THE LINEAR INDEPENDENCE OF SETS OF TWO AND THREE CANONICAL ALGEBRAIC CURVATURE TENSORS

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Abstract. Provided certain basic rank requirements are met, we establish a converse of the classical fact that if A is symmetric, then R_A is an algebraic curvature tensor. This allows us to establish a simultaneous diagonalization result in the event that three algebraic curvature tensors are linearly dependent. We use these results to establish necessary and sufficient conditions that a set of two or three algebraic curvature tensors be linearly.

1. Introduction

Let V be a real vector space of finite dimension n. Let R be a real valued function whose domain is $V \times V \times V \times V$. Then R is called an *algebraic curvature tensor* [4] if it is linear in all four of its entries and if it satisfies the following three properties for all $x, y, z, w \in V$:

$$\begin{array}{rcl} R(x,y,z,w) & = & -R(y,x,z,w), \\ R(x,y,z,w) & = & R(z,w,x,y), \text{and} \\ 0 & = & R(x,y,z,w) + R(x,z,w,y) \\ & & \quad +R(x,w,y,z)\,. \end{array} \tag{1.a}$$

The last property is known as the *Bianchi identity.* Let $\mathcal{A}(V)$ be the vector space of all algebraic curvature tensors on V.

Let φ be a bilinear form on V. We say φ is *symmetric* if $\varphi(v,w) = \varphi(w,v)$ for all $v, w \in V$, and we say φ is *positive definite* if for all $v \in V$, $\varphi(v,v) \geq 0$, and equal to zero only when $v = 0$.

Let φ be a symmetric bilinear form on V. Define R_φ by

$$R_\varphi(x,y,z,w) = \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w) \tag{1.b}$$

It can be easily verified that R_φ satisfies the properties of Equation (1.a), and is thus an algebraic curvature tensor. Furthermore, it is known that $\mathcal{A}(V) = \text{Span}\{R_\varphi\}$ [Fiedler [1, 2]]. In other words, every algebraic curvature tensor can be expressed as a linear combination of R_φ's. This leads to the question: given $\varphi_1, \ldots, \varphi_k$, when is the set $\{R_{\varphi_1}, \ldots, R_{\varphi_k}\}$ linearly independent? In this paper, we use linear algebraic methods to present new results related to sets of two and three algebraic curvature tensors as defined in Equation (1.b).

A brief outline of the paper is as follows. Throughout, we assume that ψ and τ are symmetric bilinear forms, so that $R_\psi, R_\tau \in \mathcal{A}(V)$. In Section 2, we prove the following result regarding the linear independence of two algebraic curvature tensors:

Theorem 1.1. *Suppose* Rank $\varphi \geq 3$. *The set* $\{R_\varphi, R_\psi\}$ *is linearly dependent if and only if* $R_\psi \neq 0$, *and* $\varphi = \lambda\psi$ *for some* $\lambda \in \mathbb{R}$.

Section 3 is devoted to proving

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Theorem 1.2. *Suppose φ is positive definite,* Rank $\tau = n$*, and* Rank $\psi \geq 3$*. If $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then ψ and τ are simultaneously orthogonally diagonalizable with respect to φ.*

In section 4 we use Theorem 1.2 to prove our main result regarding the linear independence of three algebraic curvature tensors. We denote the *spectrum* of φ, $\text{Spec}(\varphi)$, as the set of eigenvalues of φ, repeated according to multiplicity, and $|\text{Spec}(\varphi)|$ as the number of distinct elements of $\text{Spec}(\varphi)$. It is understood that if any of the quantities do not make sense in Condition (2) below, then Condition (2) is not satisfied.

Theorem 1.3. *Suppose* $\dim(V) \geq 4$*, φ is positive definite,* Rank $\tau = n$*, and* Rank $\psi \geq 3$*. The set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent if and only if one of the following is true:*

(1) $|\text{Spec}(\psi)| = |\text{Spec}(\tau)| = 1$.
(2) $\text{Spec}(\tau) = \{\eta_1, \eta_2, \eta_2, \ldots\}$*, and* $\text{Spec}(\psi) = \{\lambda_1, \lambda_2, \lambda_2, \ldots\}$*, with $\eta_1 \neq \eta_2$, $\lambda_2^2 = \epsilon(\delta\eta_2^2 - 1)$, and $\lambda_1 = \frac{\epsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1)$.*

It is interesting to note that in Theorem 1.1 and Theorem 1.2, we assume the rank of a certain object to be at least 3. However, in Theorem 1.3 we require that $\dim(V) \geq 4$, but there is no corresponding requirement that all objects involved have a rank of at least 4. Indeed, there exists examples in dimension 3 where Theorem 1.3 does not hold, although the situation is more complicated. See Theorem 5.1 for a detailed description of this situation.

2. Preliminaries

The goal of this section is to give the necessary background needed to understand the proofs in later sections.

