Summer 2009 REU:Knot Theory, Lower Bound for Rope Length of m-almost alternating knots

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Abstract

For any given link conformation L that admits an alternating, almost-alternating, or paired projection, we establish that $Rop(L)$ has a linear lower bound in terms of crossing number for alternating and almost-alternating projections, and in terms of bridge number for paired projections. We arrive at this conclustion by considering the height of the links and reducing them to a minimal height representation.

1 Introduction

We shall introduce some key concepts that are helpful to understanging the paper. A **knot** is a closed curve in space. A **link**, L, is a knot with multiple components. A **projection** or diagram is a two dimensional representation of a link. One knot has many projections. The **crossing number**, Cr(L), of a diagram is the number of crossings on the projection of the link. The **ropelength** of a link conformation L, denoted $Rop(L)$, is the ratio of the arclength, $l(L)$, to the radius, $ra(L)$. In other words, $Rop(L) = \frac{l(L)}{ra(L)}$. Note that Ropelent is scale-invariant since $l(L)$ and $ra(L)$ increase by the same factor. Without loss of generality, we assume $ra(L) = 1$ which implies $Rop(L) = l(L)$. This also implies that given two non-local points on the link. there must be a distance of 2 units seperating them. An **alternating** link is a link that admits a projection that alternates from overcrossing to undercrossing as you traverse the link in a fixed direction.

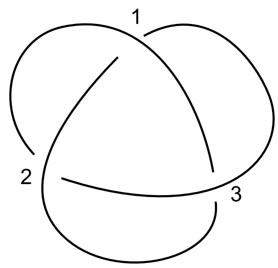
There is a section dedicated to defining paired links so we shall exclude those definitions here. We will first discuss the ropelength of alternating and almost-alternating conformations of L and discover that both have a linear lower bound in terms of crossing number.

2 Alternating Rope length

Let L be an alternating conformation with n crossings labeled 1 through n. At the i^{th} crossing of the link there is an overcrossing p_i and an undercrossing q_i that share the same x and y coordinates. We can let the height of p_i be denoted as o_i and the height of q_i as u_i. We encounter a cyclic repeating pattern of overcrossings with undercrossing as we traverse the link in a fixed direction.

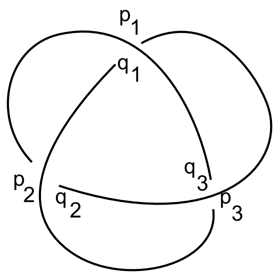
For example, label the crossings of a trefoil 1, 2, and 3 as you traverse the knot in a fixed direction(see Figure 2.1. Then label the undercrossings and overcrossings appropriately as seen in 2.2.

Figure 2.1.



An alternating projection of a trefoil with 3 crossings

Figure 2.2.



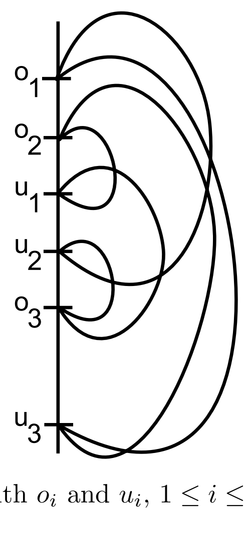
p_i and q_i, $1 \leq i \leq 3$.

Definition 2.3. Let L be an alternating conformation. The height function, $h : L \to \mathbb{R}$, is the projection $h(x, y, z) = z$ which takes every point on L to its z-coordinate. The **image** of L under the projection, denoted by $h(L)$, is a path that goes up and down along the z-axis. Furthermore the length of $h(L)$ denoted $l(h(L))$ is the length of all the edges of $h(L)$.

Note that under the projection, $h(p_i) = o_i$ and $h(q_i) = u_i$ so that the values of o_i and u_i partition the z-axis into subintervals.

Continuing the example, Figure 2.4 is a diagram of $h(L)$ along the z-axis where o_i and u_i represent p_i and q_i respectively. The series of the path runs as $o_1u_2o_3u_1o_2u_3$. and the paths are bent away from the z-axis for ease of viewing.

Figure 2.4.



$h(L)$ with o_i and u_i, $1 \leq i \leq 3$

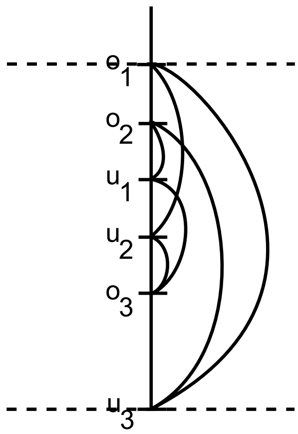
Note that for each crossing, $o_i > u_i$. Furthermore, there is at least a distance of 2 separating o_i and u_i because we normalized the radius of each strand to be 1; Therefore $|o_i - u_i| \geq 2$.

***Remark* 2.5.** If L is an alternating conformation, $Rop(L) \geq l(h(L))$ since the projection h does not increase the distance between points.

The goal is to use the height function h to obtain a lower bound on the ropelength of L. It is difficult to measure the length directly so we will simplify it by replacing the arcs of $h(L)$ with straight line segments that connect the overcrossings to the undercrossings. We will call this new image of L the **taut** image of L, denoted by $t(L)$ and has a length denoted $l(t(L))$. The edges of $t(L)$ connect successive over- and under-crossings. each o_i and u_i has exactly two edges incident with it in $t(L)$, and each edge has an endpoint of one overcrossing and one undercrossing.

Continuing the example:

Figure 2.6.



The taut image $t(L)$.

Note that the straight line segments have been slightly bent for viewing purposes.

***Remark* 2.7.** Let L be an alternating conformation. $Rop(L) \geq l(t(L))$

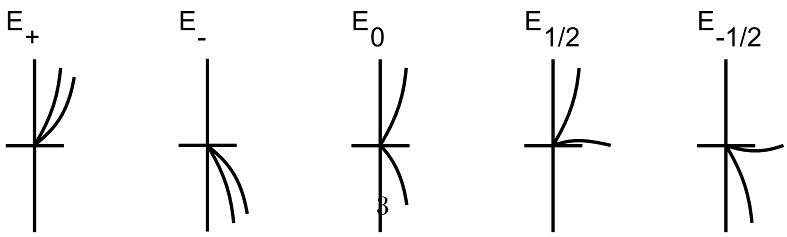
Notice that both of the paths $h(L)$ and $t(L)$ connect the heights o_i and u_j in the same cyclic order. Since the arc in $t(L)$ connecting consecutive o_i and u_j is a straight line, its length is at most the length of the corresponding piece in $h(L)$. Summing over all consecutive points, the sum of the straight line segments in $t(L)$ is at most the sum of the arcs from $h(L)$. Using Remark 2.5 we see the remark now follows.

In the example $Rop(L) \geq l(h(L)) \geq l(t(L)) = |o_1 - u_2| + |u_2 - o_3| + |o_3 - u_1| + |u_1 - o_2| + |o_2 - u_3| + |u_3 - o_1|$.

Since the next step requires us to manipulate the taut image of L, we shall first introduce some terminology and opperational definitions in order to simplify the process.

First, notice that each height of a crossing can conform to one of five configurations as seen in Figure 2.8

Figure 2.8.

