A Spanning Set for Algebraic Covariant Derivative Curvature Tensors

Bronson Lim

August 21, 2009

Abstract

We provide a spanning set for the set of algebraic covariant derivative curvature tensors on a m-dimensional real vector space.

1 Introduction

A spanning set for the algebraic covariant derivative curvature tensors on a vector space is known to be the set of algebraic covariant derivative curvature tensors built from totally symmetric p-forms [2]. This paper presents an elementary proof of this fact. We will begin with a definition of the vector space of algebraic curvature tensors and algebraic covariant derivative curvature tensors. Then we will introduce building blocks for specific elements of the aforementioned spaces.

Definition 1.1 *Let V be an m-dimensional real vector space. Let $A_0(V) \subset \otimes^4 V^*$ and $A_1(V) \subset \otimes^5 V^*$ be the spaces of all algebraic curvature tensors and all algebraic covariant derivative tensors, respectively. If we have $R \in A_0(V)$ and $\nabla R \in A_1(V)$ then by definition R and ∇R are multilinear and satisfy the following symmetries:*

$$R(x,y,z,w) = R(z,w,x,y) = -R(y,x,z,w),$$

$$R(x,y,z,w) + R(x,w,y,z) + R(x,z,w,y) = 0,$$

$$\nabla R(x,y,z,w;v) = \nabla R(z,w,x,y;v) = -\nabla R(y,x,z,w;v),$$

$$\nabla R(x,y,z,w;v) + \nabla R(x,w,y,z;v) + \nabla R(x,z,w,y;v) = 0,$$

$$\nabla R(x,y,z,w;v) + \nabla R(x,y,v,z;w) + \nabla R(x,y,w,v;z) = 0.$$

Definition 1.2 *Let $\phi \in S^p(V)$ then ϕ is a totally symmetric multilinear p-form. That is, $\phi(x_1,...,x_n) = \phi(x_{\sigma(1)},...,x_{\sigma(n)})$ where σ is any permutation of p elements.*

The next two propositions will give a way to build an algebraic curvature tensor and an algebraic covariant derivative curvature tensor using symmetric two and three forms respectively [1]. The set of the algebraic curvature tensors and the set of the covariant derivatives built from these symmetric forms are interesting in that they form a spanning set.

Proposition 1.3 *If $\phi \in S^2(V)$, then*

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$$

is an algebraic curvature tensor.

Proof. We need only show that R_ϕ satisfies the symmetries from Definition 1.1.

$$\begin{aligned} R_\phi(y, x, z, w) &= \phi(y, w)\phi(x, z) - \phi(x, w)\phi(y, z) \\ &= -(\phi(x, w)\phi(y, z) - \phi(y, w)\phi(x, z) \\ &= -R_\phi(x, y, z, w), \end{aligned}$$

$$\begin{aligned} R_\phi(z, w, x, y) &= \phi(z, y)\phi(w, x) - \phi(z, x)\phi(w, y) \\ &= \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w) \\ &= R_\phi(x, y, z, w), \end{aligned}$$

and the first Bianchi Identity:

$$\begin{aligned} &R_\phi(x, y, z, w) + R_\phi(x, w, y, z) + R_\phi(x, z, w, y) \\ &= \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w) + \phi(x, z)\phi(y, w) \\ &-\phi(x, y)\phi(w, z) + \phi(x, y)\phi(z, w) - \phi(x, w)\phi(y, z) \\ &= 0. \end{aligned}$$

□

Proposition 1.4 *If* $\phi \in S^2(V), \psi \in S^3(V)$, *then*

$\nabla R_{\phi,\psi}(x,y,z,w;s) = \phi(x,w)\psi(y,z,s)+\phi(y,z)\psi(x,w,s)-\phi(x,z)\psi(y,w,s)-\phi(y,w)\psi(x,z,s)$

is an algebraic covariant derivative curvature tensor.

Proof. We need only show that $\nabla R_{\phi,\psi}$ satisfies the symmetries from Definition 1.1.

$$\begin{aligned}
\nabla R_{\phi,\psi}(y,x,z,w;s) &= \phi(y,w)\psi(x,z,s) + \phi(x,z)\psi(y,w,s) \\
&\quad -\phi(y,z)\psi(x,w,s) - \phi(x,w)\psi(y,z,s) \\
&= -(\phi(x,w)\psi(y,z,s) + \phi(y,z)\psi(x,w,s) \\
&\quad -\phi(x,z)\psi(y,w,s) - \phi(y,w)\psi(x,z,s)) \\
&= -\nabla R_{\phi,\psi}(x,y,z,w;s), \\
\nabla R_{\phi,\psi}(z,w,x,y;s) &= \phi(z,y)\psi(w,x,s) + \phi(w,x)\psi(z,y,s) \\
&\quad -\phi(z,x)\psi(w,y,s) - \phi(w,y)\psi(z,x,s) \\
&= \phi(x,w)\psi(y,z,s) + \phi(y,z)\psi(x,w,s) \\
&\quad -\phi(x,z)\psi(y,w,s) - \phi(y,w)\psi(x,z,s) \\
&= \nabla R_{\phi,\psi}(x,y,z,w;s),
\end{aligned}$$

the first Bianchi Identity:

$$\begin{aligned}
&\nabla R_{\phi,\psi}(x,y,z,w;s) + \nabla R_{\phi,\psi}(x,w,y,z;s) + \nabla R_{\phi,\psi}(x,z,w,y;s) \\
&= \phi(x,w)\psi(y,z,s) + \phi(y,z)\psi(x,w,s) - \phi(x,z)\psi(y,w,s) - \phi(y,w)\psi(x,z,s) \\
&+\phi(x,z)\psi(y,w,s) + \phi(w,y)\psi(x,z,s) - \phi(x,y)\psi(w,z,s) - \phi(w,z)\psi(x,y,s) \\
&+\phi(x,y)\psi(z,w,s) + \phi(z,w)\psi(x,y,s) - \phi(x,w)\psi(z,y,s) - \phi(z,y)\psi(x,w,s) \\
&= 0,
\end{aligned}$$

and the second Bianchi Identity:

$$\begin{aligned}
&\nabla R_{\phi,\psi}(x,y,z,w;s) + \nabla R_{\phi,\psi}(x,y,s,z;w) + \nabla R_{\phi,\psi}(x,y,w,s;z) \\
&= \phi(x,w)\psi(y,z,s) + \phi(y,z)\psi(x,w,s) - \phi(x,z)\psi(y,w,s) - \phi(y,w)\psi(x,z,s) \\
&+\phi(x,z)\psi(y,s,w) + \phi(y,s)\psi(x,z,w) - \phi(x,s)\psi(y,z,w) - \phi(y,z)\psi(x,s,w) \\
&+\phi(x,s)\psi(y,w,z) + \phi(y,w)\psi(x,s,z) - \phi(x,w)\psi(y,s,z) - \phi(y,s)\psi(x,w,z) \\
&= 0.
\end{aligned}$$

□

Now that we have shown that we can build algebraic curvature tensors and algebraic covariant derivative curvature tensors, we can develop notation to indicate when a ∇R is in fact a $\nabla R_{\phi,\psi}$.

Definition 1.5 *If V is a vector space, then $S(V^*) = span\{R_\phi : \phi \in S^2(V^*)\}$ and $S_1(V^*) = span\{\nabla R_{\phi,\psi} : \phi \in S^2(V^*), \psi \in S^3(V^*)\}$*

The following theorem states that the set of algebraic curvature tensors on a vector space is the set of those built from symmetric bilinear forms [2, 3].