Let φ be a bilinear form on V. We say φ is *nondegenerate* if and only if for all $v \neq 0 \in V$, there exists $w \in V$ such that $\varphi(v, w) \neq 0$. It is easy to check that if φ is positive definite, then φ is also nondegenerate, however the converse does not hold.

Let ψ be some other bilinear form on V. Then $\psi(v, w) = \varphi([\psi_{ij}]v, w)$ for some unique linear transformation $[\psi_{ij}]$. We say ψ can be represented by the linear transformation so that $\psi(e_i, e_j) = \psi_{ij}$ is the (i, j) entry of the matrix $[\psi_{ij}]$, where $e_1, \ldots, e_n$ are basis vectors on V. If ψ is symmetric, then the matrix representation is also symmetric, that is to say the matrix $[\psi_{ij}] = [\psi_{ij}]^T$, where $[\psi_{ij}]^T$ is the transpose of $[\psi_{ij}]$.

The *eigenvalues* of $[\psi_{ij}]$ denoted λ_i are the roots of the equation $\det(\lambda\mathcal{I} - [\psi_{ij}]) = 0$ where $\mathcal{I}$ is the $n \times n$ identity matrix.

A diagonal matrix is a square matrix in which entries outside the main diagonal are all zero. A matrix $[\psi_{ij}]$ is diagonalizable if and only if there exists a basis of V consisting of eigenvalues of $[\psi_{ij}]$. The entries of the main diagonal are then the eigenvalues of $[\psi_{ij}]$.

An *orthonormal basis* is a basis $\{f_1, \ldots, f_n\}$ so that $f_1, \ldots, f_n$ are mutually orthogonal vectors with magnitude 1. If φ is positive definite and $\varphi([\psi_{ij}]v, w) = \varphi(v, [\psi_{ij}]w)$ for some linear transformation $[\psi_{ij}]$, then there exists a symmetric ψ defined by $\psi(v, w) = \varphi([\psi_{ij}]v, w)$. There also exists an orthonormal basis on V that simultaneously diagonalizes $[\psi_{ij}]$ and $[\varphi_{ij}]$ with respect to φ.

Let $\pi = \text{Span}\{e_1, e_2, \ldots, e_k\}$ where $1 \leq k < n$ and $e_1, \ldots, e_k$ are basis vectors for V. Denote φ restricted to π as $\varphi|_\pi$ to mean the $k \times k$ minor of $[\varphi_{ij}]$ corresponding to $e_1, \ldots, e_k$.

The R_φ's have the following additional properties:

$$\begin{aligned} R_\varphi &= R_{-\varphi} \\ \alpha R_\varphi &= R_{\sqrt{|\alpha|}\varphi} \end{aligned} \tag{2.a}$$

These are gathered from direct computations in Equation (1.b).

3. Linear independence of two algebraic curvature tensors

This section begins our study of linear independence of algebraic curvature tensors. Some preliminary remarks are in order before we begin this study.

Let $\varphi, \varphi_i : V \to V$ be a collection of symmetric bilinear forms. By Properties (2.a), for any real number c, we have $cR_\varphi = \epsilon R_{|c|^{\frac{1}{2}}\varphi}$, where $\epsilon = \text{sign}(c) = \pm 1$. Let $c_i \in \mathbb{R}$, and let $\epsilon_i = \text{sign}(c_i) = \pm 1$. By replacing φ_i with $\psi_i = |c|^{\frac{1}{2}}\varphi_i$, we may express any linear combination of algebraic curvature tensors

$$\sum_{i=1}^{k} c_i R_{\varphi_i} = \sum_{i=1}^{k} \epsilon_i R_{\psi_i}.$$

Thus the study of linear independence of algebraic curvature tensors amounts to a study of when a sum or difference of R_{φ_i} equal another canonical algebraic curvature tensor. This would always be the case if each of the φ_i are multiples of one another– this possibility is discussed here. Proceeding systematically from the case of two algebraic curvature tensors, we would assume that each of the constants c_i are nonzero, leading us to study the equation $R_{\varphi_1} \pm R_{\varphi_2} = 0$ in this section, and, for ϵ and δ a choice of signs, $R_{\varphi_1} + \epsilon R_{\varphi_2} = \delta R_{\varphi_3}$ in Section 4. The following result found in [5] will be of use, and we state it here for completeness.

Lemma 3.1. *If* Rank $\varphi \geq 3$, *then* $R_\varphi = R_\psi$ *if and only if* $\varphi = \pm\psi$.

We explore the possibility that $R_\varphi = -R_\psi$ with a lemma. The proof follows similarly to the proof in [5].

Lemma 3.2. *Suppose* Rank $\varphi \geq 3$. *There does not exist a symmetric* ψ *so that* $R_\varphi = -R_\psi$.

