Cr(\mathscr{L}))$.

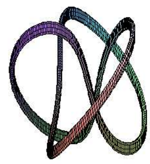

Figure 1: A non-alternating projection of the trefoil.

Figure 2: An alternating projection of the trefoil.

2 Ropelength of Alternating Links

Let L be an alternating conformation with n crossings labelled 1 through n. The i^{th} crossing consists of an overcrossing point p_i and an undercrossing point q_i which share the same x and y coordinates. Let o_i denote the height of p_i and u_i denote the height of q_i. In other words, o_i and u_i are the z-coordinates of p_i and q_i, respectively. As we traverse the link in a fixed direction, we encounter a cyclic ordered sequence of overcrossings and undercrossings along the path (which alternate *ouou...ou*) that reaches completion when the inception point is revisited.

$\odot$ For example, begin with a projection of the trefoil. Label the crossings 1, 2, and 3 as you traverse the knot in a fixed direction (see Fig. 3). The over- and under- crossing points are then labelled accordingly as in Fig. 4.

Definition 2: Let L be an alternating conformation. The height function, $h : L \to \mathbf{R}$, is the projection $h(x, y, z) = z$ which takes every point on L to its z-coordinate. The **image** of L under the projection, denoted by $h(L)$, is a path that goes up and down along the z-axis.

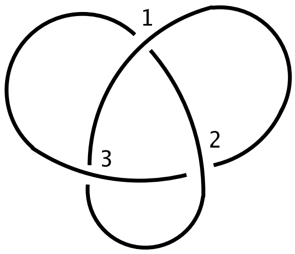

Figure 3: An alternating projection of the trefoil with 3 crossings.

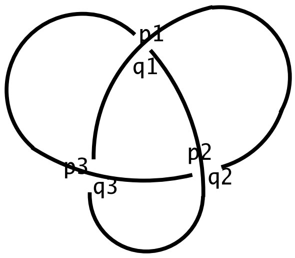

Figure 4: p_i and q_i, $1 \leq i \leq 3$.

Note that under the projection, $h(p_i) = o_i$ and $h(q_i) = u_i$ so that the values o_i and u_i partition the z-axis into subintervals.

$\odot$ Continuing with our example, Fig. 5 is a diagram of $h(L)$ along the z-axis where we now represent the p_i by their respective heights o_i, and q_i by u_i. The series of the path runs as $o_1u_2o_3u_1o_2u_3$.

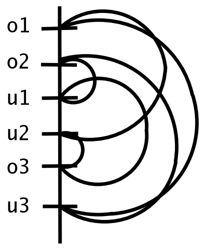

Figure 5: $h(L)$ with o_i and u_i, $1 \leq i \leq 3$.

Observation 1: Note that for each crossing, $o_i > u_i$. In fact, there is at least a distance of 2 separating o_i and u_i since the radius of each strand is 1; therefore $(o_i - u_i) \geq 2$.

Lemma 1: If L is an alternating conformation, the ropelength of L is greater than or equal to the length of $h(L)$.
Proof: Since the projection h does not increase the distance between points, the arclength of L is at least the length of $h(L)$. ■

Our goal is to use the height function h to obtain a lower bound on the ropelength of L. It is difficult to measure the length of $h(L)$ directly. Consequently, we will replace the potentially unwieldy arclengths that compose $h(L)$ with straight line segments that connect heights of overcrossings to those of undercrossings that precede and succeed them on the link. We will call this new image of L, where the arcs between successive overcrossings and undercrossings have been tightened or straightened, the **taught** image of L, denoted by $t(L)$. An **edge** is the line segment in $t(L)$ that connects successive over- and undercrossings. Note that each o_i and u_j has exactly two edges incident with it in $t(L)$, and each edge has one endpoint of each type.

⊙ Again, referring to our example, Fig. 6 is the translation of the diagram of $h(L)$ into a *taught* diagram of the connections between o_i and u_j. Note that the straight line segments have been bent slightly for viewing purposes in Fig. 6.

Lemma 2: Let L be an alternating conformation. The ropelength of L is at least the length of $t(L)$.
Proof: Both of the paths $h(L)$ and $t(L)$ connect the heights o_i, u_j in the same cyclic order. Since the arc in $t(L)$ connecting consecutive o_i and u_i is a straight line, its length is at most the length of the corresponding piece in $h(L)$. Summing over all consecutive points, the sum of the straight line segments in $t(L)$ is at most the sum of the arcs composing $h(L)$. The result now follows from Lemma 1. ■

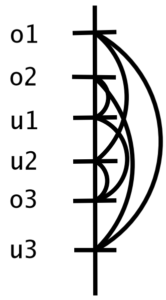

Figure 6: *Taught* image $t(L)$.

⊙ In our example $Rop(L) \geq$ length of $h(L) \geq$ length of $t(L) = |o_1 - u_2| + |u_2 - o_3| + |o_3 - u_1| + |u_1 - o_2| + |o_2 - u_3| + |u_3 - o_1|$ by Lemma 2.