Theorem 1.6 *$A_0(V^*) = S(V^*)$.*

An elementary proof has been provided [3] and a proof using representation theory [2]. The main result is that the set of algebraic covariant derivative curvature tensors is spanned by those built from totally symmetric two and three forms. A proof using representation theory [2] exists; however, an elementary proof does not. The following two lemmas will be helpful when proving this:

Lemma 1.7 *If $T_{ijkl;n} \in \otimes^5 V^*$, such that $T_{ijkl;n}$ satisfies the first two symmetries of Defintion 1.1 and if σ is a permutation on five letters,then*

$$\sum_{ijkln} c_{ijkln} T_{\sigma(i)\sigma(j)\sigma(k)\sigma(l);\sigma(n)} = \sum_{ijkln} c_{\sigma(i)\sigma(j)\sigma(k)\sigma(l);\sigma(n)} T_{ijkl;n}.$$

Lemma 1.8 *If $T_{ijkl;n} \in \otimes^5 V^*$, such that $T_{ijkl;n}$ satisfies the first two symmetries of Defintion 1.1 and if σ is a permutation on five letters, then*

$$\sum_{ijkln} c_{ijkln} T_{xyzw;s} = \sum_{ijkln} c_{\sigma(i)\sigma(j)\sigma(k)\sigma(l),\sigma(n)} T_{\sigma(x)\sigma(y)\sigma(z)\sigma(w);\sigma(s)}$$

where $\{i, j, k, l, n\} = \{x, y, z, w, s\}$.

Proof. Let σ be a permutation on $\{i, j, k, l, n\} = \{x, y, z, w, s\}$. Consider,

$$\sum_{ijkln} c_{\sigma(i)\sigma(j)\sigma(k)\sigma(l),\sigma(n)} T_{\sigma(x)\sigma(y)\sigma(z)\sigma(w);\sigma(s)}.$$

Since $\{i, j, k, l, n\}$ are dummy indices, we can rename the indices in the previous summation to obtain:

$$\sum_{ijkln} c_{ijkln} T_{xyzw;s}.$$

$\square$

2 A Spanning Set for $A_1(V^*)$

Theorem 2.1 *$S_1(V^*) = A_1(V^*)$.*

Let V be a vector space of dimension m and $\{e_i\}$ be a basis for V, $\{e^i\}$ be a basis for V^* and $\{e^i \otimes e^j \otimes e^k \otimes e^l \otimes e^n\}$ be a basis for $\otimes^5 V^*$. We will define a tensor $T_{ijkl;n}$ that satisfies the first two symmetries from Definition 1.1.

$$\begin{aligned} T_{ijkl;n} = \quad & e^i \otimes e^j \otimes e^k \otimes e^l \otimes e^n + e^k \otimes e^l \otimes e^i \otimes e^j \otimes e^n \\ - \quad & e^j \otimes e^i \otimes e^k \otimes e^l \otimes e^n - e^i \otimes e^j \otimes e^l \otimes e^k \otimes e^n \\ - \quad & e^l \otimes e^k \otimes e^i \otimes e^j \otimes e^n - e^k \otimes e^l \otimes e^j \otimes e^i \otimes e^n \\ + \quad & e^j \otimes e^i \otimes e^l \otimes e^k \otimes e^n + e^l \otimes e^k \otimes e^j \otimes e^i \otimes e^n \end{aligned}$$

Let $\nabla R \in A_1(V^*)$. Since $\nabla R \in \otimes^5 V^*$ and satisfies the first two symmetries of Definition 1.1, ∇R is a linear combination of the $T_{ijkl;n}$'s. Therefore,

$$\nabla R = \sum_{i,j \ distinct} c_{ijjii} T_{ijji;i} + \tag{1}$$

$$\sum_{i,j,k \ distinct} c_{ijkii} T_{ijki;i} + c_{ijkij} T_{ijki;j} + c_{ijjik} T_{ijji;k} + \tag{2}$$

$$\sum_{i,j,k,l \ distinct} c_{ijkli} T_{ijkl;i} + c_{ijkil} T_{ijki;l} + \tag{3}$$

$$\sum_{i,j,k,l,n \ distinct} c_{ijkln} T_{ijkl;n}. \tag{4}$$

Since $\nabla R \in A_1(V^*)$ we can choose the constants, c_{ijkln}, to satisfy the first and second Bianchi identities from Definition 1.1. That is $c_{ijkln} + c_{ijnkl} + c_{ijlnk} = 0, c_{ijkln} + c_{iljkn} + c_{ikljn} = 0, c_{ijkln} = -c_{jikln} = c_{klijn}$. We are going to show that each term (1), (2), (3) and (4)$\in S_1(V^*)$ individually.

2.1 Two Distinct Indices

Let $\{i, j\}$ be distinct indices and $\{a, b\} \neq \{j\}$, $\{x, y, z\} \neq \{i\}$. Choose $\phi \in S^2(V^*)$ and $\psi \in S^3(V^*)$ so that:

$$\phi(e_j, e_j) = 1 \text{ and } \phi(e_a, e_b) = 0,$$

$$\psi_(e_i, e_i, e_i) = 1 \text{ and } \psi(e_x, e_y, e_z) = 0.$$

Up to our symmetries, the only nontrivial entry for $\nabla R_{\phi,\psi}$ is

$$\nabla R_{\phi,\psi}(e_i, e_j, e_j, e_i; e_i) = 1.$$

We have $T_{ijji;i} = 2\nabla R_{\phi,\psi}$. Thus $T_{ijji;i} \in S_1(V^*)$ and (1)$\in S_1(V^*)$.

2.2 Three Distinct Indices

Let $\{i, j, k\}$ be distinct indices and $\{a, b\} \neq \{j, k\}$, $\{x, y, z\} \neq \{i\}$. Choose $\phi \in S^2(V^*)$ and $\psi \in S^3(V^*)$ so that:

$$\phi(e_j, e_k) = \phi(e_k, e_j) = 1 \text{ and } \phi(e_a, e_b) = 0,$$
$$\psi(e_i, e_i, e_i) = 1 \text{ and } \psi(e_x, e_y, e_z) = 0.$$

Up to our symmetries, the only nontrivial entry for $\nabla R_{\phi,\psi}$ is

$$\nabla R_{\phi,\psi}(e_i, e_j, e_k, e_i; e_i) = 1.$$

We have $\nabla R_{\phi,\psi} = T_{ijki;i}$. Thus $T_{ijki;i} \in S_1(V^*)$.

Choose $\phi \in S^2(V^*)$ and $\psi \in S^3(V^*)$ so that:

$$\phi(e_i, e_i) = 1 \text{ and } \phi(e_a, e_b) = 0,$$
$$\psi(e_j, e_j, e_k) = \psi(e_j, e_k, e_j) = \psi(e_k, e_j, e_j) = 1 \text{ and } \psi(e_a, e_b, e_c) = 0.$$

Up to our symmetries, the nontrivial entries for $\nabla R_{\phi,\psi}$ are:

$$\nabla R_{\phi,\psi}(e_i, e_j, e_k, e_i; e_j) = 1, \nabla R_{\phi,\psi}(e_i, e_j, e_j, e_i; e_k) = 1.$$

Up to the symmetries, the only nonzero entries for $T_{ijji;k}$ and $T_{ijki;j}$ are:

$$T_{ijji;k}(e_i, e_j, e_j, e_i; e_k) = 2 = 2\nabla R_{\phi,\psi}(e_i, e_j, e_j, e_i; e_k),$$
$$T_{ijki;j}(e_i, e_j, e_k, e_i; e_j) = 1 = \nabla R_{\phi,\psi}(e_i, e_j, e_k, e_i; e_j).$$