Proof. Suppose to the contrary that there is such a solution. By replacing φ with $-\varphi$ if need be, we may assume that there are vectors e_1, e_2, e_3 with the relations $\varphi(e_1, e_1) = \varphi(e_2, e_2) = \epsilon\varphi(e_3, e_3) = 1$, where $\epsilon = \pm 1$. Thus on $\pi = \text{Span}\{e_1, e_2\}$, the form $\varphi|_\pi$ is positive definite, and we may diagonalize $\psi|_\pi$ with respect to $\varphi|_\pi$. Therefore the matrix $[(\psi|_\pi)_{ij}]$ of $\psi|_\pi$ has $(\psi|_\pi)_{12} = (\psi|_\pi)_{21} = 0$, and $(\psi|_\pi)_{ii} = \lambda_i$ for $i = 1, 2, 3$. Now we compute

$$1 = R_\varphi(e_1, e_2, e_2, e_1) = -\lambda_1\lambda_2, \tag{3.a}$$

so λ_1 and $\lambda_2 \neq 0$. We compute

$$0 = R_\varphi(e_1, e_2, e_3, e_1) = -\lambda_1(\psi|_\pi)_{23},$$

so $(\psi|_\pi)_{23} = 0$. Similarly,

$$0 = R_\varphi(e_2, e_1, e_3, e_2) = -\lambda_2(\psi|_\pi)_{13},$$

and so $(\psi_\pi)_{13} = 0$. Now, for $j = 1, 2$, we have

$$\epsilon = R_\varphi(e_j, e_3, e_3, e_j) = -\lambda_j\lambda_3.$$

We conclude $\lambda_3 \neq 0$, and that $\lambda_1 = \lambda_2$. This contradicts Equation (3.a), since it would follow that $1 = -\lambda_1^2 < 0$. □

We now use Lemma 3.1 and Lemma 3.2 to establish Theorem 1.1.

Proof. (Proof of Theorem 1.1.) Suppose $c_1 R_\varphi + c_2 R_\psi = 0$, and at least one of c_1 or c_2 is not zero. Since $R_\psi \neq 0$, and φ is of rank 3 or more (which implies $R_\varphi \neq 0$), we conclude that both $c_1, c_2 \neq 0$. Thus, we may write

$$c_1 R_\varphi + c_2 R_\psi = 0 \Leftrightarrow R_\varphi = \epsilon R_{\tilde{\lambda}\psi}$$

for some $\tilde{\lambda} \neq 0$, and for ϵ a choice of signs. If $\epsilon = 1$, then we use Lemma 3.1 to conclude that $\varphi = \pm\tilde{\lambda}\psi$, in which case $\varphi = \lambda\psi$ for $0 \neq \lambda = \pm\tilde{\lambda}$. Lemma 3.2 eliminates the possibility that $\epsilon = -1$.

Conversely, suppose $\varphi = \lambda\psi$ for some $\lambda \neq 0$. Then we have

$$\begin{aligned} R_\varphi + (-\lambda^2)R_\psi &= R_{\lambda\psi} + (-\lambda^2)R_\psi \\ &= \lambda^2 R_\psi + (-\lambda^2)R_\psi \\ &= 0. \end{aligned}$$

This demonstrates the linear dependence of the tensors R_φ and R_ψ and completes the proof. □

4. A study of the tensors R_A and commuting symmetric endomorphisms

Suppose φ is a symmetric bilinear form on V which is nondegenerate, and let A be an endomorphism of V. Let A^* be the adjoint of A with respect to φ, characterized by the equation $\varphi(Ax, y) = \varphi(x, A^*y)$. We say that A is *symmetric* if $A^* = A$, and we say that A is *skew-symmetric* if $A^* = -A$. For the remainder of this section, we will consider the adjoint A^* of a linear endomorphism A of V with respect to the form φ.

The goal of this section is to prove Theorem 1.2. We begin by constructing the following object.

Definition 4.1. *If $A : V \to V$ is a linear map, then we may create the element R_A by*

$$R_A(x, y, z, w) = \varphi(Ax, w)\varphi(Ay, z) - \varphi(Ax, z)\varphi(Ay, w). \tag{4.a}$$

In the event that A is the identity map, $R_A = R_\varphi$. The object R_A satisfies the first property in Equation (1.a), although one requires A to be symmetric to ensure $R_A \in \mathcal{A}(V)$. In the event that $A^* = -A$, there is a different construction [4].