Definition 3: Let z_0 be any height along the z-axis. Any pair (o_i, u_i) is **split** if z_0 lies between the pairing, i.e., $o_i > z_0 > u_i$; otherwise the pair is **unsplit**. Observe in Fig. 8 that o_1 and u_1 are split by z_0. The pair (o_1, u_1) is unsplit by z_0 in Fig. 10.

We now discover a linear lower bound for the length of an alternating conformation of L by establishing a relationship between the length of $t(L)$ and the crossing number of L. First we find a lower bound for the length of $t(L)$, which can be manipulated in such a way as to provide an inequality regarding the crossing number of L.

Lemma 3: Let L be an alternating conformation, z_0 a particular height on the z-axis, and b the number of pairs (o_i, u_i) split by z_0. In $t(L)$ there are at least $2b$ edges from overcrossings above z_0 that must run by z_0 to connect to undercrossings below.
Proof: Let a be the number of unsplit pairs that lie above z_0 ($u_i > z_0$), b the number of pairs split by z_0, and c the number of unsplit pairs beneath z_0 ($o_i < z_0$). Note that $a + b + c = n$. There are $a + b$ overcrossings and a undercrossings above z_0. The case that results in the least number of edges that run by z_0 occurs when every undercrossing above z_0 connects to an overcrossing above z_0. Thus there are $2(a + b)$ total edges incident with the overcrossings above z_0 and $2a$ total edges incident with the undercrossings above z_0. Once all the edges leaving from the undercrossings above z_0 are connected to overcrossings above z_0, $2(a + b) - 2a = 2b$ total edges leaving the overcrossings remain. These edges must cross z_0 to connect to the undercrossings below z_0. ■

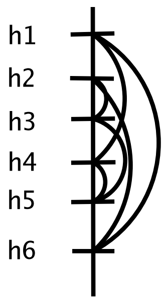

Figure 7: $h_1, ..., h_{2n}, 1 \leq n \leq 3$.

Lemma 4: Let L be an alternating conformation and h_i, $1 \leq i \leq 2n$, a representation of the overcrossing and undercrossing heights in descending order. The length of $t(L) \geq \sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1})$.
Proof: Represent the overcrossing and undercrossing heights as an ordered sequence of points, $h_1, ..., h_{2n}$, along the z-axis. We will define b_m to be the number of split pairs that corresponds to z_0 cutting the z-axis between the point h_m and the point directly below, h_{m+1}. On the interval (h_m, h_{m+1}), we have at least $2b_m$ edges spanning the length of the interval, therefore every interval contributes a distance of at least $2b_m(h_m - h_{m+1})$ and the total length of $t(L)$. Thus the length of the taught image is greater than or equal to $\sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1})$. ■

⊙ Back to our example, represent the heights o_1, o_2, u_1, u_2, o_3, and u_3 as $h_1, ..., h_{2n}, 1 \leq n \leq 3$ along the z-axis(see Fig.7). Let z_0 intersect every interval between consecutive crossing heights (see Figures 8, 9, 10, 11, and 12).

Lemma 5: If L is an alternating conformation, then $\sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1}) = \sum_{i=1}^{n} 2(o_i - u_i)$. ■
Proof: We will show that the sum $\sum_{i=1}^{n}(o_i - u_i)$ can be subdivided and rearranged to form $\sum_{m=1}^{2n-1} b_m(h_m - h_{m+1})$. Let o_i and u_i be represented by h_x and h_y respectively. The interval $[h_y, h_x]$ is partitioned by intermediate heights $h_x \geq h_{x+1} \geq ... \geq h_{y-1} \geq h_y$. Therefore the length of the in-

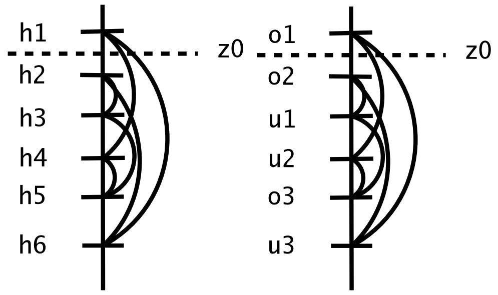

Figure 8: b_1=1.

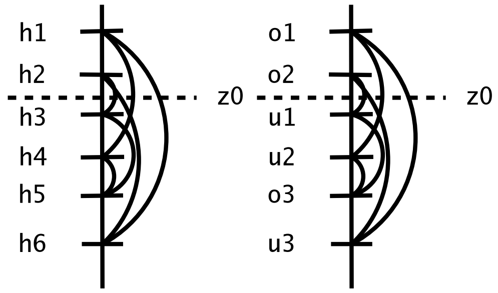

Figure 9: b_2=2.

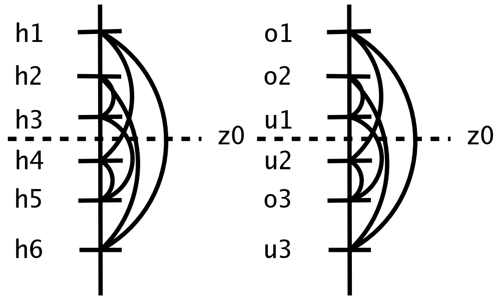

Figure 10: b_3=1.