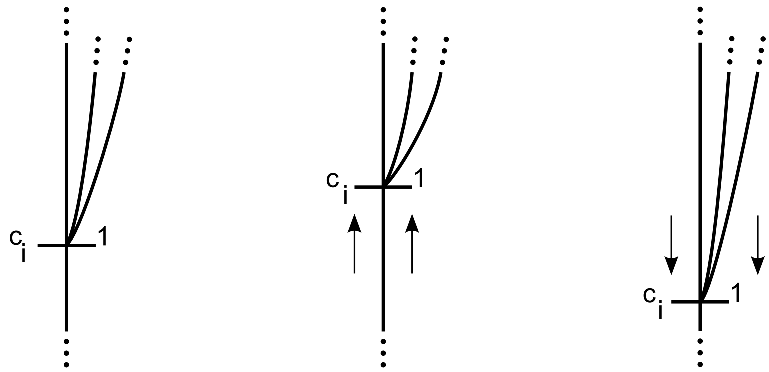


The possible configurations of two edges extending off of $t(L)$.

Let c_i be an arbitrary crossing that may either be over- or under-crossing. We can assign each crossing height an **edge value**, denoted $e(c_i)$, of 1, -1, $\frac{1}{2}$, -$\frac{1}{2}$ or 0 corresponding to an E_+, E_-, $E_{\frac{1}{2}}$, $E_{-\frac{1}{2}}$ and E_0 configurations respectively.

We will be moving crossings up and down along the z-axis so we must consider the effects on ropelength when we shift a given crossing up or down. Now observe the effects of the movement of a crossing with edge value 1 on $l(t(L))$ in Figure 2.9.

Figure 2.9.

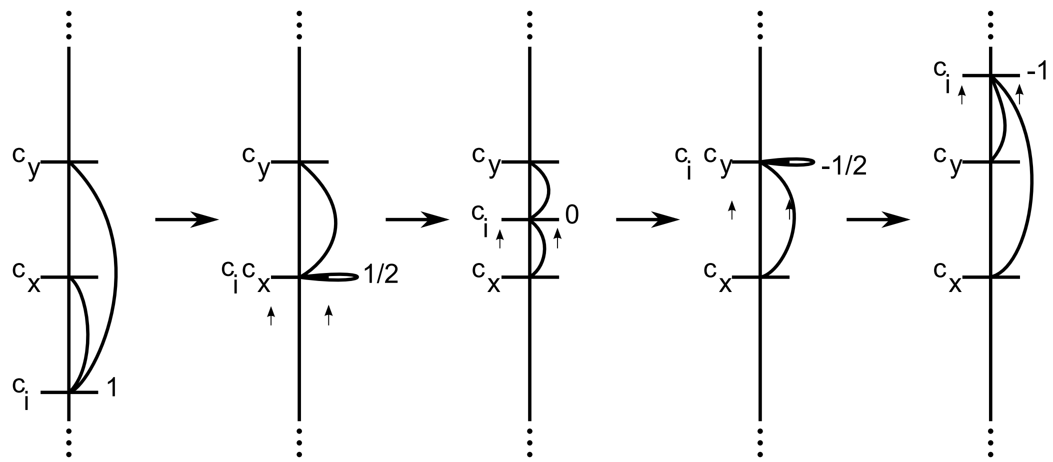


The movement of a crossing with edge value 1.

Notice that a positive edge value decreases $l(t(L))$ when shifted in the positive direction and increases $l(t(L))$ when shifting in the negative direction. Similarly, a negative edge value will increase $l(t(L))$ when shifted up and decrease $l(t(L))$ when shifted down, and a zero edge value will move in any direction with no effect on $l(t(L))$.

During the movement of heights the edge value of a crossing may change when it changes its relative height to a crossing it is connected with, as seen in Figure 2.10 when c_i changes height. Note that during an upward shift the edge value can never increase and during a downward shift the edge value can never decrease

Figure 2.10.



The change of edge value when shifted along z-axis.

When shifting a crossing across a height in which the edge number changes we must be careful to use the right edge value when calculating the change of ropelength during the shift.

Suppose that we shift a crossing between two heights, h_1 and h_2, along the z-axis of $t(L)$ such that $h_1 > h_2$. If we shift c_i from h_1 to h_2 (downward shift), we must use the maximal edge value of c_i in the range $[h_1, h_2)$. Similarly if $h_1 < h_2$ (upward shift) we must use the minimal edge value of c_i in the range $[h_1, h_2)$. In our example, if we shift c_i from c_x to c_y we see that $e(c_i) = \frac{1}{2}$ initally, then passes through a range where $e(c_i) = 0$ and $e(c_i) = -\frac{1}{2}$ when the shift stops. Thus $e(c_i)$ during the shift would be $\min(\frac{1}{2}, 0) = 0$. This conceptually makes sense because looking at Figure 2.10, when we shift c_i from c_x to c_y we do not increase or decrease the ropelength during an upward shift.

We have sufficiently characterized the effect on $l(t(L))$ when we shift a single height. The following lemma will address the effect when we shift multiple heights simultaneously.

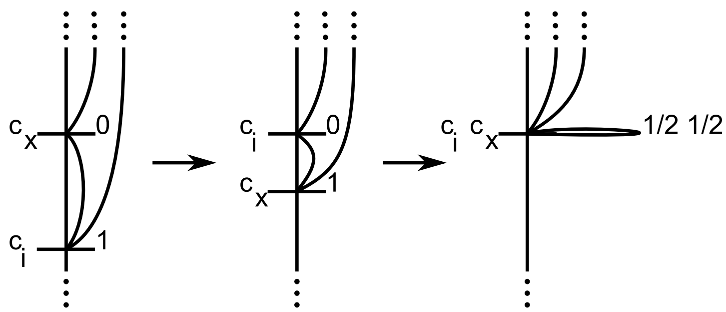
Definition 2.11. The **shifting set**, σ, is the set of all crossings that are moved simultaneously along the taught image.

Definition 2.12. The edge value of the shifting set, $e(\sigma)$ is the sum of the edge values of all the crossing in the set. In other words, $e(\sigma) = \sum_{c_i \in \sigma} e(c_i)$.

Since we are moving all crossings of the set simultaneously and the same distance each, any effect on $l(t(L))$ caused by one crossing will be propegated with the effect on $l(t(L))$ by all the other crossings at the same rate. Thus we can sum together all the edge values and this produces the edge value of σ. Thus we have a well defined edge value for our shifting set.

For an example, we can view in Figure 2.13 that if our shifting set contains c_x and c_i, then $e(\sigma) = e(c_x) + e(c_i) = 1$ which holds for all positions of c_i relative to c_x.

Figure 2.13.



$\sigma = \{c_i, c_x\}$ and $e(\sigma) = 1$.

When we move σ there are a total of two edges extending up from the two crossings to the rest of the link so together they are in an E_+ configuration and thus $e(\sigma) = 1$.

In general if $e(\sigma) \geq 0$, we can shift the shifting set up without increasing $l(t(L))$ and if $e(\sigma) \leq 0$, we can shift the shifting set down without increasing $l(t(L))$.

Now that we have the tools to work with our taut image, we can get back to show a linear lower bound on ropelength.

Definition 2.14. Let the height of the highest o_i on $t(L)$ be called the **o-line** and the height of the lowest u_j of $t(L)$ be called the **u-line**. When all the overcrossings are at the o-line and all the undercrossings are at the u-line we have a graph with only two heights.