Lemma 4.2. *Suppose $A, B, \bar{A} : V \to V$.*

1. $R_{A+B} + R_{A-B} = 2R_A + 2R_B$.
2. *If $R_A \in \mathcal{A}(V)$, then $R_{A^*} \in \mathcal{A}(V)$, and $R_A = R_{A^*}$.*
3. *If $\bar{A}^* = -\bar{A}$ and $R_{\bar{A}} \in \mathcal{A}(V)$, then $R_{\bar{A}}(x, y, z, w) = \varphi(\bar{A}x, y)\varphi(\bar{A}w, z)$.*

Proof. Assertion (1) follows from direct computation using Definition 4.1. To prove Assertion (2), let $x, y, z, w \in V$, and we compute

$$\begin{aligned} R_A(x, y, z, w) &= R_A(z, w, x, y) \\ &= \varphi(Az, y)\varphi(Aw, x) - \varphi(Az, x)\varphi(Aw, y) \\ &= \varphi(A^*y, z)\varphi(A^*x, w) - \varphi(A^*x, z)\varphi(A^*y, w) \\ &= R_{A^*}(x, y, z, w). \end{aligned}$$

Now we prove Assertion (3). Note that since $\bar{A}^* = -\bar{A}$, for all $u, v \in V$ we have $\varphi(\bar{A}v, u) = -\varphi(v, \bar{A}u) = -\varphi(\bar{A}u, v)$. We use the Bianchi identity to see that

$$\begin{aligned} 0 &= R_{\bar{A}}(x, y, z, w) + R_{\bar{A}}(x, w, y, z) + R_{\bar{A}}(x, z, w, y) \\ &= \quad \varphi(\bar{A}x, w)\varphi(\bar{A}y, z) - \varphi(\bar{A}x, z)\varphi(\bar{A}y, w) \\ &\quad +\varphi(\bar{A}x, z)\varphi(\bar{A}w, y) - \varphi(\bar{A}x, y)\varphi(\bar{A}w, z) \\ &\quad +\varphi(\bar{A}x, y)\varphi(\bar{A}z, w) - \varphi(\bar{A}x, w)\varphi(\bar{A}z, y) \\ &= 2\varphi(\bar{A}x, w)\varphi(\bar{A}y, z) - 2\varphi(\bar{A}x, z)\varphi(\bar{A}y, w) \\ &\qquad +2\varphi(\bar{A}x, y)\varphi(\bar{A}z, w) \\ &= 2R_{\bar{A}}(x, y, z, w) + 2\varphi(\bar{A}x, y)\varphi(\bar{A}z, w). \end{aligned}$$

It follows that $R_{\bar{A}}(x, y, z, w) = -\varphi(\bar{A}x, y)\varphi(\bar{A}z, w) = \varphi(\bar{A}x, y)\varphi(\bar{A}w, z)$. □

Remark 4.3. *With regards to Assertion (2) of Lemma 4.2, we will only require that $R_{A^*} \in \mathcal{A}(V)$ if $R_A \in \mathcal{A}(V)$. The fact that $R_{A^*} = R_A$ is not needed, although it simplifies some calculations (see Equation (4.b)).*

We now use Lemma 4.2 to prove

Lemma 4.4. *Let $A : V \to V$, and $R_A \in \mathcal{A}(V)$. If* Rank $A \geq 3$*, then $A^* = A$.*

Proof. Define $\bar{A} = A - A^*$. Then $\bar{A}^* = -\bar{A}$. Then using $B = A^*$ in Assertion (1) of Lemma 4.2 we have $R_A = R_{A^*} \in \mathcal{A}(V)$ and

$$R_{\bar{A}} = 4R_A - R_{A+A^*}. \tag{4.b}$$

Since $(A + A^*)^* = A + A^*$, we have $R_{A+A^*} \in \mathcal{A}(V)$, and so, as the linear combination of algebraic curvature tensors, $R_{\bar{A}} \in \mathcal{A}(V)$. Thus by Lemma 4.2, we conclude that $R_{\bar{A}}(x, y, z, w) = \varphi(\bar{A}x, y)\varphi(\bar{A}w, z)$.

Since $\bar{A}$ is skew-symmetric with respect to φ, Rank $\bar{A}$ is even. We note that if Rank $\bar{A} = 0$, then $\bar{A} = 0$, and $A = A^*$. So we break the remainder of the proof up into two cases: Rank $\bar{A} \geq 4$, and Rank $\bar{A} = 2$.

Suppose Rank $\bar{A} \geq 4$. Then there exist $x, y, z, w \in V$ with

$$\varphi(\bar{A}x, y) = \varphi(\bar{A}w, z) = 1, \text{ and } \varphi(\bar{A}x, z) = \varphi(\bar{A}x, w) = 0.$$

Then we compute $R_{\bar{A}}(x, y, z, w)$ in two ways. First, we use Definition 4.1, and next we use Assertion (3) of Lemma 4.2:

$$\begin{aligned}
R_{\bar{A}}(x, y, z, w) &= \varphi(\bar{A}x, w)\varphi(\bar{A}y, z) - \varphi(\bar{A}x, z)\varphi(\bar{A}y, w) \\
&= 0. \\
R_{\bar{A}}(x, y, z, w) &= \varphi(\bar{A}x, y)\varphi(\bar{A}w, z) \\
&= 1.
\end{aligned}$$

This contradiction shows that Rank $\bar{A}$ is not 4 or more.

