The decomposability of full and weak model spaces

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Abstract

This paper involves looking at when model spaces are decomposable. Building off of some previous results, we show when a full model space is indecomposable. Given a vector space, metric, and an algebraic curvature tensor, with stipulations on how (V, R) decomposes, we can show that the full model space $(V, \langle\cdot,\cdot\rangle, R)$ is indecomposable. The next part of the paper discusses when $(V, R_\phi \pm R_\psi)$ is decomposable. One way to decompose a model space is to consider $\ker(R_\phi \pm R_\psi)$ (This is the case in Theorem 2.1). The $\dim(\ker(R_\phi \pm R_\psi))$ is considered in this case in order to eventually know how the weak model space can decompose.

Keywords: Indecomposability, decomposability, model space

1 Introduction

I started this project by looking at a full model space, $(V, \langle\cdot,\cdot\rangle, R)$, and worked with indecomposability of that model space. Then my project took a different route; I began to look at a weak model space, $(V, R_\phi \pm R_\psi)$, mostly with a positive definite ϕ. This is where my more interesting results were found dealing with the dimension of $ker(R_\phi \pm R_\psi)$. The main point of the project was to use $ker(R_\phi \pm R_\psi)$ to come up with a decomposition of V.

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Knowing this next definition will help understand what a model space consists of.

Definition 1.1 *Let V be a vector space and $R : V \times V \times V \times V \to \mathbb{R}$ satisfying:*

1. *R is linear in each slot*
2. *$R(x, y, z, w) = -R(y, x, z, w)$*
3. *$R(x, y, z, w) = R(z, w, x, y)$*
4. *(Bianchi Identity) $R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$*

*Then R is an **algebraic curvature tensor**.*

Definition 1.2 *If ϕ is a symmetric, bilinear form which is positive definite then a basis $\{e_1, e_2, \ldots, e_k\}$ is called **orthonormal** if $\phi(e_i, e_i) = 1$ and $\phi(e_i, e_j) = 0$ (for $i \neq j$).*

Definition 1.3 *If ϕ is a symmetric bilinear form, then ϕ is **nondegenerate** if for all $v \in V$, $v \neq 0$, there exists $w \in V$ such that, $\phi(v, w) \neq 0$.*

If ϕ is positive definite then this implies that ϕ is nondegenerate, although the converse is not true. In the "Future Work" section, we discuss open questions when considering ϕ is nondegenerate.

Definition 1.4 *We can define an algebraic curvature tensor, R, on some symmetric bilinear form, ϕ by:*
$R_\phi(x, y, z, w) = \phi(x, w) \cdot \phi(y, z) - \phi(x, z) \cdot \phi(y, w)$

We will use the next definition in our theorem to show when a full model space is indecomposable.

Definition 1.5 *Let V be a vector space and let $\langle \cdot, \cdot \rangle$ be a metric on V. Let $W \subseteq V$ and $x, y \in W$. W is **totally isotropic** if $\langle x, y \rangle = 0, \forall\ x, y \in W$.*

Definition 1.6 *$(V, \langle \cdot, \cdot \rangle, R)$ is a **full model space**, which means that there is a vector space, V, with a metric, $\langle \cdot, \cdot \rangle$, and an algebraic curvature tensor, R.*
*(V, R) is a **weak model space**, which means that there is a vector space, V, with an algebraic curvature tensor, R.*

Knowing what a model space is, we can define what it means for a model space to be decomposable.

Definition 1.7 *Let V be a vector space, with R, an algebraic curvature tensor on V, along with a nondegenerate metric, $\langle \cdot, \cdot \rangle$. If $V = V_1 \oplus V_2$, $R = R_1 \oplus R_2$, and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_2$ where $R_i|_{V_j} = 0$ for $i \neq j$, then $(V, \langle \cdot, \cdot \rangle, R)$ is said to be **decomposable**. If $(V, \langle \cdot, \cdot \rangle, R)$ is not decomposable, then $(V, \langle \cdot, \cdot \rangle, R)$ is said to be **indecomposable** (without the metric, one would define decomposability and indecomposability for a weak model space in the same way).*

2 Indecomposability

In order to show that $(V, \langle \cdot, \cdot \rangle, R)$ is indecomposable, the definitions of key subspaces are listed below.

$$\ker(R) = \{v \in V | R(v, \eta_1, \eta_2, \eta_3) = 0, \; \forall \; \eta_i \in V\}$$
$$V/\ker(R) = \{v + \ker(R) | v \in V\}$$

$\ker(R)$ is a linear subspace of V.

If $v \in \ker(R)$, $R(v, x_2, x_3, x_4) = 0$ for all $x_i \in V$. Also, v can be located in any of the four slots because of the symmetries of an algebraic curvature tensor.