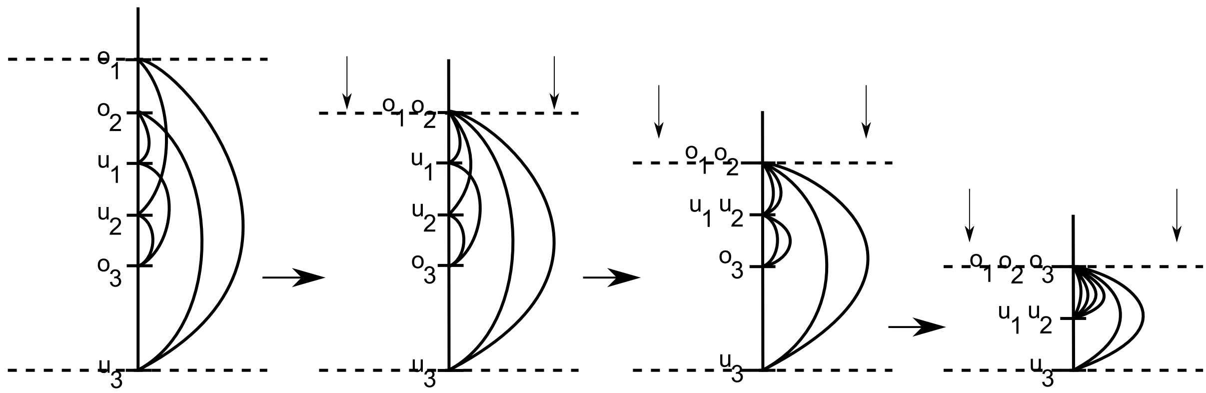
Definition 2.15. The n**-reduced** image of L, $r(L)$, is the taut image of L that has all crossings at n distinct heights.

Definition 2.16. The **squished** image of L, $\tilde{s}(L)$, is the 2-reduced image of L. In an alternating case all overcrossings are at the same height and all undercrossings are at the same height and the two heights are at least 2 units apart.

Lemma 2.17. *Let L be an alternating conformation, $l(t(L)) \geq l(\tilde{s}(L))$.*

*Proof.*Let the shifting set, σ, of the taut image, $t(L)$, contain all o_i and u_j such that o_i is on the o-line and u_j is two units below the o-line. Let this line of u_j be called the u_o-line. Since every overcrossing is connected to two undercrossings, the edge value of each overcrossing on the o-line is -1, in other words $e(o_i) = -1$ for all o_i on o-line. By Definition 2.12 $e(\sigma) = \sum_{o_i, u_i \in \sigma} e(o_i) + e(u_j)$. We see that $e(o_i) = -1$ and the largest value of $e(u_j)$ is 1. Since there cannot be more undercrossings in σ than overcrossings, otherwise there will be an overcrossing below its associated undercrossing, it follows that $e(\sigma) \leq 0$. Thus we can shift σ down along the z-axis without increasing $l(t(L))$. This allows us to continually reduce the height of the o-line until all overcrossings are at the same height as we can see in our example of the trefoil in Figure 2.18. Note that if there are undercrossings further than 2 units below the lowest overcrossing, then all the undercrossings will not be at the same height.

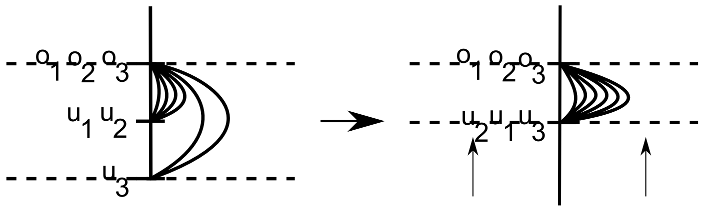
Figure 2.18.



Notice in Figure 2.18 that the dashed lines correspond to the o-line and the u-line.

To get the squished image, let the shifting set, τ, contain all undercrossings on the u-line. Each undercrossing on the u-line is connected to two overcrossings above it, so $e(u_k) = 1$ for all u_k on the $u - line$. Clearly, by Definition 2.12, $e(\tau) \geq 0$. In fact $e(\tau) = k$ where k is the number of undercrossings at the u-line. Thus we can continually increase the height of the u-line until all undercrossings are at the same height as we can see in Figure 2.19. Thus we have reduced the taut image of L to the squished image of L while strictly maintaining or reducing the length of $t(L)$. Therefore, $l(t(L)) \geq l(\tilde{s}(L))$. □

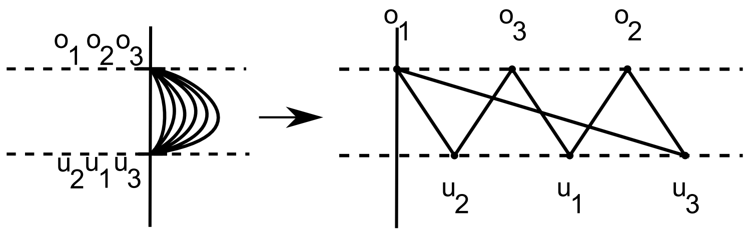
Figure 2.19.



Note that $|o\text{-line}-u\text{-line}| = 2$ since we stopped moving the u-line when all undercrossing were contained on it, which occured at the u_o-line that was set 2 units away from the o-line.

For ease of viewing we can extend $\tilde{s}(L)$ horizontally to keep track of the path in the link as seen in Figure 2.20

Figure 2.20.



Theorem 2.21. *If L is an alternating conformation, then $Rop(L) \geq 4Cr(L)$.*

*Proof.*Since $\tilde{s}(L)$ is a collection of straight edges running along the z-axis and the length of each edge is 2 units and there are two edges per crossing we know that $\tilde{s}(L) \geq 2(2Cr(L))$. Combining this observation with Remarks 2.5 and 2.7 and Lemma 2.17, we see that $Rop(L) \geq l(h(L)) \geq l(t(L)) \geq l(\tilde{s}(L)) = 4Cr(L)$. □

3 Almost-Alternating Ropelength

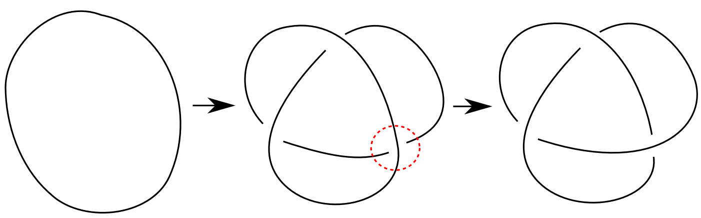
3.1 Introduction

Definition 3.2. An **m-Alternating** knot has a projection that requires m crossing changes to become alternating.

Definition 3.3. An **almost-Alternating** knot has a projection that requires 1 crossing change to become alternating.

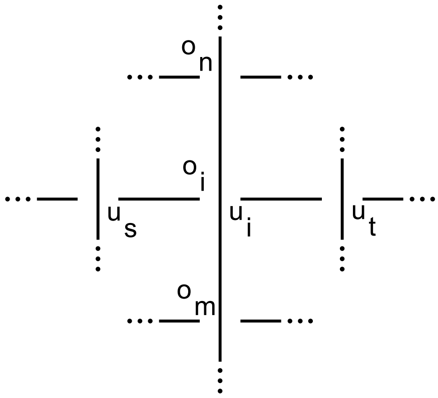
For example the unknot is almost alternating as seen in Figure 3.4.

Figure 3.4.



Let L be an almost-alternating conformation of a link. Notice that if we consider the sequence of crossings as we traverse the knot in a fixed direction we will encounter an alternating pattern, *ououou*, and two breaks in that pattern; one break occurs where an overcrossing connects to two overcrossings, *ooo*, and the other where an undercrossing connects to two undercrossings, *uuu*, both occurances are unique. The central o and u correspond to the same crossing on L. Let us label the three overcrossings in a row as o_m, o_i, and o_n and label the three undercrossings in a row as u_s, u_i, and u_t as seen in Figure 3.5.