Finally, we assume Rank $\bar{A} = 2$. There exists a basis $\{e_1, e_2, \ldots, e_n\}$ that is orthonormal with respect to φ, where $\ker \bar{A} = \mathrm{Span}\{e_3, \ldots, e_n\}$, and we have the relations $\varphi(\bar{A}e_2, e_1) = -\varphi(\bar{A}e_1, e_2) = \lambda \neq 0$. Let $A_{ij} = \varphi(Ae_i, e_j)$ be the (j, i) entry of the matrix A with respect to this basis, similarly for $\bar{A}$, A^*, and $A + A^*$. With respect to this basis, the only nonzero entries $\bar{A}_{ij}$ are $A_{12} - A_{12} = \bar{A}_{12} = -\bar{A}_{21} = \lambda$. Thus, unless $\{i, j\} = \{1, 2\}$, we have $A_{ij} = A_{ji}$, and in such a case, we have $(A + A^*)_{ij} = 2A_{ij}$.

Now suppose that $\{i, j\} \not\subseteq \{1, 2\}$. We compute $R_{\bar{A}}(e_i, e_2, e_j, e_1)$ in two ways. According to Assertion (3) of Lemma 4.2, we have

$$R_{\bar{A}}(e_i, e_2, e_j, e_1) = \varphi(\bar{A}e_i, e_2)\varphi(\bar{A}e_1, e_j) = 0. \tag{4.c}$$

Now according to Equation (4.b), we have

(4.d)

$$\begin{aligned}
R_{\bar{A}}(e_i, e_2, e_j, e_1) &= (4R_A - R_{A+A^*})(e_i, e_2, e_j, e_1) \\
&= 4A_{i1}A_{2j} - 4A_{ij}A_{21} - (2A_{i1})(2A_{2j}) + (2A_{ij})(A_{21} + A_{12}) \\
&= 2A_{ij}(A_{12} - A_{21}) \\
&= 2\lambda A_{ij}.
\end{aligned}$$

Comparing Equations (4.c) and (4.d) and recalling that $\lambda \neq 0$, we see that $A_{ij} = 0$ if one or both of i or j exceed 2. This shows that Rank $A \leq 2$, which is a contradiction to our original hypothesis. □

Remark 4.5. *If* Rank $A = 1$ *or* 0*, then $R_A = 0$, and in the rank 1 case, A^* need not equal A. In addition, if A is* any *rank 2 endomorphism, then there exists examples of $R_A \neq 0$ where $A^* \neq A$. Thus, Lemma 4.4 can fail if* Rank $A \leq 2$.

We may now prove the following as a corollary to Lemma 4.4.

Lemma 4.6. *Let $A : V \to V$, and $R_A = R_\psi \in \mathcal{A}(V)$. If* Rank $A \geq 3$*, then A is symmetric, and $A = \pm\psi$.*

Proof. According to Lemma 4.4, we conclude A is symmetric. Since Rank $A \geq 3$ we apply Lemma 3.1 to conclude $A = \pm\psi$. □

The following lemma is easily verified using Definition 4.1.

Lemma 4.7. *Suppose $\theta : V \to V$. For all $x, y, z, w \in V$, we have*

$$R_\theta(x, y, z, w) = R_\varphi(\theta x, \theta y, z, w) = R_\varphi(x, y, \theta^* z, \theta^* w).$$

We may now provide a proof to Theorem 1.2.

Proof. (Proof of Theorem 1.2.) Suppose $c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0$. According to the discussion at the beginning of the previous section, we reduce the situation to one of two cases. If one or more of the c_i are zero, then since none of φ, ψ or τ have a rank less than 3, Theorem 1.1 applies, and the result holds. Otherwise, we have all $c_i \neq 0$, and we are reduced to the case that $R_\varphi + \epsilon R_\psi = \delta R_\tau$, where ϵ and δ are a choice of signs.

Let $x, y, z, w \in V$. By hypothesis, τ^{-1} exists. Note first that τ is self-adjoint with respect to φ, so that according to Lemma 4.7

$$\begin{aligned} R_\varphi(\tau x, \tau y, \tau^{-1} z, \tau^{-1} w) &= R_\varphi(x, y, z, w), \text{and} \\ R_\tau(\tau x, \tau y, \tau^{-1} z, \tau^{-1} w) &= R_\varphi(\tau x, \tau y, z, w) \\ &= R_\tau(x, y, z, w). \end{aligned}$$