We must have $\nabla R_{\phi,\psi} = \frac{1}{2}T_{ijji;k} + T_{ijki;j}$, and $\frac{1}{2}T_{ijji;k} + T_{ijki;j} \in S_1(V^*)$. By permuting $\{i, j\}$ we have $\frac{1}{2}T_{ijji;k} + T_{ijjk;i} \in S_1(V^*)$. By adding the previous terms, we arrive at $T_{ijji;k} + T_{ijki;j} + T_{ijjk;i} \in S_1(V^*)$. By subtracting the same terms, we arrive at $T_{ijki;j} - T_{ijjk;i} \in S_1(V^*)$. By the second Bianchi identity,

$$c_{ijjik} = c_{ijkij} + c_{ijjki}.$$

We will show that $\sum_{ijk\ distinct} c_{ijjik} T_{ijji;k} \in S_1(V^*)$:

$$\begin{aligned}
& c_{ijjik}(T_{ijji;k} + T_{ijki;j} + T_{ijjk;i}) \in S_1(V^*) \\
\Rightarrow & \sum_{ijk\ distinct} c_{ijjik}(T_{ijji;k} + T_{ijki;j} + T_{ijjk;i}) \in S_1(V^*) \\
= & \sum_{ijk} c_{ijji;k} T_{ijji;k} + \sum_{ijk} c_{ijji;k} T_{ijki;j} + \sum_{ijk} c_{ijji;k} T_{ijjk;i}
\end{aligned}$$

We can interchange the indices of the constants and the tensor of the second and third summations by Lemma 1.7. Then we factor out the constants and use the Bianchi identities.

$$\begin{aligned}
= & \sum_{ijk} c_{ijjik} T_{ijji;k} + \sum_{ijk} c_{ijkij} T_{ijji;k} + \sum_{ijk} c_{ijjki} T_{ijji;k} \\
= & \sum_{ijk} c_{ijjik} T_{ijji;k} + \sum_{ijk} (c_{ijkij} + c_{ijjki}) T_{ijji;k} \\
= & \sum_{ijk} c_{ijjik} T_{ijji;k} + \sum_{ijk} c_{ijjik} T_{ijji;k} \\
= & \; 2\sum_{ijk} c_{ijjik} T_{ijji;k}. \\
\Rightarrow & \sum_{ijk} c_{ijjik} T_{ijji;k} \in S_1(V^*).
\end{aligned}$$

Now we will show that $\sum_{ijk\ distinct} c_{ijkij} T_{ijki;j} \in S_1(V^*)$:

$$\begin{aligned}
& c_{ijkij}(T_{ijkij} - T_{ijjki}), 2c_{ijjki}(\frac{1}{2}T_{ijji;k} + T_{ijjk;i}) \in S_1(V^*) \\
\Rightarrow & \sum_{ijk\ distinct} c_{ijkij}(T_{ijki;j} - T_{ijjk;i}) + 2c_{ijjki}(\frac{1}{2}T_{ijji;k} + T_{ijjk;i}) \in S_1(V^*) \\
= & \sum_{ijk} c_{ijkij} T_{ijki;j} - \sum_{ijk} c_{ijkij} T_{ijjk;i} + \sum_{ijk} c_{ijjki} T_{ijji;k} + 2\sum_{ijk} c_{ijjki} T_{ijjk;i}
\end{aligned}$$

We use the Bianchi identity on the second summand. Switch the third and fourth entry in both the constant and the tensor in the fourth summation and permute $\{i, j\}$.

$$= \; 3\sum_{ijk} c_{ijkij} T_{ijki;j} + \sum_{ijk} c_{ijjki} T_{ijjk;i} - \sum_{ijk} c_{ijji;k} T_{ijjk;i} + \sum_{ijk} c_{ijjki} T_{ijji;k}$$

We interchange indices with the constants and the tensor in the third summand. We also switch the third and fourth entry in both the constant and the tensor for the second summations.

$$\begin{aligned} &= 4\sum_{ijk} c_{ijkij}T_{ijki;j} - \sum_{ijk} c_{ijjik}T_{ijjk;i} + \sum_{ijk} c_{ijji;k}T_{ijjk;i} \\ &= 4\sum_{ijk} c_{ijkij}T_{ijki;j} \\ &\Rightarrow \sum_{ijk} c_{ijkij}T_{ijki;j} \in S_1(V^*). \end{aligned}$$

Thus the terms in (2)$\in S_1(V^*)$.

2.3 Four Distinct Indices

Let $\{i, j, k, l\}$ be distinct indices and $\{a, b\} \neq \{j, k\}$, $\{x, y, z\} \neq \{i, l\}$. Choose $\phi \in S^2(V^*)$ and $\psi \in S^3(V^*)$ so that:

$$\phi(e_j, e_k) = \phi(e_k, e_j) = 1 \text{ and } \phi(e_a, e_b) = 0,$$

$$\psi(e_i, e_i, e_l) = \psi(e_i, e_l, e_i) = \psi(e_l, e_i, e_i) = 1 \text{ and } \psi(e_x, e_y, e_z) = 0.$$

Up to the symmetries, the nontrivial entries for $\nabla R_{\phi,\psi}$ are:

$$\nabla R_{\phi,\psi}(e_i, e_j, e_k, e_l; e_i) = 1, \nabla R_{\phi,\psi}(e_i, e_k, e_j, e_l; e_i) = 1, \nabla R_{\phi,\psi}(e_i, e_k, e_j, e_i; e_l) = 1.$$

Up to our symmetries, the only nonzero entries for $T_{ijkl;i}$, $T_{ijki;l}$, and $T_{ikjl;i}$ are as follows:

$$\begin{aligned} T_{ijkl;i}(e_i, e_j, e_k, e_l; e_i) &= 1 = \nabla R_{\phi,\psi}(e_i, e_j, e_k, e_l; e_i), \\ T_{ijki;l}(e_i, e_j, e_k, e_i; e_l) &= 1 = \nabla R_{\phi,\psi}(e_i, e_j, e_k, e_i; e_l), \\ T_{ikjl;i}(e_i, e_k, e_j, e_l; e_i) &= 1 = \nabla R_{\phi,\psi}(e_i, e_k, e_j, e_l; e_i). \end{aligned}$$

Since, these are the only nonzero entries, we must have $\nabla R_{\phi,\psi} = T_{ijkl;i} + T_{ijki;l} + T_{ikjl;i}$ and $T_{ijkl;i} + T_{ijki;l} + T_{ikjl;i} \in S_1(V^*)$. Now since $T_{ijkl;i} + T_{ijki;l} + T_{ikjl;i} \in S_1(V^*)$, we can permute the indices $\{j, k, l\}$ to get more linear combinations:

$$T_{ijkl;i} + T_{ijki;l} + T_{ikjl;i} \in S_1(V^*), \quad (5)$$

$$T_{iklj;i} + T_{ikli;j} + T_{ilkj;i} \in S_1(V^*), \quad (6)$$

$$T_{iljk;i} + T_{ilji;k} + T_{ijlk;i} \in S_1(V^*). \quad (7)$$

By subtracting (7) from (5), we have:

$$\begin{aligned} & T_{ijkl;i}+T_{ijki;l}+T_{ikjl;i}-T_{iljk;i}-T_{ilji;k}-T_{ijlk;i}\in S_1(V^*) \\ = \; & T_{ijkl;i}+T_{ijki;l}+T_{ikjl;i}-T_{iljk;i}+T_{ilij;k}+T_{ijkl;i} \\ = \; & 2T_{ijkl;i}+T_{ijki;l}+T_{ikjl;i}-T_{iljk;i}+T_{ilij;k}\in S_1(V^*). \end{aligned} \tag{8}$$

By the first and second Bianchi identities:

$$\begin{aligned} c_{ijkli}+c_{ijikl}+c_{ijlik}&=0, \\ c_{ijkli}+c_{iljki}+c_{iklji}&=0. \end{aligned}$$