Figure 3.5.



3 overcrossings in a row and 3 undercrossings in a row, centered around o_i and u_i.

We find that we can also achieve a linear lower bound on ropelength for almost-alternating diagrams. Let L be an almost-alternating conformation of a link. We will find $Rop(L) \geq 4(Cr(L) - 4)$. We shall employ Remarks 2.5 and 2.7 from the previous section and again opperate on $t(L)$ to get a reduced image of L. This reduced image will be achieved at 4 heights as opposed to the two heights of the squished image for the alternating links.

3.6 Almost-Alternating Ropelength

Let L be a conformation of the link which admits an almost-alternating diagram. $t(L)$ is the taut image of L and $\bar{r}(L)$ is the 4-reduced image of L.

We established in the previous section that the $Rop(L) \geq l(t(L))$ with an alternating conformation. The same reasoning as before applies to the almost-alternating case. Thus all we need to show is $l(t(L)) \geq \bar{r}(L)$ for the almost-alternating case and the result of a linear lower bound on ropelength will follow.

Recall that the o-line is the height of the highest overcrossing in $t(L)$ and the u-line is the height of the lowest undercrossing.

For the following lemmas, σ is the shifting set that contains all crossings at the o-line, as well as all undercrossings 2 units below the o-line which we will call the u_o-line. The analogous shifting set for the u-line is the σ_u shifting set. The height 2 units above the u-line in σ_u is the o^u-line. We will operate on the o-line to get results that can also be applied to the u-line.

***Remark* 3.7.** All undercrossings at the u_o-line must have associated overcrossings at the o-line.

If there is an undercrossing at the u_o-line that does not have it's associated overcrossing on the o-line, then the associated overcrossing is above the o-line which is false by definition or it is below the o-line but then $|o_j - u_j| \leq 2$ which is a contradiction.

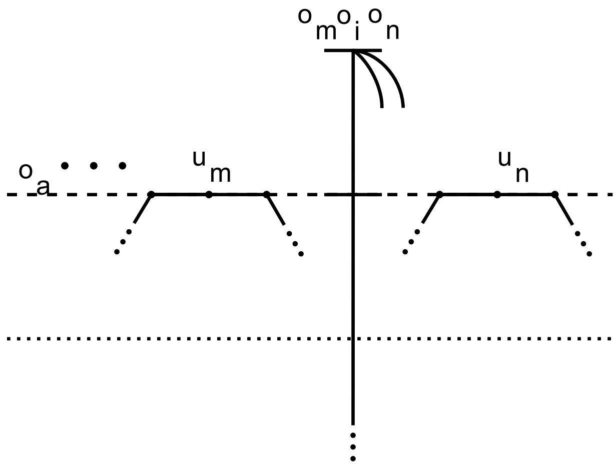
***Remark* 3.8.** L is an almost alternating conformation of a link and $e(\sigma_0) = n$ when $u_i \notin \sigma_0$ and all undercrossing on the u_o line have an edge value of 1. If $\sigma = \{\sigma_o, u_i\}$ then $e(\sigma) = n - 1$.

By definition u_i is connected to two undercrossings that are at or below the u_o-line. Thus, if $u_i \in \sigma$, then $e(\sigma) = e(\sigma_0) + e(u_i)$. If $e(u_i) = -1$ then $e(\sigma) = n - 1$. If $e(u_i) = 0$ then two undercrossings on the u_o line are reduced from an edge value of 1 to an edge value of $\frac{1}{2}$ and thus $e(\sigma) = n - 1$.

Definition 3.9. The **bumped image**,$b(L)$, of L is the configuration of the taut image, seen in Figure 3.10, such that all overcrossings local to o_i are at the same height, above the o-line, and the associated u_m and u_n are at the o-line. The **bumped structure** is the 3 overcrossings and 2 undercrossings elevated from the o-line and u_o-line.

For the bumped images we allow the heights o_i, o_n, and o_m to be excluded from the o-line. The crossings u_m and u_n are also excluded from the u_o line. This structure stays on top of the o-line and is shifted as a part of σ.

Figure 3.10.



The bumped structure of L.

Lemma 3.11. *If L is an almost-alternating conformation, $l(t(L)) \geq l(b(L))$.*

*Proof.*We will show that we can start with $t(L)$ and reduce it down to $b(L)$ without lengthening the rope and thus proving the lemma.

Let us begin with the taut image of L.

Since the image is alternating everywhere except locally around o_i and u_i we can apply the process established in Lemma 2.17 and reduce the o-line down using the σ shifting set to the height at which o_i is at the same height as, without loss of generality, o_m. At this point $e(o_i) = -\frac{1}{2}$, $e(o_m) = -\frac{1}{2}$, but $e(u_i) \leq 0$ and $e(u_m) \leq 1$. Combining this with remarks 3.7 and 3.8 we see that $e(\sigma) \leq 0$ and we can shift σ down.

We can continue to shift σ down until o_i, o_n, and o_m are at the o-line. Now considering the edge value of σ, we see that $e(\{o_m, o_i, o_n\}) = -1$ and $e(u_m) \leq 1$ and $e(u_n) \leq 1$. If there are k overcrossings on the o-line and $k-1$ undercrossings on the u_o-line all with edge value of 1 then $e(\sigma) = 1$ and we cannot shift down. However we consider the edge value of all the elements of the bumped structure, $e(\{o_i, o_m, o_n, u_n, u_m\}) = 1$ thus we can shift those crossings up and we achieve the bumped image of L while only reducing ropelength. $\square$

Now that we have this new configuration, we need to clarify the conditions in which we cannot shift it down.

Lemma 3.12. *Given $b(L)$ with k overcrossings on the o-line. $e(\sigma) > 0$ if and only if there are k undercrossings on the u_o-line and there are 2 overcrossings connected to σ between the o-line and the u_o-line.*

*Proof.*First note that if there are k undercrossings on the u_o-line all with edge value of 1 then the edge value of the bumped structure $e\{o_i, o_m, o_n, u_n, u_m\} = -1$, the edge value of all the overcrossings on the o-line $e(o_k \in o\text{-line}) = k+2$ and $e(\sigma) = 1$. In order to conform to a link the two extra edges extending up from the u_o-line must connect with two overcrossings above the u_o-line but below the o-line. This situation is seen in Figure 3.13 We will show that this is the only condition that $e(\sigma) > 0$.

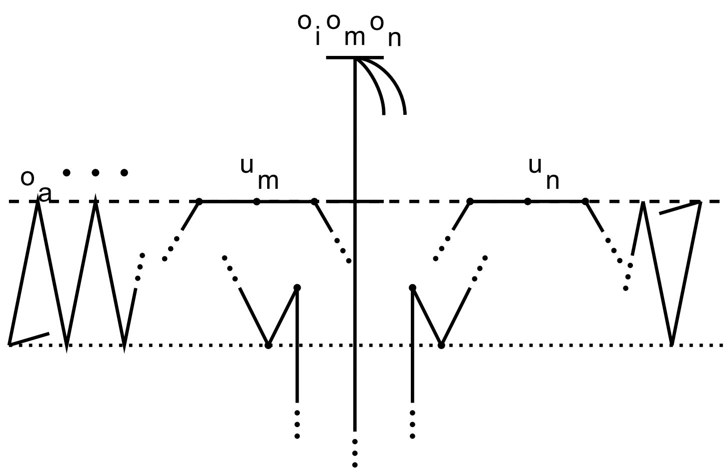
Suppose that there are more than k crossings on the u_o-line. Since we cannot have any undercrossings on the u_o-line that do not have their associated overcrossings on the o-line by Remark 3.7, the only crossing that can increase the number of crossings on the u_o-line is u_i, which by Remark 3.8 implies that $e(\sigma) = 0$. If there are less than k crossings on the u_o-line then clearly $e(sigma) \leq 0$. Thus there must be k undercrossings on the u_o-line in order for $e(sigma) > 0$.