Theorem 2.1 ([?]) *Let $\mathcal{M} := (V, \langle \cdot, \cdot \rangle, R)$. If:*

1. *$\bar{\mathcal{M}} = (V/ker(R), \bar{R})$ is indecomposable, and*
2. *$ker(R)$ is totally isotropic,*

*then $\mathcal{M}$ is indecomposable. [**?**]*

Lemma 2.1 *If V is a vector space with R, an algebraic curvature tensor, on V, let $V/\ker(R) = \bar{V}$ and define, $\pi^* \bar{R} = R$, then $\bar{R}$ is well-defined.*

Proof . In order to show that $\bar{R}$ is well-defined we must show that if $\pi v = \pi w$, for $v, w \in V$, then $R(v, x_1, x_2, x_3) = R(w, x_1, x_2, x_3)$ for all $x_i \in V$. Let $\pi v = \pi w$. Since $\pi := V \to \bar{V}$, then $v - w \in \ker(R)$. This implies, $R(v - w, x_2, x_3, x_4) = 0$, for all $x_i \in V$. Since R is an algebraic curvature tensor,
$R(v - w, x_2, x_3, x_4) = 0 \Rightarrow R(v, x_2, x_3, x_4) - R(w, x_2, x_3, x_4) = 0 \Rightarrow$
$R(v, x_2, x_3, x_4) = R(w, x_2, x_3, x_4)$. Therefore, $\bar{R}$ is well-defined.

Lemma 2.2 *If V is a vector space with R, an algebraic curvature tensor, on V. If $V = V_1 \oplus V_2$ and $R = R_1 \oplus R_2$, let $V/V_1 = \bar{V_2}$ and define, $\pi^* \bar{R}_2 = R$, then $\bar{R}_2$ is well-defined.*

Proof Since $\pi^* \bar{R}_2 = R$, by definition, $\bar{R}_2(\pi x_1, \pi x_2, \pi x_3, \pi x_4) = R(x_1, x_2, x_3, x_4)$.

In order to show that $\bar{R}_2$ is well-defined we must show that if $\pi v_1 = \pi w_1$ then $R(v_1, x_1, x_2, x_3) = R(w_1, x_1, x_2, x_3)$ for all $x_i \in V$

Let $\pi v_1 = \pi w_1$. Since $\pi := V \to \bar{V_2}$, then $\pi(v) = v + V_1$ for all $v \in V$. If $v + V_1 = 0$, then this implies that $v \in V_1$. Having $\pi v_1 = \pi w_1$, implies that $v_1 - w_1 \in V_1 = \ker(\pi)$. So therefore, $\bar{R}_2(\pi v_1 - \pi w_1, x_2, x_3, x_4) =$
$R(v_1 - w_1, x_2, x_3, x_4) = 0$ for all $x_i \in V$. Since R is an algebraic curvature tensor,

$R(v_1 - w_1, x_2, x_3, x_4) = R(v_1, x_2, x_3, x_4) - R(w_1, x_2, x_3, x_4) = 0$ Which is the same as, $R(v_1, x_2, x_3, x_4) = R(w_1, x_2, x_3, x_4)$. Therefore, $\bar{R}_2$ is well-defined.

Theorem 2.3 *Let $(V, R) \cong (W, S)$. (V, R) is (in)decomposable if and only if (W, S) is (in)decomposable.*

Proof Let ϕ be a vector space isomorphism defined by: $\phi : V \to W$ and let (V, R) be decomposable by, $(V, R) = (V_1, R_1) \oplus (V_2, R_2)$. Since ϕ is an isomorphism between V and W, then $V \cong W$, and $\phi^* S = R$.

Set $W_i = \phi(V_i)$. It needs to be shown that $W = W_1 \oplus W_2$. Let $w \in W$, then there exists $v \in V$ such that $\phi(v) = w$, since ϕ is onto. $v = v_1 + v_2$ (where $v_i \in V_i$), since $v \in V$ and V is decomposable. $w = \phi(v) = \phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$, where $\phi(v_i) \in W_i$. Then $w = \phi(v_1) + \phi(v_2) \in W_1 + W_2$, for all $w \in W$. It needs to be shown that $W_1 \cap W_2 = \{0\}$ in order for W to decompose as stated. Assume $0 \neq w \in W_1 \cap W_2$. Then there exists $v_1 \in V_1$, $v_2 \in V_2$, such that $\phi(v_1) = \phi(v_2) = w$. Since ϕ is one-to-one, $v_1 = v_2 = 0 = w$, but $w \neq 0$, which means that $W_1 \cap W_2 = \{0\}$. Therefore, W is decomposable as $W = W_1 \oplus W_2$.