We can now show $\sum_{ijkl\ distinct} c_{ijkli}T_{ijkl;i}\in S_1(V^*)$, by considering c_{ijkli} multiplied by the equation linear combination from (8):

$$\begin{aligned} & c_{ijkli}(2T_{ijkl;i}-T_{iklj;i}-T_{iljk;i}+T_{ijki;l}+T_{ijil;k})\in S_1(V^*) \\ \Rightarrow \; & \sum_{ijkl\ distinct} c_{ijkli}(2T_{ijkl;i}-T_{iklj;i}-T_{iljk;i}+T_{ijki;l}+T_{ijil;k})\in S_1(V^*) \\ = \; & 2\sum_{ijkl} c_{ijkli}T_{ijkl;i}-\sum_{ijkl} c_{ijkli}T_{iklj;i}-\sum_{ijkl} c_{ijkli}T_{iljk;i}+\sum_{ijkl} c_{ijkli}T_{ijki;l}+\sum_{ijkl} c_{ijkl;i}T_{ijil;k}. \end{aligned}$$

We use Lemma 1.7 on the second, third, fourth, and fifth summations.

$$\begin{aligned} = \; & 2\sum_{ijkl} c_{ijkli}T_{ijkl;i}-\sum_{ijkl} c_{iklji}T_{ijkl;i}-\sum_{ijkl} c_{iljki}T_{ijkl;i}+\sum_{ijkl} c_{ijkil}T_{ijkl;i}+\sum_{ijkl} c_{ijilk}T_{ijkl;i} \\ = \; & 2\sum_{ijkl} c_{ijkli}T_{ijkl;i}-\sum_{ijkl}(c_{iklji}+c_{iljki})T_{ijkl;i}+\sum_{ijkl}(c_{ijkil}+c_{ijilk})T_{ijkl;i}. \end{aligned}$$

We use the Bianchi identities on the constants.

$$\begin{aligned}
&= 2\sum_{ijkl} c_{ijkli}T_{ijkl;i} - \sum_{ijkl}(-c_{ijkli})T_{ijkl;i} + \sum_{ijkl}(c_{ijkli})T_{ijkl;i} \\
&= 4\sum_{ijkl} c_{ijkli}T_{ijkl;i}. \\
&\Rightarrow \sum_{ijkl} c_{ijkli}T_{ijkl;i} \in S_1(V^*).
\end{aligned}$$

To show $\sum_{ijkl\ distinct} c_{ijkil}T_{ijki;l} \in S_1(V^*)$ is somewhat more complicated. Again we consider the constant c_{ijkli} multiplied by the linear combination from (8):

$$\begin{aligned}
&\sum_{ijkl} c_{ijkli}(2T_{ijkl;i} + T_{ijki;l} - T_{iklj;i} + T_{ijil;k} - T_{iljk;i}) \in S_1(V^*) \\
&= \sum_{ijkl} 2c_{ijkli}T_{ijkl;i} + \sum_{ijkl} c_{ijkli}T_{ijki;l} - \sum_{ijkl} c_{ijkli}T_{iklj;i} + \sum_{ijkl} c_{ijkli}T_{ijil;k} - \sum_{ijkl} c_{ijkli}T_{iljk;i}.
\end{aligned}$$

In the third and fifth summations, we use Lemma 1.7, then factor out the tensor. In the fourth summation, we permute $\{k, l\}$ and use the usual symmetries to obtain the second summation.

$$\begin{aligned}
&= \sum_{ijkl} 2c_{ijkli}T_{ijkl;i} + 2\sum_{ijkl} c_{ijkli}T_{ijki;l} + \sum_{ijkl}(-c_{iklji} - c_{iljki})T_{ijkl;i} \\
&= \sum_{ijkl} 3c_{ijkli}T_{ijkl;i} + 2\sum_{ijkl} c_{ijkli}T_{ijki;l} \in S_1(V^*).
\end{aligned}$$

But the first summand $\in S_1(V^*)$,

$$\begin{aligned}
&\Rightarrow \sum_{ijkl} 2c_{ijkli}T_{ijki;l} \in S_1(V^*). \\
&\Rightarrow \sum_{ijkl} 2(c_{ijkil} + c_{ijilk})T_{ijki;l} \in S_1(V^*). \\
&\Rightarrow \sum_{ijkl} 2c_{ijkil}T_{ijki;l} + 2c_{ijilk}T_{ijki;l} \in S_1(V^*). \qquad (9)
\end{aligned}$$

Now consider (5)+(6)+(7):

$$\begin{aligned}
& T_{ijkl;i} + T_{ijki;l} + T_{ikjl;i} + T_{iklj;i} + T_{ikli;j} + T_{ilkj;i} + T_{iljk;i} + T_{ilji;k} + T_{ijlk;i} \in S_1(V^*) \\
=\ & T_{ijkl;i} + T_{ijki;l} + T_{ikjl;i} - T_{ikjl;i} + T_{ikli;j} + T_{ilkj;i} - T_{ilkj;i} + T_{ilji;k} - T_{ijkl;i} \\
=\ & T_{ijkl;i} - T_{ijkl;i} + T_{ijki;l} + T_{ikjl;i} - T_{ikjl;i} + T_{ikli;j} + T_{ilkj;i} - T_{ilkj;i} + T_{ilji;k} \\
=\ & +T_{ijki;l} + T_{ikli;j} + T_{ilji;k} \in S_1(V^*). \qquad (10)
\end{aligned}$$

By (10) we consider the following:

$$\begin{aligned}
& \sum_{ijkl} c_{ijkil}(T_{ijki;l} - T_{ijil;k} - T_{ilik;j}) \in S_1(V^*) \\
= & \sum_{ijkl} c_{ijkil}T_{ijki;l} - \sum_{ijkl} c_{ijkil}T_{ijil;k} - \sum_{ijkl} c_{ijkil}T_{ilik;j}.
\end{aligned}$$

We now permute $\{j, k\}$ in the third summation,

$$\begin{aligned}
= & \sum_{ijkl} c_{ijkil}T_{ijki;l} - \sum_{ijkl} c_{ijkil}T_{ijil;k} - \sum_{ijkl} c_{ikjil}T_{ilij;k} \\
= & \sum_{ijkl} c_{ijkil}T_{ijki;l} - 2\sum_{ijkl} c_{ijkil}T_{ijil;k}. \\
\Rightarrow & \sum_{ijkl} c_{ijkil}T_{ijki;l} - 2\sum_{ijkl} c_{ijilk}T_{ijki;l} \in S_1(V^*). \qquad (11)
\end{aligned}$$

Let us consider the sum between (9) and (11):

$$\begin{aligned}
& \sum_{ijkl} c_{ijkil}T_{ijkil} - 2\sum_{ijkl} c_{ijilk}T_{ijkil} + 2\sum_{ijkl} c_{ijkil}T_{ijkil} + 2\sum_{ijkl} c_{ijilk}T_{ijkil} \in S_1(V^*) \\
= & \ 3\sum_{ijkl} c_{ijkil}T_{ijkil}. \\
\Rightarrow & \sum_{ijkl} c_{ijkil}T_{ijkil} \in S_1(V^*).
\end{aligned}$$

We have the terms in (3)$\in S_1(V^*)$, all that remains is when ∇R has five distinct indices.