Suppose there are less than 2 overcrossings connected to σ between the o-line and the u_o-line. Either both edges extending up from u_o-line connect to the one overcrossing, which implies a disjoint height image, or one edge connects with an overcrossing on the o-line which implies there are $k+1$ crossings on the o-line. Both cases are contradictions.

Suppose there are more than 2 overcrossings connected to σ between the o-line and the u_o-line. This implies there are more undercrossings on the u_o-line than overcrossings on the o-line which contradicts Remark 3.7.

Therefore, the only conditions in which $e(\sigma) > 0$ is when there are k undercrossings on the u_o-line all with edge value of 1 and there are exactly 2 overcrossings connected to σ between the o-line and the u_o-line. Furthermore this shows that $e(\sigma)$ cannot be greater than 1. □

Figure 3.13.



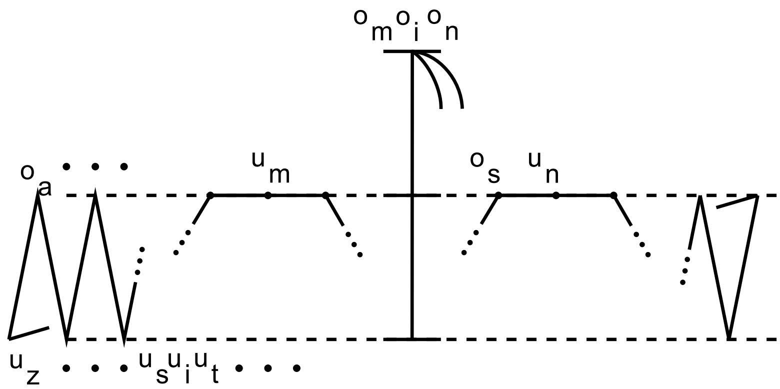
k undercrossings all with edge value 1 and two crossings between the o-line and u_o-line.

Now that we have the bumped image and the conditions in which the bumped image cannot shift we may now proceed to find the linear lower bound on ropelength for almost-alternating conformations of links. Let $\tilde{r}$ be the 3-reduced image of L and let $\bar{4}$ be the 4-reduced image of L.

Theorem 3.14. *L is an almost alternating conformation of a link and without loss of generality u_n is connected by an edge to o_s. $l(t(L) \geq l(\tilde{r}(L))$.*

*Proof.*By Lemma 3.11 we can reduce $t(L)$ to $b(L)$ where the bumped structure is on the o-line. Since $e(\sigma) \leq 1$ we cannot shift σ down but we can shift the u-line compliment, σ_u, up. Recall that σ_u is the shifting set that contains all crossings on the u-line and all crossings 2 units above it on the o^u-line. Since u_n is connected to o_s then o_s is on the o-line. When all undercrossings local to u_i are on the u-line and there are k undercrossings on the u-line then there must be $k-2$ crossings on the o^u-line unless the $h(o^u\text{-line}) = h(u\text{-line})$. Thus by Lemma 3.12 we can always shift σ_u up until we achieve $\tilde{r}(L)$ as seen in Figure 3.15. Since we achieved this image strictly maintaining or shortening the rope then $l(t(L) \geq l(\tilde{r}(L))$. □

Figure 3.15.

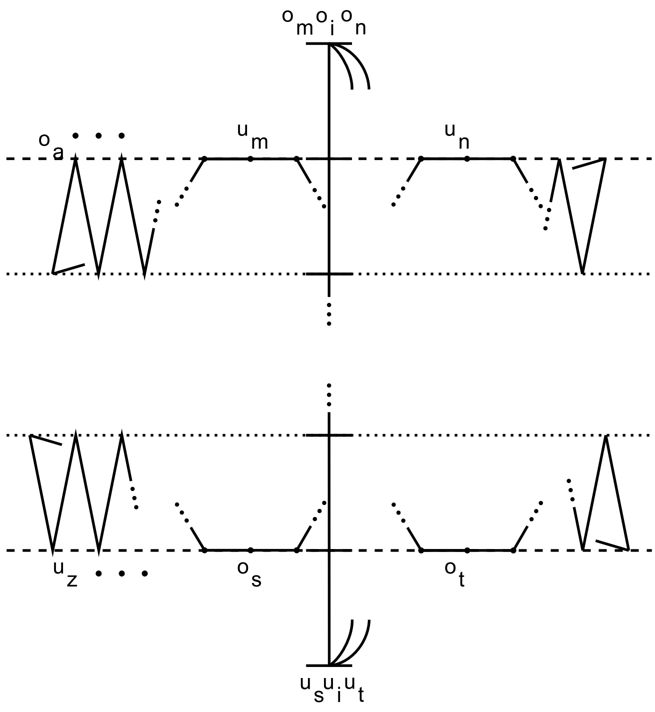


The 3-reduced image of L, $\tilde{r}(L)$.

Theorem 3.16. *L is an almost-alternating conformation of a link and all crossings associated with the almost-alternating pattern are not connected to each other. $l(t(L) \geq l(\bar{r}(L))$.*

*Proof.*We can begin by reducing $t(L)$ to $b(L)$ with bumped structures on the o-line and the u-line by Lemma 3.11 as seen in Figure 3.17. Since $e(\sigma) \leq 1$ and $e(\sigma_u) \geq -1$, then we cannot shift either closer to each other so we must consider a different shifting set.

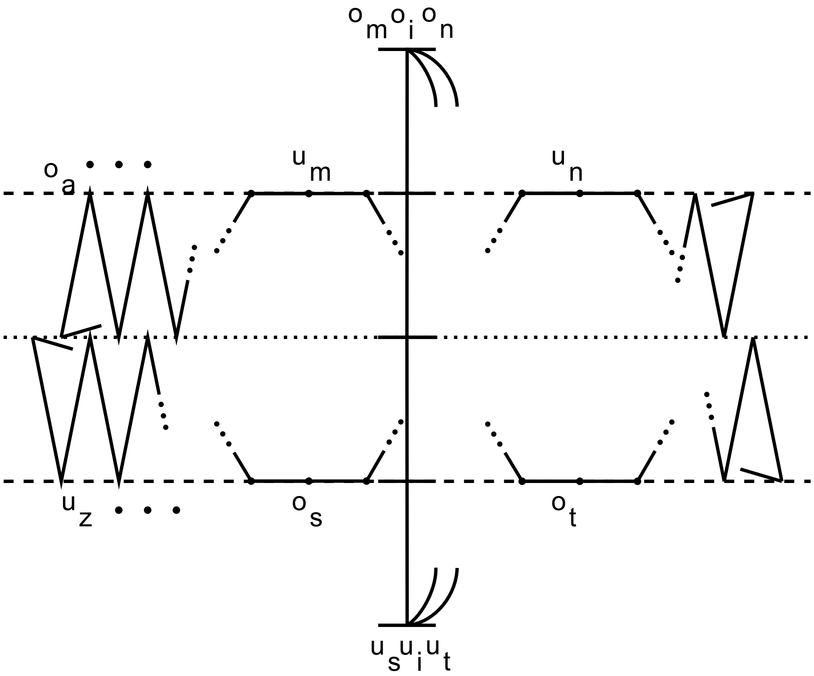
Figure 3.17.