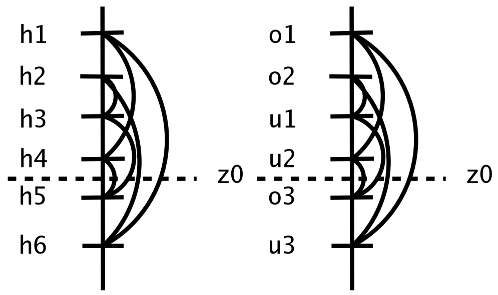

Figure 11: b_4=0.

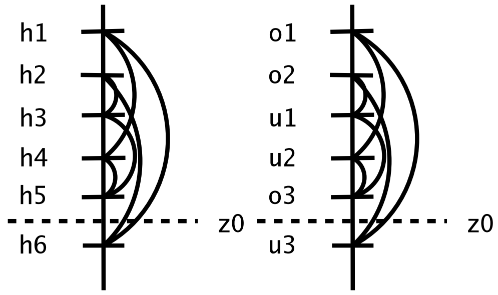

Figure 12: b_5=1.

terval $[u_i, o_i]$ can be written as $(o_i - u_i) = \sum_{j=x}^{y-1}(h_j - h_{j+1})$, and we have $\sum_{i=1}^{n}(o_i - u_i) = \sum_{i=1}^{n}(\sum_{j=x_i}^{y_i-1}(h_j - h_{j+1}))$. We now determine how often the length $h_m - h_{m+1}$ occurs in the double sum $\sum_{i=1}^{n}(\sum_{j=x_i}^{y_i-1}(h_j - h_{j+1}))$. Note that $h_m - h_{m+1}$ occurs once in the expansion of $o_i - u_i$ if and only if the heights between $[h_{m+1}, h_m]$ split the pair (o_i, u_i). Thus $h_m - h_{m+1}$ occurs exactly b_m times in the double sum, and we have shown $\sum_{i=1}^{n}(o_i - u_i) = \sum_{i=1}^{n}(\sum_{j=x_i}^{y_i-1}(h_j - h_{j+1})) = \sum_{m=1}^{2n-1} b_m(h_m - h_{m+1})$. ■

⊙ We now illustrate Lemmas 4 and 5 using our example. We want $|o_1 - u_2| + |u_2 - o_3| + |o_3 - u_1| + |u_1 - o_2| + |o_2 - u_3| + |u_3 - o_1| \geq 2[(o_1 - u_1) + (o_2 - u_2) + (o_3 - u_3)]$. By Lemma 4, $|o_1 - u_2| + |u_2 - o_3| + |o_3 - u_1| + |u_1 - o_2| + |o_2 - u_3| + |u_3 - o_1| \geq 2b_1(h_1 - h_2) + 2b_2(h_2 - h_3) + 2b_3(h_3 - h_4) + 2b_4(h_4 - h_5) + 2b_5(h_5 - h_6) = 2(h_1 - h_2) + 4(h_2 - h_3) + 2(h_3 - h_4) + 0 + 2(h_5 - h_6) = 2(o_1 - u_1) + 2(o_2 - u_2) + 2(o_3 - u_3)$.

Theorem 1: If L is an alternating conformation, then $Rop(L) \geq 4Cr(L)$.
$Proof$: Since $(o_i - u_i) \geq 2$ we have $\sum_{i=1}^{n} 2(o_i - u_i) \geq \sum_{i=1}^{n} 2(2) = 4n = 4Cr(L)$. Combining this observation with Lemmas 1 through 5 we have $Rop(L) \geq l(h(L)) \geq l(t(L)) \geq \sum_{i=1}^{n} 2(o_i - u_i) \geq 4Cr(L)$. ■

In our final corollaries we offer a partial positive answer to the question posed by Diao, Ernst, and Thistlethwaite.

Corollary 1: If $\mathscr{L}$ is an alternating link type with an alternating conformation that realizes $Rop(\mathscr{L})$, then $Rop(\mathscr{L})$ is at least $O(Cr(K))$.

Corollary 2: If $\mathscr{L}$ is an alternating link type and $Rop(\mathscr{L}) < 4Cr(\mathscr{L})$, then ropelength minimizers for $\mathscr{L}$ do not admit alternating projections.
$Proof$: This is the contrapositive of Theorem 1. ■

3 Conclusion

By looking at an alternating link conformation in terms of overcrossing and undercrossing heights and cutting intervals formed by successive points, we formed a relation between split intervals and the distance between o_i and u_i. We established this distance to be at least 2, thereby enabling us to form an inequality between $Rop(L)$ and $Cr(L)$. Suggestions for further research include discovering examples of alternating link types ($\mathscr{L}$) with $Rop(\mathscr{L}) < 4Cr(\mathscr{L})$, analyzing non-alternating projections of alternating links to completely resolve the question posed by Diao, Ernst, and Thistlethwaite, and finally, applying similar techniques to bridge and superbridge numbers.

4 Acknowledgements

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