2.4 Five Distinct Indices

Let $\{i, j, k, l, n\}$ be distinct indices and $\{a, b\} \neq \{i, l\}$, $\{j, k, n\} \neq \{x, y, z\}$. Choose $\phi \in S^2(V^*)$ and $\psi \in S^3(V^*)$ so that:

$$\phi(e_i, e_l) = \phi(e_l, e_i) = 1 \text{ and } \phi(e_a, e_b) = 0,$$

$$\psi(e_j, e_k, e_n) = \psi(e_j, e_n, e_k) = \psi(e_k, e_j, e_n) = \psi(e_k, e_n, e_j) = 1,$$
$$\psi(e_n, e_j, e_k) = \psi(e_n, e_k, e_j) = 1 \text{ and } \psi(e_x, e_y, e_z) = 0.$$

Up to the symmetries, the nontrivial terms for $\nabla R_{\phi,\psi}$ are:

$$\nabla R_{\phi,\psi}(e_i, e_j, e_k, e_l, e_n) = \nabla R_{\phi,\psi}(e_i, e_j, e_n, e_l, e_k) = \nabla R_{\phi,\psi}(e_i, e_k, e_j, e_l, e_n) = 1$$

$$\nabla R_{\phi,\psi}(e_i, e_n, e_j, e_l, e_k) = \nabla R_{\phi,\psi}(e_i, e_n, e_k, e_l, e_j) = \nabla R_{\phi,\psi}(e_i, e_k, e_n, e_l, e_j) = 1.$$

Up to the usual symmetries, the only nontrivial entries for $T_{ijkl;n}, T_{ijnl;k}, T_{ikjl;n}, T_{injl;k}, T_{inkl;j}$ and $T_{iknl;j}$ are:

$$\begin{aligned}
T_{ijkl;n}(e_i, e_j, e_k, e_l; e_n) &= 1 = \nabla R_{\phi,\psi}(e_i, e_j, e_k, e_l; e_n) \\
T_{ijnl;k}(e_i, e_j, e_n, e_l; e_k) &= 1 = \nabla R_{\phi,\psi}(e_i, e_j, e_n, e_l; e_k), \\
T_{ijkl;n}(e_i, e_k, e_j, e_l; e_n) &= 1 = \nabla R_{\phi,\psi}(e_i, e_k, e_j, e_l; e_n), \\
T_{ijkl;n}(e_i, e_n, e_j, e_l; e_k) &= 1 = \nabla R_{\phi,\psi}(e_i, e_n, e_j, e_l; e_k), \\
T_{ijkl;n}(e_i, e_n, e_k, e_l; e_j) &= 1 = \nabla R_{\phi,\psi}(e_i, e_n, e_k, e_l; e_j), \\
T_{ijkl;n}(e_i, e_k, e_n, e_l; e_j) &= 1 = \nabla R_{\phi,\psi}(e_i, e_k, e_n, e_l; e_j).
\end{aligned}$$

Since these are the only nonzero entries, we must have $\nabla R_{\phi,\psi} = T_{ijkl;n} + T_{ijnl;k} + T_{ikjl;n} + T_{injl;k} + T_{inkl;j} + T_{iknl;j}$, and $T_{ijkl;n} + T_{ijnl;k} + T_{ikjl;n} + T_{injl;k} + T_{inkl;j} + T_{iknl;j} \in S_1(V^*)$. We can permute the indices $\{i, j, k, l, n\}$ to get the following linear combinations:

$$T_{ijkl;n} + T_{ijnl;k} + T_{ikjl;n} + T_{injl;k} + T_{inkl;j} + T_{iknl;j} \in S_1(V^*), \tag{12}$$
$$T_{jkln;i} + T_{jkin;l} + T_{jlkn;i} + T_{jikn;l} + T_{jiln;k} + T_{jlin;k} \in S_1(V^*), \tag{13}$$
$$T_{klni;j} + T_{klji;n} + T_{knli;j} + T_{kjli;n} + T_{kjni;l} + T_{knji;l} \in S_1(V^*), \tag{14}$$
$$T_{lnij;k} + T_{lnkj;i} + T_{linj;k} + T_{lknj;i} + T_{lkij;n} + T_{likj;n} \in S_1(V^*), \tag{15}$$
$$T_{nijk;l} + T_{nilk;j} + T_{njik;l} + T_{nlik;j} + T_{nljk;i} + T_{njlk;i} \in S_1(V^*). \tag{16}$$

We can get more linear combinations in $S_1(V^*)$ by constructing $\nabla R_{\tilde{\phi},\tilde{\psi}}$:

$$\tilde{\phi}(e_i, e_n) = \tilde{\phi}(e_n, e_i) = 1 \text{ and } \tilde{\phi}(e_a, e_b) = 0,$$

$$\begin{gathered}
\tilde{\psi}(e_j, e_k, e_l) = \tilde{\psi}(e_j, e_l, e_k) = \tilde{\psi}(e_k, e_j, e_l) = \tilde{\psi}(e_k, e_l, e_j) = 1, \\
\tilde{\psi}(e_l, e_j, e_k) = \tilde{\psi}(e_l, e_k, e_j) = 1 \text{ and } \tilde{\psi}(e_x, e_y, e_z) = 0.
\end{gathered}$$

Up to the symmetries, the nontrivial terms for $\nabla R_{\phi,\psi}$ are:

$$\nabla R_{\phi,\psi}(e_i, e_j, e_k, e_n, e_l) = \nabla R_{\phi,\psi}(e_i, e_j, e_l, e_n, e_k) = \nabla R_{\phi,\psi}(e_i, e_k, e_j, e_n, e_l) = 1,$$

$$\nabla R_{\phi,\psi}(e_i, e_l, e_j, e_n, e_k) = \nabla R_{\phi,\psi}(e_i, e_l, e_k, e_n, e_j) = \nabla R_{\phi,\psi}(e_i, e_k, e_l, e_n, e_j) = 1.$$

Up to the usual symmetries, the only nontrivial entries for $T_{ijkn;l}, T_{ikjn;l}, T_{ijln;k}, T_{iljn;k}, T_{ikln;j}$ and $T_{ilkn;j}$ are:

$$\begin{aligned}
T_{ijkn;l}(e_i, e_j, e_k, e_l; e_n) = 1 = \nabla R_{\tilde{\phi},\tilde{\psi}}(e_i, e_j, e_k, e_l; e_n),\\
T_{ikjn;l}(e_i, e_j, e_n, e_l; e_k) = 1 = \nabla R_{\tilde{\phi},\tilde{\psi}}(e_i, e_j, e_n, e_l; e_k),\\
T_{ijln;k}(e_i, e_k, e_j, e_l; e_n) = 1 = \nabla R_{\tilde{\phi},\tilde{\psi}}(e_i, e_k, e_j, e_l; e_n),\\
T_{iljn;k}(e_i, e_n, e_j, e_l; e_k) = 1 = \nabla R_{\tilde{\phi},\tilde{\psi}}(e_i, e_n, e_j, e_l; e_k),\\
T_{ikln;j}(e_i, e_n, e_k, e_l; e_j) = 1 = \nabla R_{\tilde{\phi},\tilde{\psi}}(e_i, e_n, e_k, e_l; e_j),\\
T_{ilkn;j}(e_i, e_k, e_n, e_l; e_j) = 1 = \nabla R_{\tilde{\phi},\tilde{\psi}}(e_i, e_k, e_n, e_l; e_j).
\end{aligned}$$

Thus $\nabla R_{\tilde{\phi},\tilde{\psi}} = T_{ijkn;l} + T_{ikjn;l} + T_{ijln;k} + T_{iljn;k} + T_{ikln;j} + T_{ilkn;j}$ and $T_{ijkn;l} + T_{ikjn;l} + T_{ijln;k} + T_{iljn;k} + T_{ikln;j} + T_{ilkn;j} \in S_1(V^*)$. We can permute the indices $\{i, j, k, l, n\}$ to get the following linear combinations:

$$T_{ijkn;l} + T_{ikjn;l} + T_{ijln;k} + T_{iljn;k} + T_{ikln;j} + T_{ilkn;j} \in S_1(V^*), \tag{17}$$
$$T_{jkli;n} + T_{jlki;n} + T_{jkni;l} + T_{jnki;l} + T_{jlni;k} + T_{jnli;k} \in S_1(V^*), \tag{18}$$
$$T_{klnj;i} + T_{knlj;i} + T_{klij;n} + T_{kilj;n} + T_{knij;l} + T_{kinj;l} \in S_1(V^*), \tag{19}$$
$$T_{lnik;j} + T_{link;j} + T_{lnjk;i} + T_{ljnk;i} + T_{lijk;n} + T_{ljik;n} \in S_1(V^*), \tag{20}$$
$$T_{nijl;k} + T_{njil;k} + T_{nikl;j} + T_{nkil;j} + T_{njkl;i} + T_{nkjl;i} \in S_1(V^*). \tag{21}$$