Note that τ is self-adjoint with respect to φ if and only if τ^{-1} is self-adjoint with respect to φ. Now we use the hypothesis $R_\varphi + \epsilon R_\psi = \delta R_\tau$ and Lemma 4.7 to see that

$$\begin{aligned} \delta R_\tau(x, y, z, w) &= \delta R_\tau(\tau x, \tau y, \tau^{-1} z, \tau^{-1} w) \\ &= R_\varphi(\tau x, \tau y, \tau^{-1} z, \tau^{-1} w) + \epsilon R_\psi(\tau x, \tau y, \tau^{-1} z, \tau^{-1} w) \\ &= R_\varphi(x, y, z, w) + \epsilon R_\varphi(\psi\tau x, \psi\tau y, \tau^{-1} z, \tau^{-1} w) \\ &= R_\varphi(x, y, z, w) + \epsilon R_\varphi(\tau^{-1}\psi\tau x, \tau^{-1}\psi\tau y, z, w) \\ &= R_\varphi(x, y, z, w) + \epsilon R_{\tau^{-1}\psi\tau}(x, y, z, w). \end{aligned}$$

It follows that $R_{\tau^{-1}\psi\tau} = R_\psi$. Now Rank $\tau^{-1}\psi\tau =$ Rank $\psi \geq 3$, so using $A = \tau^{-1}\psi\tau$ in Lemma 4.6 gives us $\tau^{-1}\psi\tau = \pm\psi$. We show presently that $\tau^{-1}\psi\tau = -\psi$ is not possible.

Suppose $\tau^{-1}\psi\tau = -\psi$. We diagonalize τ with respect to φ with the basis $\{e_1, \ldots, e_n\}$. Suppose i, j, and k are distinct indices. With respect to this basis, for all $v \in V$ we have

$$R_\varphi(e_i, e_k, v, e_j) = R_\tau(e_i, e_k, v, e_j) = 0.$$

Let ψ_{ij} be the (j, i) entry of ψ with respect to this basis. Since τ and ψ anti-commute, and τ is diagonal, $\psi_{ii} = 0$. Thus there exists an entry $\psi_{ij} \neq 0$. Fix this i and j for the remainder of the proof. We must have $i \neq j$. Then for indices i, j, k, ℓ with i, j, k distinct, we have

$$\begin{aligned} 0 = \delta R_\tau(e_i, e_k, e_\ell, e_j) &= (R_\varphi + \epsilon R_\psi)(e_i, e_k, e_\ell, e_j) \\ &= \epsilon R_\psi(e_i, e_k, e_\ell, e_j) \\ &= \epsilon(\psi_{ij}\psi_{k\ell} - \psi_{i\ell}\psi_{kj}). \end{aligned} \tag{4.e}$$

If $\ell = i$, then Equation (4.e) with $\psi_{ii} = 0$ and $\psi_{ij} \neq 0$ shows that $\psi_{ki} = \psi_{ik} = 0$ for all $k \neq j$. Exchanging the roles of i and j and setting $\ell = j$ shows that $\psi_{jk} = \psi_{kj} = 0$ as well. Finally, for i, j, k, ℓ distinct, we use Equation (4.e) again to see that $\psi_{k\ell} = \psi_{\ell k} = 0$. Thus, under the assumption that there is at least one nonzero entry in the matrix $[\psi_{ab}]$ leads us to the conclusion that there are at most two nonzero entries ($\psi_{ij} = \psi_{ji} \neq 0$) in $[\psi_{ab}]$. This contradicts the assumption that Rank $\psi \geq 3$. We conclude that ψ and τ must not anticommute.

Otherwise, we have $\tau^{-1}\psi\tau = \psi$, and so $\psi\tau = \tau\psi$. Thus, we may simultaneously diagonalize ψ and τ. □

5. Linear independence of three algebraic curvature tensors

We may now use our previous results to establish our main results concerning the linear independence of three algebraic curvature tensors. This section is devoted to the proof of Theorem 1.3, and to the description of the exceptional setting when $\dim(V) = 3$.

Proof. (Proof of Theorem 1.3.) We assume first that $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, and show that one of Conditions (1) or (2) must be satisfied. As such, we suppose there exist c_i (not all zero) so that $c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0$. As in the proof of Theorem 1.2 in Section 4, if any of the c_i are zero, then this case reduces to Theorem 1.1, and all of the forms involved are real multiples of one another. Namely, $|\mathrm{Spec}(\psi)| = |\mathrm{Spec}(\tau)| = 1$, and Condition (1) holds.

So we consider the situation that none of the c_i are zero. This question of linear dependence reduces to the equation

$$R_\varphi + \epsilon R_\psi = \delta R_\tau. \tag{5.a}$$

We use Theorem 1.2 to simultaneously diagonalize ψ and τ with respect to φ to find a basis $\{e_1, \ldots, e_n\}$ which is orthonormal with respect to φ. Therefore, if $\mathrm{Spec}(\psi) = \{\lambda_1, \ldots, \lambda_n\}$, and $\mathrm{Spec}(\tau) = \{\eta_1, \ldots, \eta_n\}$, evaluating Equation (5.a) at (e_i, e_j, e_j, e_i) gives us the equations

$$1 + \epsilon\lambda_i\lambda_j = \delta\eta_i\eta_j \tag{5.b}$$

for any $i \neq j$. The remainder of this portion of the proof eliminates all possibilities except those found in Conditions (1) and (2).