Bumped structures on the u-line and o-line.

Let ω be the shifting set that contains all elements of σ as well as the two crossings that connect to σ between the o-line and u_o-line and the associated undercrossings that remain a distance ≥ 2 units below the two overcrossings. If $e(\omega) > 0$ then we can expand ω to include the undercrossings connected to the two overcrossings and continue to expand the shifting set. Note that if $e(\omega) = 0$ and $u_i \notin \omega$ then we have a disjoint height image which is a contradiction. Since there must be two edges between σ and σ_u then $e(\omega) = -1$ as long as $u_i \notin \omega$. Thus we can shift ω down until $h(u_o\text{-line}) = h(o^u\text{-line})$ as seen in Figure 3.18.

Figure 3.18.

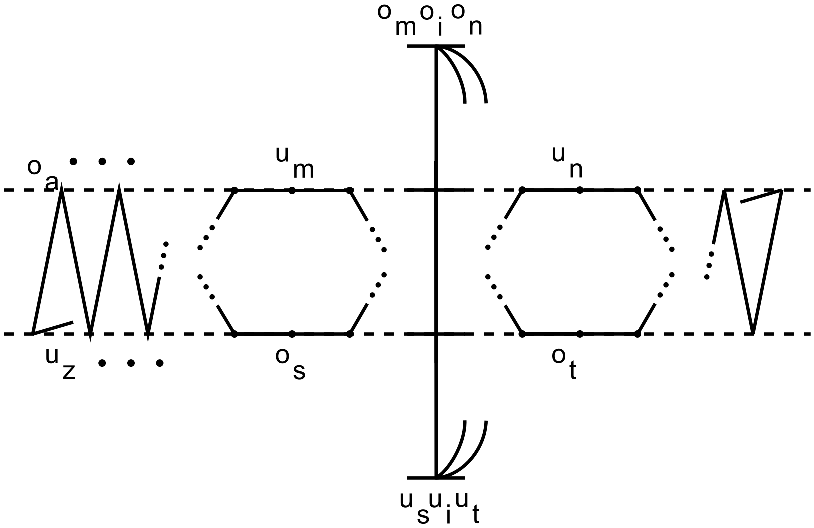


$h(u_o\text{-line}) = h(o^u\text{-line})$.

In order to sweep up any crossings that may be at a height between the o-line and u-line that are not on the u_o-line we can do the following two operations. Let τ be the shifting set that contains all overcrossings on the o-line that are not connected to the bumped structure and all associated undercrossings two units below. Since all overcrossings have an edge value of -1 we can $e(\tau) \leq 0$ and we can shift τ down so that all overcrossings are at the u_o line. Since all overcrossings have an edge value of -1 we can $e(\tau) \leq 0$ Similarly we can do the same to the u-line now and shift all undercrossings not a part of the bumped structure on the u-line up. This forces all crossings that are associated to any crossings not a part of any bumped structure to either the o-line or the u_o-line.

Now by Lemma 3.12 if we cannot shift σ_u up there are 4 overcrossings on the u_o-line with crossing number of -1 when shifting up. Since the crossings need to be consistent with a link the edges must connect with the crossings below which are only the crossings that are a part of the bumped structure and two crossings that were between but now must be on the u_o-line. The same conditions cannot exist for σ because the 2 crossings required between the u_o-line and the o-line must also be on the u_o line and connect down must connect down and thus there must exist crossings on the u-line that are not connected to the 4 overcrossings not mentioned earlier so there are at least $k+1$ crossings on the u-line and only k overcrossings on the u_o-line which contradicts the conditions needed for the bumped image to remain fixed. Therefore, either σ or σ_u can shift down/up and we thus can move them closer until we achieve the 4-reduced image of L as seen Figure 3.19. Since the process strictly decreased or maintained ropelength we have thus proved the result. $\square$

Figure 3.19.



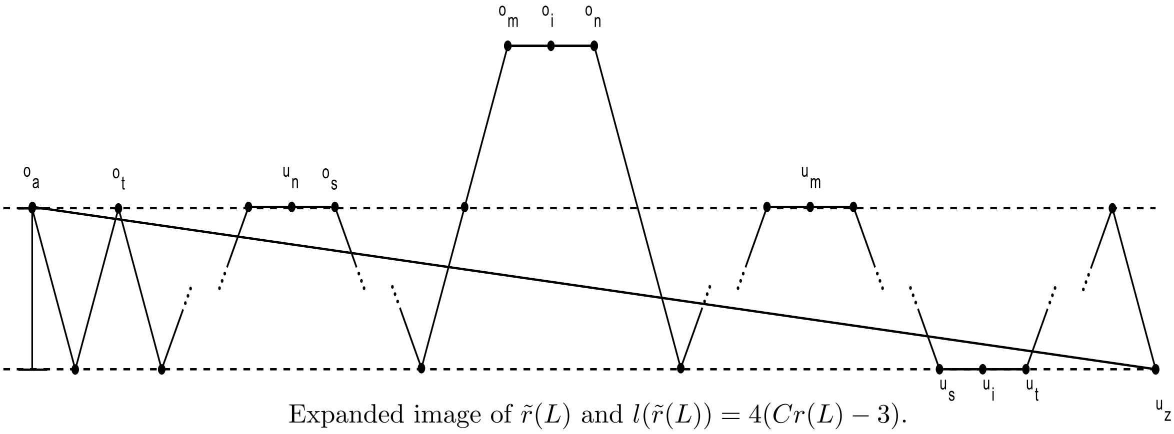
The 4-reduced image of L $\bar{r}(L)$.

Now that we have two images of L that have a ropelength less than or equal to the taut image we need to determine which one is shorter so that we use that image when determining the lower bound on ropelength.

Lemma 3.20. *L is an almost alternating conformation of a link, $l(\tilde{r}(L)) < l(\bar{r}(L))$.*

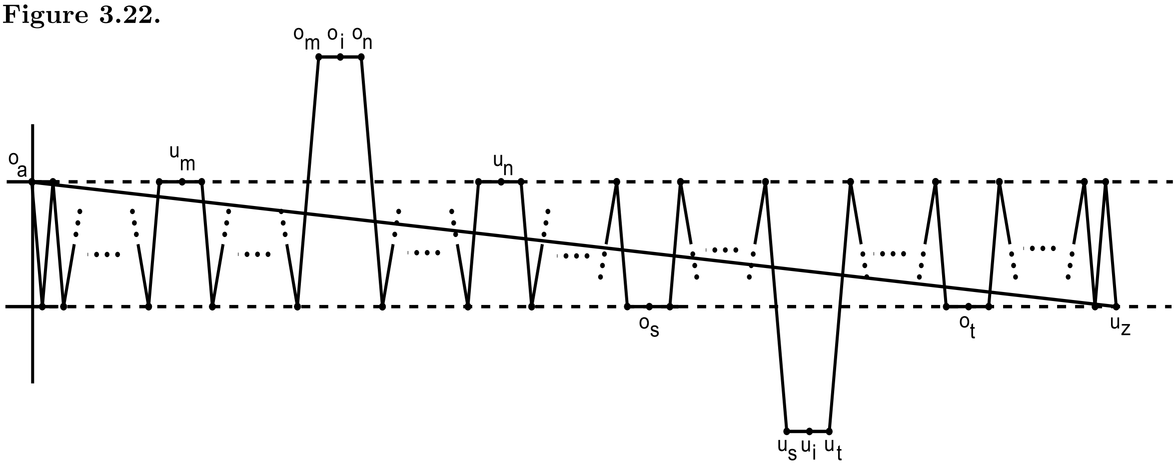
*Proof.*we when we expand out horizontally $\tilde{r}(L)$ we get the image seen in Figure 3.21 and when we expand out $\bar{r}(L)$ we get the image seen in Figure 3.22. Since each edge is a straight line along the z-axis that is equal to 2 units. we can see that $l(\tilde{r}(L)) = 4(Cr(L) - 3)$ and $l(\bar{r}(L)) = 4(Cr(L) - 4)$. Thus $l(\tilde{r}(L)) < l(\bar{r}(L))$. $\square$

Figure 3.21.