Consider the linear combinations from (12)+(13)+(14)+(15)+(16):

$$\begin{aligned}
& T_{ijkl;n} + T_{ijnl;k} + T_{ikjl;n} + T_{injl;k} + T_{inkl;j} + T_{iknl;j} + T_{jkln;i} + T_{jkin;l} + T_{jlkn;i}\\
& +T_{jikn;l} + T_{jiln;k} + T_{jlin;k} + T_{klni;j} + T_{klji;n} + T_{knli;j} + T_{kjli;n} + T_{kjni;l} + T_{knji;l}\\
& +T_{lnij;k} + T_{lnkj;i} + T_{linj;k} + T_{lknj;i} + T_{lkij;n} + T_{likj;n} + T_{nijk;l} + T_{nilk;j} + T_{njik;l}\\
& +T_{nlik;j} + T_{nljk;i} + T_{njlk;i}\\
= \; & T_{ijkl;n} + T_{ijnl;k} + T_{ikjl;n} + T_{injl;k} + T_{inkl;j} + T_{iknl;j} + T_{jkln;i} + T_{injk;l} + T_{jlkn;i}\\
& +T_{ijnk;l} + T_{ijnl;k} + T_{injl;k} - T_{inkl;j} - T_{ijkl;n} + T_{knli;j} + T_{iljk;n} + T_{injk;l} + T_{ijnk;l}\\
& -T_{ijnl;k} - T_{lnjk;i} + T_{linj;k} + T_{lknj;i} - T_{ijkl;n} + T_{iljk;n} - T_{injk;l} + T_{inkl;j} + T_{njik;l}\\
& +T_{iknl;j} - T_{jkln;i} + T_{lknj;i}\\
= \; & T_{ikjl;n} + T_{injl;k} + T_{iknl;j} + T_{injk;l} + T_{jlkn;i}\\
& +T_{ijnk;l} + T_{ijnl;k} + T_{injl;k} + T_{knli;j} + T_{iljk;n} + T_{ijnk;l}\\
& -T_{lnjk;i} + T_{linj;k} + T_{lknj;i} - T_{ijkl;n} + T_{iljk;n} + T_{inkl;j} + T_{njik;l}\\
& +T_{iknl;j} + T_{lknj;i}
\end{aligned}$$

$$\begin{aligned}
&= T_{ikjl;n} - T_{ijkl;n} + T_{iljk;n} + T_{iljk;n} + T_{ijnk;l} + T_{injk;l} + T_{ijnk;l} + T_{njik;l} \\
&\quad +T_{ijnl;k} + T_{injl;k} + T_{injl;k} + T_{linj;k} + T_{knli;j} + T_{iknl;j} + T_{inkl;j} + T_{iknl;j} \\
&\quad -T_{lnjk;i} + T_{lknj;i} + T_{jlkn;i} + T_{lknj;i} \\
&= T_{ikjl;n} - T_{ijkl;n} + 2T_{iljk;n} + 2T_{ijnk;l} + T_{injk;l} + T_{njik;l} \\
&\quad +T_{ijnl;k} + 2T_{injl;k} + T_{linj;k} + T_{knli;j} + 2T_{iknl;j} + T_{inkl;j} \\
&\quad -T_{lnjk;i} + 2T_{lknj;i} + T_{jlkn;i}. \qquad (22)
\end{aligned}$$

Now consider c_{ijknl} multiplied by the linear combination from (22):

$$\begin{aligned}
&= \sum_{ijkln} c_{ijknl}(T_{ikjl;n} - T_{ijkl;n} + 2T_{iljk;n} - 2T_{ijkn;l} + T_{injk;l} + T_{njik;l} \\
&\quad +T_{ijnl;k} + 2T_{injl;k} + T_{linj;k} + T_{knli;j} + 2T_{iknl;j} + T_{inkl;j} \\
&\quad -T_{lnjk;i} + 2T_{lknj;i} + T_{jlkn;i}) \\
&= \sum_{ijkln} c_{ijknl}T_{ikjl;n} - \sum_{ijkln} c_{ijknl}T_{ijkl;n} + 2\sum_{ijkln} c_{ijknl}T_{iljk;n} - 2\sum_{ijkln} c_{ijknl}T_{ijkn;l} \\
&\quad + \sum_{ijkln} c_{ijknl}T_{injk;l} + \sum_{ijkln} c_{ijknl}T_{njik;l} + \sum_{ijkln} c_{ijknl}T_{ijnl;k} + 2\sum_{ijkln} c_{ijknl}T_{injl;k} \\
&\quad + \sum_{ijkln} c_{ijknl}T_{linj;k} + \sum_{ijkln} c_{ijknl}T_{knli;j} + 2\sum_{ijkln} c_{ijknl}T_{iknl;j} + \sum_{ijkln} c_{ijknl}T_{inkl;j} \\
&\quad - \sum_{ijkln} c_{ijknl}T_{lnjk;i} + 2\sum_{ijkln} c_{ijknl}T_{lknj;i} + \sum_{ijkln} c_{ijknl}T_{jlkn;i}.
\end{aligned}$$

We will use Lemma 1.7 on the first, second, fifth, sixth, seventh, ninth, tenth, twelfth, thirteenth and fifteenth summations.

$$\begin{aligned}
&= \sum_{ijkln} c_{ikjln}T_{ijkn;l} - \sum_{ijkln} c_{ijkln}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iljk;n} - 2\sum_{ijkln} c_{ijknl}T_{ijkn;l} \\
&\quad + \sum_{ijkln} c_{injkl}T_{ijkn;l} + \sum_{ijkln} c_{njikl}T_{ijkn;l} + \sum_{ijkln} c_{ijnlk}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{injl;k} \\
&\quad + \sum_{ijkln} c_{linjk}T_{ijkn;l} + \sum_{ijkln} c_{knlij}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iknl;j} + \sum_{ijkln} c_{inklj}T_{ijkn;l} \\
&\quad - \sum_{ijkln} c_{lnjki}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{lknj;i} + \sum_{ijkln} c_{jlkni}T_{ijkn;l}.
\end{aligned}$$

We will use the usual symmetries on the constants, to obtain a Bianchi-like pattern so that we may simplify the equation with the Bianchi identities.

$$\begin{aligned}
= \quad & -\sum_{ijkln} c_{ikljn}T_{ijkn;l} - \sum_{ijkln} c_{ijkln}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iljk;n} - 2\sum_{ijkln} c_{ijknl}T_{ijkn;l} \\
& + \sum_{ijkln} c_{injkl}T_{ijkn;l} + \sum_{ijkln} c_{iknjl}T_{ijkn;l} - \sum_{ijkln} c_{ijlnk}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{injl;k} \\
& - \sum_{ijkln} c_{ilnjk}T_{ijkn;l} - \sum_{ijkln} c_{ilknj}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iknl;j} - \sum_{ijkln} c_{inlkj}T_{ijkn;l} \\
& - \sum_{ijkln} c_{lnjki}T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{lknj;i} - \sum_{ijkln} c_{ljkni}T_{ijkn;l} \\
= \quad & -\sum_{ijkln} (c_{ikljn} + c_{ijkln})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iljk;n} - 2\sum_{ijkln} c_{ijknl}T_{ijkn;l} \\
& + \sum_{ijkln} (c_{injkl} + c_{iknjl})T_{ijkn;l} - \sum_{ijkln} (c_{ijlnk} + c_{ilnjk})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{injl;k} \\
& - \sum_{ijkln} (c_{ilknj} + c_{inlkj})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iknl;j} \\
& - \sum_{ijkln} (c_{lnjki} + c_{ljkni})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{lknj;i} \\
= \quad & -\sum_{ijkln} (-c_{iljkn})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iljk;n} - 2\sum_{ijkln} c_{ijknl}T_{ijkn;l} \\
& + \sum_{ijkln} (-c_{ijknl})T_{ijkn;l} - \sum_{ijkln} (-c_{injlk})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{injl;k} \\
& - \sum_{ijkln} (-c_{iknlj})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{iknl;j} \\
& - \sum_{ijkln} (-c_{lknji})T_{ijkn;l} + 2\sum_{ijkln} c_{ijknl}T_{lknj;i}.
\end{aligned}$$