If $|\mathrm{Spec}(\tau)| \geq 3$, then we permute the basis vectors so that η_1, η_2 and η_3 are distinct. Then we have, according to Equation (5.b) for i, j, and k distinct:

$$\begin{array}{rcl} 1 + \epsilon\lambda_i\lambda_j &=& \delta\eta_i\eta_j \\ 1 + \epsilon\lambda_i\lambda_k &=& \delta\eta_i\eta_k, \text{ so subtracting,} \\ \epsilon\lambda_i(\lambda_j - \lambda_k) &=& \delta\eta_i(\eta_j - \eta_k). \end{array} \tag{5.c}$$

All $\eta_i \neq 0$ since $\det\tau \neq 0$, and so since $\eta_j \neq \eta_k$, the above equation shows that all $\lambda_i \neq 0$, and that $\lambda_j \neq \lambda_k$ for $\{i, j, k\} = \{1, 2, 3\}$. Since $\dim(V) \geq 4$, we may compute

$$\begin{array}{rcl} (1 + \epsilon\lambda_1\lambda_2)(1 + \epsilon\lambda_3\lambda_4) &=& \eta_1\eta_2\eta_3\eta_4 \\ &=& (1 + \epsilon\lambda_1\lambda_3)(1 + \epsilon\lambda_2\lambda_4). \end{array}$$

Multiplying the above and cancelling, we have $\lambda_1\lambda_2 + \lambda_3\lambda_4 = \lambda_1\lambda_3 + \lambda_2\lambda_4$. In other words, $\lambda_2(\lambda_1 - \lambda_4) = \lambda_3(\lambda_1 - \lambda_4)$. Since $\lambda_2 \neq \lambda_3$, we conclude $\lambda_1 = \lambda_4$. Performing the same manipulation, we have

$$\begin{array}{rcl} (1 + \epsilon\lambda_1\lambda_4)(1 + \epsilon\lambda_2\lambda_3) &=& \eta_1\eta_2\eta_3\eta_4 \\ &=& (1 + \epsilon\lambda_1\lambda_3)(1 + \epsilon\lambda_2\lambda_4). \end{array}$$

One then concludes, similarly to above, that $\lambda_4 = \lambda_3$. This is a contradiction, since $\lambda_4 = \lambda_1 \neq \lambda_3 = \lambda_4$.

Now suppose that $|\mathrm{Spec}(\tau)| = 2$, and that there are at least two pairs of repeated eigenvalues of τ. We may assume that $\mathrm{Spec}(\tau) = \{\eta_1, \eta_1, \eta_3, \eta_3, \ldots\}$, and that $\eta_1 \neq \eta_3$. Proceeding as in Equation (5.b), we have

$$1 + \epsilon\lambda_1\lambda_3 = 1 + \epsilon\lambda_1\lambda_4 = \delta\eta_1\eta_3 = 1 + \epsilon\lambda_2\lambda_3 = 1 + \epsilon\lambda_2\lambda_4.$$

Now, as in Equation (5.c) we have the equations

$$\begin{array}{ll} \lambda_1(\lambda_3 - \lambda_4) = 0, & \lambda_3(\lambda_1 - \lambda_2) = 0, \\ \lambda_2(\lambda_3 - \lambda_4) = 0, & \lambda_4(\lambda_1 - \lambda_2) = 0. \end{array} \tag{5.d}$$

Now according to Equation (5.b), we have

$$1 + \epsilon\lambda_1\lambda_2 = \delta\eta_1^2, \text{and } 1 + \epsilon\lambda_1\lambda_3 = \delta\eta_1\eta_2. \tag{5.e}$$

Subtracting, we conclude $\epsilon\lambda_1(\lambda_2 - \lambda_3) = \delta\eta_1(\eta_1 - \eta_3) \neq 0$. Thus $\lambda_1 \neq 0$, and $\lambda_2 \neq \lambda_3$. We use a similar argument to conclude λ_2, λ_3, and $\lambda_4 \neq 0$. According to Equation (5.d), we have $\lambda_3 = \lambda_4$, and $\lambda_1 = \lambda_2$.

We find a contradiction after performing one more calculation from Equation (5.b). Note that

$$(1 + \epsilon\lambda_1^2)(1 + \epsilon\lambda_3^2) = \eta_1^2\eta_1^2 = (1 + \epsilon\lambda_1\lambda_3)(1 + \epsilon\lambda_1\lambda_3).$$

After multiplying out and cancelling the common constant and quartic terms, we conclude

$$\begin{aligned} & \lambda_1^2 + \lambda_3^2 = 2\lambda_1\lambda_3 \\ \Leftrightarrow \quad & (\lambda_1 - \lambda_3)^2 = 0 \\ \Leftrightarrow \quad & \lambda_1 = \lambda_3. \end{aligned}$$

This is a contradiction since $\lambda_1 = \lambda_2 \neq \lambda_3$.