Expanded image of $\tilde{r}(L)$ and $l(\tilde{r}(L)) = 4(Cr(L) - 3)$.

Figure 3.22.



Expanded image of $\bar{r}(L)$ and $l(\bar{r}(L)) = 4(Cr(L) - 4)$.

Theorem 3.23. *L is an almost-alternating conformation of a link. $Rop(L) \geq 4(Cr(L) - 4)$.*

*Proof.*By Remarks 2.5 2.7 and Theorem 3.16 and Lemma 3.20 we see that $Rop(L) \geq l(h(L)) \geq l(t(L)) \geq l(\bar{r}(L)) = 4(Cr(L) - 4)$.

□

This concludes the almost-alternating case and now we move on to Paired Links.

4 Paired Links

4.1 Algorithm for Converting Knot to Bipartite Graph

This algorithm starts with a minimal crossing projection of a link and produces a bipartite graph that gives us the needed vocabulary to continue in the same way as Shahla's proof.

Definition 4.2. A **Maximal Overpass** is a section of an arc that cannot be extended any further in order to incorporate additional overcrossings. simmilarly a **Maximal Underpass** is a section of an arc that cannot be extended any further in order to incorporate additional undercrossings.

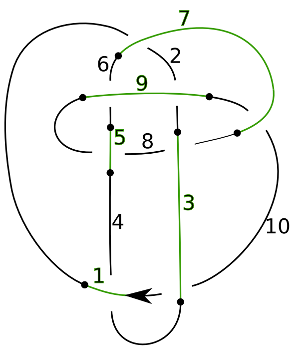
Note: when traversing the knot in a set direction there is an alternating pattern of maximal overpassing to maximal underpasses

Definition 4.3. A **Graph** is a collection of **edges** and **vertices** where the vertices are points in space and edges are lines that connect one vertex to another. A **Bipartite Graph** denoted $G(X, Y, E)$ is a special type of graph that has two sets of vertices, X and Y, and the only edges E that exist connect a vertex from X and a vertex from Y. No vertex from X can have an edge to another vertex in X and similarly with vertices in Y.

Start with a minimal crossing diagram, D of a prime knot L.

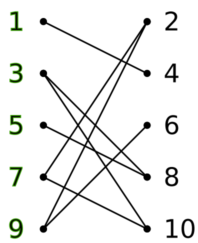
Choose a starting point and direction to traverse D. Label the maximal overpasses and maximal underpasses sequentially as you traverse the knot. See Figure 4.4 for an example where the maximal underpasses are highlighted in green and the maximal underpasses are black.

Figure 4.4. Labeled Diagram of 8_{20}.



Next, create a bipartite graph $G_D(X, Y, E)$ such that $X = \{x \in X | x \text{ is a maximal overpass of } D\}$ and $Y = \{y \in Y | y \text{ is a maximal underpass of } D\}$ as seen below. Edges on the bipartite graph correspond to all the possible pairings of an overpass with an underpass; these edges represent the crossings of the knot.

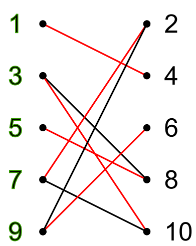
Figure 4.5. Bipartite graph G_D.



A **Matching** is a collection of independent edges from our graph. A vertex may only be matched with one other vertex. The **Maximal-Matching** representation of a graph is the representation of the graph in which the largest number of vertices are matched. A graph is called **X-Matching** if every vertex of the graph is matched.

Definition 4.6. A diagram is **Paired** if the maximal-matching of bipartite graph G_D is also X-matching. A Link is **Paired** if it admits a paired diagram. Similarly, A diagram is **Non-Paired** if the maximal-matching of bipartite graph, G_D is not X-matching and A Link is **Non-Paired** if it does not admit a Paired diagram.

Figure 4.7. The Maximal-Matching of $G_{8_{20}}$.

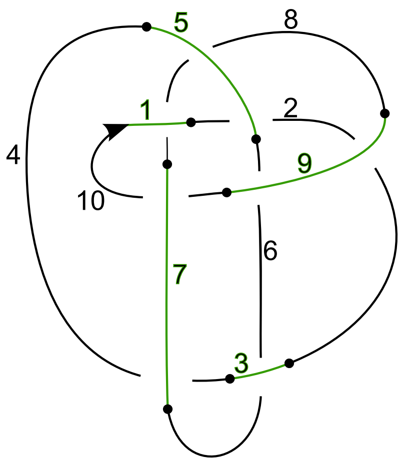


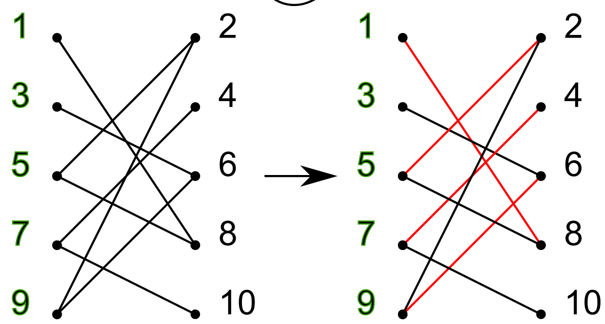
Definition 4.8. The **Matching Number** of a knot, $m(L)$, is the number of matchings on its corresponding maximal-matching bipartite graph.

In Figure 4.7, we can see that $m(L) = 5$. Also we can see that every vertex is matched so we know that 8_{20} is a paired knot.

As an example of a non-paired knot consider 8_{19}.

Example 4.9. A Non-Paired Knot.





Notice that the vertices 3 and 10 are not matched and any reconfigurations of matching will result in two vertices that are not matched. The the maximal-matching of 8_{19} is not X-matching and thus non-paired.

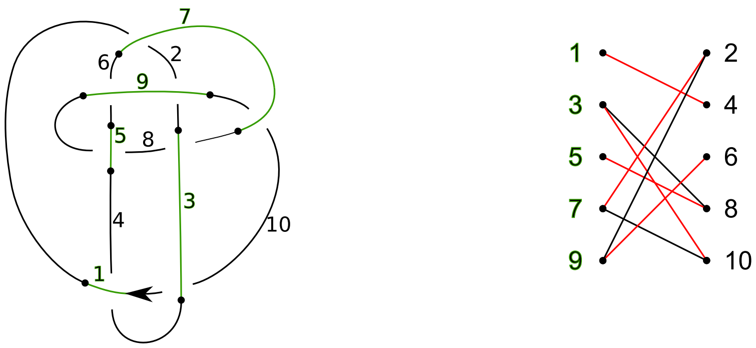
4.10 Ropelength For Paired Links

The following proof opperates identically to our earlier proof for alternating Links and is a recreation for this new class of knots. Instead of crossing number we will find a linear lowerbound in terms of the bridge number.