We will use Lemma 1.7 on the first, fourth, fifth, seventh and ninth summations.

$$\begin{aligned}
= & \sum_{ijkln}(c_{ijknl})T_{iljk;n} + 2\sum_{ijkln} c_{ijknl}T_{iljk;n} - 2\sum_{ijkln} c_{ijknl}T_{ijkn;l} \\
& - \sum_{ijkln}(c_{ijknl})T_{ijkn;l} + \sum_{ijkln}(c_{ijknl})T_{injl;k} + 2\sum_{ijkln} c_{ijknl}T_{injl;k} \\
& + \sum_{ijkln}(c_{ijknl})T_{iknl;j} + 2\sum_{ijkln} c_{ijknl}T_{iknl;j} \\
& + \sum_{ijkln}(c_{ijknl})T_{lknj;i} + 2\sum_{ijkln} c_{ijknl}T_{lknj;i} \\
= & \; 3\sum_{ijkln} c_{ijknl}T_{iljk;n} - 3\sum_{ijkln} c_{ijknl}T_{ijkn;l} + 3\sum_{ijkln} c_{ijknl}T_{injl;k} + 3\sum_{ijkln} c_{ijknl}T_{iknl;j} \\
& + 3\sum_{ijkln} c_{ijknl}T_{lknj;i} \in S_1(V^*). \qquad (23)
\end{aligned}$$

Using Lemma 1.8 we can find permutations that give us equalities for $\sum_{ijkln} c_{ikljn}T_{inkl;j}$. Among the permutations of $\sum_{ijkln} c_{ikljn}T_{inkl;j}$, there are a few of interest:

$$\sigma = (jnlk) \Rightarrow \sum_{ijkln} c_{ikljn}T_{inkl;j} = \sum_{ijkln} c_{ijknl}T_{iljk;n}, \qquad (24)$$

$$\sigma = (ln)(ijk) \Rightarrow \sum_{ijkln} c_{ikljn}T_{inkl;j} = -\sum_{ijkln} c_{ijknl}T_{inlj;k}, \qquad (25)$$

$$\sigma = (kn)(inlj) \Rightarrow \sum_{ijkln} c_{ikljn}T_{inkl;j} = -\sum_{ijkln} c_{ijknl}T_{njkl;i}, \qquad (26)$$

$$\sigma = (inl) \Rightarrow \sum_{ijkln} c_{ikljn}T_{inkl;j} = \sum_{ijkln} c_{ijknl}T_{iknl;j}, \qquad (27)$$

$$\sigma = (kl) \Rightarrow \sum_{ijkln} c_{ikljn}T_{inkl;j} = \sum_{ijkln} c_{ilkjn}T_{inlk;j}, \qquad (28)$$

$$\sigma = (kl)(il)(jn) \Rightarrow \sum_{ijkln} c_{ikljn}T_{inkl;j} = \sum_{ijkln} c_{iljkn}T_{iknl;j}. \qquad (29)$$

Notice that (24) through (29) are all equal, for convenience we will use the rightmost summand on (24). Looking at (23) with our new relations we have:

$$\begin{aligned}
& 3\sum_{ijkln} c_{ijknl}T_{iljk;n} - 3\sum_{ijkln} c_{ijknl}T_{ijkn;l} + 3\sum_{ijkln} c_{ijknl}T_{injl;k} + 3\sum_{ijkln} c_{ijknl}T_{iknl;j} \\
& +3\sum_{ijkln} c_{ijknl}T_{lknj;i} \\
= \;& 3\sum_{ijkln} c_{ijknl}T_{iljk;n} - 3\sum_{ijkln} c_{ijknl}T_{ijkn;l} - 3\sum_{ijkln} c_{ijknl}T_{inlj;k} + 3\sum_{ijkln} c_{ijknl}T_{iknl;j} \\
& -3\sum_{ijkln} c_{ijknl}T_{njkl;i}.
\end{aligned}$$

We will use (25) on the third summand, (27) on the fourth summation and (26) on the fifth summation.

$$\begin{aligned}
= \;& 3\sum_{ijkln} c_{ijknl}T_{iljk;n} - 3\sum_{ijkln} c_{ijknl}T_{ijkn;l} + 3\sum_{ijkln} c_{ijknl}T_{iljk;n} + 3\sum_{ijkln} c_{ijknl}T_{iljk;n} \\
& 3\sum_{ijkln} (c_{ijknl})T_{iljk;n} \\
= \;& 12\sum_{ijkln} (c_{ijknl})T_{iljk;n} - 3\sum_{ijkln} c_{ijknl}T_{ijkn;l} \in S_1(V^*). \qquad (30)
\end{aligned}$$

Now we are going to use the first Bianchi identity to get a relation:

$$\begin{aligned}
& (c_{ijkln} + c_{iljkn} + c_{ikljn})T_{iljk;n} = 0 \cdot T_{iljk;n} = 0. \\
\Rightarrow \;& c_{ijkln}T_{iljk;n} + c_{iljkn}T_{iljk;n} + c_{ikljn}T_{iljk;n} = 0. \\
\Rightarrow \;& \sum_{ijkln} c_{ijkln}T_{iljk;n} + c_{iljkn}T_{iljk;n} + c_{ikljn}T_{iljk;n} = 0. \\
\Rightarrow \;& \sum_{ijkln} c_{iljkn}T_{iljk;n} = -\sum_{ijkln} c_{ijkln}T_{iljk;n} - \sum_{ijkln} c_{ikljn}T_{iljk;n}.
\end{aligned}$$

Consider $\sigma = (il)$, then $\sum_{ijkln} c_{ijkln}T_{iljk;n} = \sum_{ijkln} c_{ikljn}T_{iljk;n}$ and we have:

$$\begin{aligned}
\Rightarrow \sum_{ijkln} c_{iljkn}T_{iljk;n} &= -\sum_{ijkln} c_{ijkln}T_{iljk;n} - \sum_{ijkln} c_{ikljn}T_{iljk;n} \\
&= -2\sum_{ijkln} c_{ikljn}T_{iljk;n}. \qquad (31)
\end{aligned}$$

Again we will use the first Bianchi identity to get another relation:

$$\begin{aligned}
& (c_{njkli} + c_{nljki} + c_{nklji})T_{iljk;n} = 0 \cdot T_{iljk;n} = 0. \\
\Rightarrow\ & c_{njkli}T_{iljk;n} + c_{nljki}T_{iljk;n} + c_{nklji}T_{iljk;n} = 0. \\
\Rightarrow\ & \sum_{njkli} c_{nljki}T_{iljk;n} = -\sum_{ijkln} c_{njkli}T_{iljk;n} - \sum_{ijkln} c_{nklji}T_{iljk;n}.
\end{aligned}$$

Consider $\sigma = (jk)$ $\sum_{ijkln} c_{njkli}T_{iljk;n} = \sum_{ijkln} c_{nklji}T_{iljk;n}$ and we have:

$$\begin{aligned}
\Rightarrow \sum_{njkli} c_{nljki}T_{iljk;n} &= -\sum_{ijkln} c_{ijkln}T_{iljk;n} - \sum_{ijkln} c_{nklji}T_{iljk;n} \\
&= -2\sum_{ijkln} c_{nklji}T_{iljk;n}. \qquad (32)
\end{aligned}$$