In order to finish the proof of one implication in Theorem 1.3, we consider the case that $\mathrm{Spec}(\tau) = \{\eta_1, \eta_2, \eta_2, \dots\}$. Using $i = 1$ in Equation (5.b), we see that $\lambda_j = \lambda_2$ for all $j \geq 2$. In that event we may solve for λ_2 and λ_1 to be as given in Condition (2) of Theorem 1.3. This concludes the proof that if the set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then Condition (1) or Condition (2) must hold.

Conversely, we suppose one of Condition (1) or (2) from Theorem 1.3 holds, and show that the set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent. If Condition (1) is satisfied, then $\psi = \lambda\varphi$, and $\tau = \eta\varphi$ for some λ and η. The set $\{R_\varphi, R_\psi\}$ is a linearly dependent set by Theorem 1.1, and so it follows that $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent as well. If Condition (2) holds, then the discussion already presented in the above paragraph shows that, for this choice of ψ and τ, that $R_\varphi + \epsilon R_\psi = \delta R_\tau$. □

The following result shows that the assumption that $\dim(V) = 4$ is necessary in Theorem 1.3 by exhibiting (in certain cases) a unique solution up to sign ψ of full rank in the case $\dim(V) = 3$. Of course, our assumptions put certain restrictions on the eigenvalues η_i of τ for there to exist a solution, in particular, since we assume ψ and τ have full rank, none of their eigenvalues can be 0.

Theorem 5.1. *Let φ be a positive definite symmetric bilinear form on a real vector space V of dimension 3. Suppose* $\mathrm{Spec}(\tau) = \{\eta_1, \eta_2, \eta_3\}$. *Set*

$$\eta(i,j,k) = (-\epsilon)\sqrt{\frac{(1 - \delta\eta_i\eta_j)(1 - \delta\eta_i\eta_k)}{(-\epsilon)(1 - \delta\eta_j\eta_k)}}.$$

If $\mathrm{Rank}\ \psi = \mathrm{Rank}\ \tau = 3$*, and* $\mathrm{Spec}(\psi) = \{\lambda_1, \lambda_2, \lambda_3\}$*, where*

$$\lambda_1 = (-\epsilon)\eta(1,2,3), \quad \lambda_2 = (-\epsilon)\eta(2,3,1), \quad \lambda_3 = \eta(3,1,2),$$

then ψ and $-\psi$ are the only solutions to the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$.

Proof. If a solution exists, we use Theorem 1.2 that orthogonally diagonalizes ψ and τ with respect to φ. The diagonal entries λ_i of ψ and η_i of τ are their respective eigenvalues. We have the following equations for $i \neq j$:

$$\begin{aligned} \epsilon R_\psi(e_i, e_j, e_j, e_i) \quad &= \quad \epsilon\lambda_i\lambda_j \quad = \quad -R_\varphi(e_i, e_j, e_j, e_i) + \delta R_\tau(e_i, e_j, e_j, e_i) \\ & \qquad\qquad\quad = \quad -1 + \delta\eta_i\eta_j. \end{aligned}$$

Since ψ has full rank, we know $\lambda_3 \neq 0$. Solving for λ_1 and λ_2 gives

$$\lambda_1 = \frac{-1 + \delta\eta_1\eta_3}{\epsilon\lambda_3}, \text{ and } \lambda_2 = \frac{-1 + \delta\eta_2\eta_3}{\epsilon\lambda_3}.$$

Substituting into $R_\psi(e_1, e_2, e_2, e_1)$ gives

$$\begin{aligned} \epsilon\left(\frac{-1 + \delta\eta_1\eta_3}{\epsilon\lambda_3}\right)\left(\frac{-1 + \delta\eta_2\eta_3}{\epsilon\lambda_3}\right) &= -1 + \delta\eta_1\eta_2, \text{ so} \\ \frac{\epsilon(-1 + \delta\eta_1\eta_3)(-1 + \delta\eta_2\eta_3)}{\epsilon^2\lambda_3^2} &= -1 + \delta\eta_1\eta_2, \text{ and so} \end{aligned}$$

$$\frac{(-1+\delta\eta_1\eta_3)(-1+\delta\eta_2\eta_3)}{(-\epsilon)(1-\delta\eta_1\eta_2)} = \eta(3,1,2)^2 = \lambda_3^2.$$

One checks that for these values of λ_1, λ_2, and λ_3, that ψ is a solution, and that these λ_i are completely determined in this way by the η_i, hence, they are the only solutions. □

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