The **bridge number** of a diagram is the number of maximal-overpasses on the diagram.

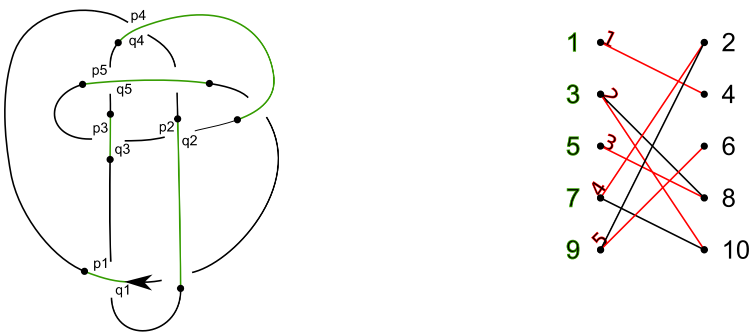
Let L be a paired conformation with n maximal arcs labeled 1 through n and Let G_L be the maximal matching bipartite graph of L with a matching number of k. Since it has been established that all alternating knots have a linear lowerbound we shall consider a non-alternating paired knot for our example, the first of which being 8_{20}.

Example 4.11. Paired Diagram and Associated Maximal-Matching Graph.



As we traverse the link in a fixed direction, we encounter a cyclic ordered sequence of maximal underpasses and maximal underpasses that is complete when we return to our starting position. The crossings represented by the maximal matching diagram are the critical points of the graph where an overpass is connected in a way to an underpass. Let the i^{th} matching correspond to the point where the p_i overpass crosses the q_i underpass. and let o_i denote the height of p_i and u_i denote the height of q_i.

Figure 4.12. p_i and q_i, $1 \leq i \leq 5$.



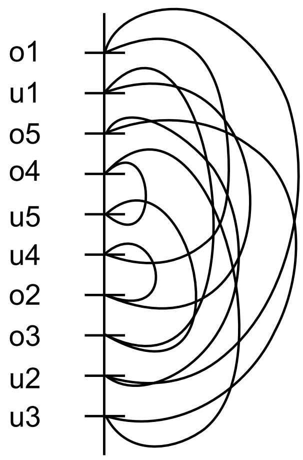
Definition 4.13. Let L be a Paired Diagram. The height function, $h : L \to R$, is the procection $h(x, y, z) = z$ which takes every point on L to its z-coordinate. The **image** of L under the projection, denoted by $h(L)$, is a path that goes up and down alont the z-axis.

Note that under the projection $h(p_i) = o_i$ $h(q_i) = u_i$ so that the values of o_i and u_i partition the z-axis into subintervals.

continuing with our example, we now represent the p_i and q_i by there respective heights o_i and u_i.The series of the path runs as $o_1u_4o_2u_1o_3u_5o_4u_3o_5u_2$.

From this point we have the initial conditions for Shahlas paper prepared and her result follows, however, we will run through the proof step by step with 8_20 and paired links in regard to matching number as opposed to the trefoil and alternating links in regard to crossing number.

Figure 4.14. $h(L)$with o_i and u_i, $1 \leq i \leq 5$.



Observation 1: Note that in alternating conformations the ordering of the hights may be arbitrary as long as $o_i > u_i$. However in non alternating conformations we must take into acount the other crossings of the diagram that are represented by the edges not in the maximal matching set. Thus we may still create this hight sceme as Shahla did but we must be careful not to place an overcrossing under an undercrossing derived from a non critical point.

Observation 2: Since the radius of each strand is 1, there must be at least a distance of 2 seperating o_i and u_i. Therefore $(o_i - u_i) \geq 2$.

***Remark* 4.15.** If L is a paired conformation, the ropelenth of L is greater than or equal to the length of $h(L)$.

To justify this, notice that the projection h does not increase the distance between critical poins, the arclength of L is at least the length of $h(L)$.

The Goal is to use the height funtion h to obtain a lower bound on the ropelength of L. It is difficult to measure the length directly so we will simplify it by replacing the arclenghts of $h(L)$ with straight line segments that connect the overpasses to the underpasses. We will call this new image of L the **taught** image of L, denoted by $t(L)$. The edges of $t(L)$ connect successive maximal over- and under-passes. each o_i and u_i has exactly two edges incident with it in $t(L)$, and each edge has an endpoint of one overpass and one underpass.

Figure 4.16. Taught image $t(L)$.

Note that the straight line segments have been slightly bent for viewing purposes.

***Remark* 4.17.** Let L be a paired conformation. The ropelength of L is at least the length of $t(L)$

Notice that both of the paths $h(L)$ and $t(L)$ connect the heights o_i and u_j in the same cyclic order. Since the arc in $t(L)$ connecting consecutive o_i and u_j is a straight line, its length is at most the length of the corresponding piece in $h(L)$. Summing over all consecutive points, the sum of the straight line segments in $t(L)$ is at most the sum of the arcs from $h(L)$. Using Remark 4.15 we see the result now follows.

In our example in Figure 4.16 $Rop(L) \geq$ length of $h(L) \geq$ length of $t(L) = |o_1 - u_4| + |u_4 - o_2| + |o_2 - u_1| + |u_1 - o_3| + |o_3 - u_5| + |u_5 - o_4| + |o_4 - u_3| + |u_3 - o_5| + |o_5 - u_2| + |u_2 - o_1|$ by Remark 4.17.

Lemma 4.18. *Let L be a paired conformation. The ropelength of L is at least the length of the 2-reduced image of L.*

*Proof.*We can use the shifting set,σ, that includes all overpasses on the o-line, which is the height of the highest overpass, and all associated underpasses two units below the o-line. It clearly follows that the same reasoning that applied to the alternating conformations applies to paired conformation because each overpass is connected to two underpasses below and each pari of associated passes must be 2 units apart. Therefore, we can achieve the 2-reduced image of L by strictly reducing or maintaining ropelength and thus $l(t(L)) \geq l(r(L))$. □

Now we have everything we need to prove a linear lower bound on ropelength for paired conformations in terms of bridge number.

Theorem 4.19. *Let L be a paired conformation. $Rop(L) \geq 4br(L)$.*

*Proof.*By by Remarks 4.15 and 4.17 and Lemma 4.18 we see that $Rop(L) \geq l(h(L)) \geq l(t(L)) \geq l(r(L))$. when we consider $l(r(L))$ we notice that each overpass is connected to two underpasses below and each edge is at least two units long. Thus $l(r(L)) \geq 4br(L)$. Therefore, $Rop(L) \geq 4br(L)$. □

5 Conclusion

This is a recap of all the results we found. In the first section we found that the lower bound on ropelength for alternating conformations is $4Cr(L)$, which is $OCr(L)$. In the second section we found that the lower bound on ropelength for almost-alternating conformations is $4(Cr(L) - 4)$ which is $OCr(L)$. In the third section we found that the lower bound on ropelength for paired conformations is $4(br(L))$ which is $Obr(L)$.

6 Acknowledgements

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7 References

[1] Adams, Colin. "'The Knot Book"'. W.H. Freeman and Company, New York. 1994

[2]Sadjadi, Shahla. "'On Ropelength of Alternating Links"'. California State Universtiy, San Bernardino. August 23, 2007.

[3]Tucker, Alan. "'Applied Combinatorics, Third Edition"'. John Wiley and Sons, Inc. New York. 1995