Use (28)(29) and the first Bianchi identity to get another relation :

$$\begin{aligned}
& \sum_{ijkln}(c_{iknlj} + c_{ilknj} + c_{inlkj})T_{iljkn} = 0 \cdot T_{iljkn} = 0, \\
& \sum_{ijkln} c_{ilknj}T_{iljkn} = -(\sum_{ijkln} c_{iknlj}T_{iljkn} + \sum_{ijkln} c_{inlkj}T_{iljkn}), \\
& \sum_{ijkln} c_{ilknj}T_{iljkn} = -(-\sum_{ijkln} c_{kilnj}T_{iljkn} - \sum_{ijkln} c_{inlkj}T_{ilkjn}), \\
& \sum_{ijkln} c_{ilknj}T_{iljkn} = 2\sum_{ijkln} c_{ikljn}T_{inklj}, \\
& \sum_{ijkln} c_{ilknj}T_{iljkn} = 2\sum_{ijkln} c_{iljkn}T_{iknlj}. \qquad (33)
\end{aligned}$$

One last relation using the second Bianchi identity:

$$\begin{aligned}
& \sum_{ijkln}(c_{nljki} + c_{nlijk} + c_{nlkij})T_{iljk;n} = 0 \cdot T_{iljk;n} = 0. \\
& \Rightarrow \sum_{ijkln} c_{nljki}T_{iljk;n} = -\sum_{ijkln} c_{nlijk}T_{iljk;n} - \sum_{ijkln} c_{nlkij}T_{iljk;n}.
\end{aligned}$$

Consider $\sigma = (ik)$, then $\sum_{ijkln} c_{njkli}T_{iljk;n} = \sum_{ijkln} c_{nklji}T_{iljk;n}$ and we have:

$$\begin{aligned}
\sum_{ijkln} c_{nljki}T_{iljk;n} &= -\sum_{ijkln} c_{nlijk}T_{iljk;n} - \sum_{ijkln} c_{nlkij}T_{iljk;n} \\
&= -2\sum_{ijkln} c_{nlkij}T_{iljk;n}. \qquad (34)
\end{aligned}$$

Now let's look at c_{iljkn}(20) and use (24) to (29) (31) and (32):

$$
\begin{aligned}
& c_{iljkn}(T_{lnik;j}+T_{link;j}+T_{lnjk;i}+T_{ljnk;i}+T_{lijk;n}+T_{ljik;n}) \in S_1(V^*) \\
\Rightarrow\ & \sum_{ijkln} c_{iljkn}(T_{lnik;j}+T_{link;j}+T_{lnjk;i}+T_{ljnk;i}+T_{lijk;n}+T_{ljik;n}) \in S_1(V^*) \\
=\ & -\sum_{ijkln} c_{iljkn}T_{iknl;j}+\sum_{ijkln} c_{iljkn}T_{ilkn;j}-\sum_{ijkln} c_{iljkn}T_{nljk;i}-\frac{1}{2}\sum_{ijkln} c_{iljkn}T_{nljk;i} \\
& -\sum_{ijkln} c_{iljkn}T_{iljk;n}-\frac{1}{2}\sum_{ijkln} c_{iljkn}T_{iljk;n}.
\end{aligned}
$$

We use (33) and Lemma 1.7 on the second summation.

$$
\begin{aligned}
=\ & -\sum_{ijkln} c_{iljkn}T_{iknl;j}+2\sum_{ijkln} c_{iljkn}T_{iknl;j}-\sum_{ijkln} c_{iljkn}T_{nljk;i}-\frac{1}{2}\sum_{ijkln} c_{iljkn}T_{nljk;i} \\
& -\sum_{ijkln} c_{iljkn}T_{iljk;n}-\frac{1}{2}\sum_{ijkln} c_{iljkn}T_{iljk;n} \\
=\ & \sum_{ijkln} c_{iljkn}T_{iknl;j}-\frac{3}{2}\sum_{ijkln} c_{iljkn}T_{nljk;i}-\sum_{ijkln} c_{iljkn}T_{iljk;n}-\frac{1}{2}\sum_{ijkln} c_{iljkn}T_{iljk;n}.
\end{aligned}
$$

We will use (34) on the second summation.

$$
\begin{aligned}
=\ & \sum_{ijkln} c_{iljkn}T_{iknl;j}-3\sum_{ijkln} c_{iljkn}T_{iknl;j}-\frac{3}{2}\sum_{ijkln} c_{iljkn}T_{iljk;n} \\
=\ & -2\sum_{ijkln} c_{iljkn}T_{iknl;j}-\frac{3}{2}\sum_{ijkln} c_{iljkn}T_{iljk;n}. \\
\Rightarrow\ & 2\sum_{ijkln} c_{iljkn}T_{iknl;j}+3\sum_{ijkln} c_{iljkn}T_{iljk;n} \in S_1(V^*),
\end{aligned}
$$

By (29) on the first summation,

$$
\Rightarrow\ 2\sum_{ijkln} c_{ijknl}T_{iknl;j}+3\sum_{ijkln} c_{ijknl}T_{ijkn;l} \in S_1(V^*). \tag{35}
$$

By (30) and (35) we have:

$$
\begin{aligned}
& 4\sum_{ijkln}(c_{ijknl})T_{iljk;n}-1\sum_{ijkln} c_{ijknl}T_{ijkn;l}-6\sum_{ijkln} c_{ijknl}T_{ijkn;l}-2\sum_{ijkln} c_{ijknl}T_{iljk;n} \\
=\ & -7\sum_{ijkln} c_{ijknl}T_{ijkn;l} \in S_1(V^*). \\
\Rightarrow\ & \sum_{ijkln} c_{ijkln}T_{ijkl;n} \in S_1(V^*).
\end{aligned}
$$

We must have (4)$\in S_1(V^*)$, therefore $\nabla R \in S_1(V^*)$ $\square$

3 Questions

Since we have a spanning set for $A_1(V^*)$ on a vector space, V, the natural questions become:

- Given any $\nabla R \in A_1(V^*)$, how do you reconstruct it using $\nabla R_{\phi,\psi}$'s?
- How many $\nabla R_{\phi,\psi}$'s are necessary to fully reconstruct a given ∇R?
- If $\nabla R_{\phi_1,\psi_1} = \nabla R_{\phi_2,\psi_2}$, then what does this tell us about ϕ_1, ϕ_2 and ψ_1, ψ_2? What conditions must we impose on ϕ_1 and ψ_1 to give $\phi_1 = \phi_2$ and $\psi_1 = \psi_2$?
- When is it the case that $\nabla R_{\phi_1,\psi_1} + \nabla R_{\phi_2,\psi_2} = \nabla R_{\phi_3,\psi_3}$?
- The dimension of the basis for $A_1(V^*)$ is known [4]. How would we construct an algorithm for generating a basis for $A_1(V^*)$ by trimming away our spanning set $S_1(V^*)$?

4 Acknowledgements

I thank California State University at San Bernardino and NSF Grant DMS-0850959 for funding the REU. I give special thanks to Dr. Rolland Trapp for directing this program and to Dr. Corey Dunn for advising my research endeavors this summer.

References

[1] J. Díaz-Ramos, B. Fiedler, E. García-Río, and P. Gilkey. The structure of algebraic covariant derivative curvature tensors. *Int. J. Geom. Methods Mod. Phys.*, 1(6):711–720, 2004.

[2] Bernd Fiedler. On the symmetry classes for the first covariant derivatives of tensor fields. *Sém. Lothar. Combin.*, 49:Art. B49f, 22 pp. (electronic), 2002/04.

[3] Peter B. Gilkey. *The geometry of curvature homogeneous pseudo-Riemannian manifolds*, volume 2 of *ICP Advanced Texts in Mathematics*. Imperial College Press, London, 2007.

[4] Robert S. Strichartz. Linear algebra of curvature tensors and their covariant derivatives. *Canad. J. Math.*, 40(5):1105–1143, 1988.