Now it must be shown that there exists a decomposition, $S = B_1 \oplus B_2$, for some algebraic curvature tensors, B_i on W_i. Let $\phi(v_i) = w_i$. Then, $S(w_1, w_2, w_3, w_4) = S(\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)) = R(v_1, v_2, v_3, v_4) = 0$ Therefore, S decomposes.

$\therefore$ if $(V, R) \cong (W, S)$, (V, R) decomposes if and only if (W, S) decomposes.

By weakening the requirements in theorem 2.1 and following the proof, a stronger theorem was created.

Theorem 2.4 *Let V be a vector space and let R be an algebraic curvature tensor on V along with $\langle\cdot,\cdot\rangle$, a metric on V. If $V = V_1 \oplus V_2$ and $R = R_1 \oplus R_2$, such that:*

1. *(V_2, R_2) is indecomposable, and*
2. *V_1 is totally isotropic,*

then $(V, \langle\cdot,\cdot\rangle, R)$ is indecomposable.

Proof Suppose that $V = W_1 \oplus W_2$, where each W_i is a nontrivial subspace of V. Let $\pi : V \to V/V_1$, and $\bar{V_2} = V/V_1$. We know that $\pi(v) = v + V_1 \ \forall\ v \in V$ and $ker(\pi) = V_1$.

Define $\bar{R_2}$ on $\bar{V_2}$ by $\pi^*\bar{R_2} = R$. By Lemma 2.2, we know that $\bar{R_2}$ is well defined.

By Theorem 2.3, if it can be shown that $(V_2, R_2) \cong (\bar{V_2}, \bar{R_2})$, then $(\bar{V_2}, \bar{R_2})$ is indecomposable and a contradiction will arise.

Define $\phi : V_2 \to \bar{V_2}$ by $\phi(v) = v + V_1$ $\ker(\phi) = \{v \in V_2 | v \in V_1\} = \{0\}$

To show ϕ is onto, define: $\rho_2 : V \to V_2$ by $\rho_2(v_1 + v_2) = v_2$ for all $v_i \in V_i$ and $v_1 + v_2 \in V_1 \oplus V_2$.

$\phi(\rho_2(v)) = v + V_1$ for all $v \in V$

To show ϕ is one-to-one, let $v_i \in V_2$ and let $\phi(v_1) = \phi(v_2)$ to show that $v_1 = v_2$. $\phi(v_i) = v_i + V_1$. $v_1 + V_1 = v_2 + V_2 \Rightarrow v_1 - v_2 \in V_1$, and since $v_1, v_2 \in V_2$, $v_1 - v_2 \in V_2$. This means that $v_1 - v_2 \in V_1 \cap V_2$ but $V_1 \cap V_2 = \{0\}$ so $v_1 - v_2 = 0 \Rightarrow v_1 = v_2$. Thus, ϕ is one-to-one.

Therefore ϕ is a vector space isomorphism and thus, $V_2 \cong \bar{V_2}$ Show that $\phi^*\bar{R_2} = R_2$. Let each $v_i \in V_2$, then

$$\begin{aligned}\phi^*\bar{R_2}(v_1, v_2, v_3, v_4) &= \bar{R_2}(\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)) \\ &= \bar{R_2}(v_1 + V_1, v_2 + V_1, v_3 + V_1, v_4 + V_1)\end{aligned}$$

From earlier: $\pi(v) = v + V_1$,

$$\begin{aligned}&= \bar{R_2}(\pi v_1, \pi v_2, \pi v_3, \pi v_4) \\ &= R(v_1, v_2, v_3, v_4) \\ &= R_2(v_1, v_2, v_3, v_4)\end{aligned}$$

Therefore, $(V_2, R_2) \cong (\bar{V_2}, \bar{R_2})$.

Consider $\bar{W_1} = \pi W_1$ and $\bar{W_2} = \pi W_2$ where $V = W_1 \oplus W_2$ and $R = A_1 \oplus A_2$ Let us look at $\bar{W_1} \cap \bar{W_2}$. Suppose $0 \neq \bar{w} \in \bar{W_1} \cap \bar{W_2}$. Then, there exists $w_i \in W_i$ such that, $\bar{w} = \pi w_1 = \pi w_2$, where $0 \neq w_i \in W_i$. This implies that $w_1 - w_2 \in \ker(\pi) = V_1$. Using Theorem 2.3, since $(V_2, R_2) \cong (\bar{V_2}, \bar{R_2})$, then $(\bar{V_2}, \bar{R_2})$ is indecomposable. This implies that $\ker(\bar{R_2}) = 0$.

Since $\pi w_2 \neq 0$ then $w_2 \notin V_1$, which implies there exists $0 \neq \bar{\eta_1}, \bar{\eta_2}, \bar{\eta_3} \in \bar{V_2}$ such that $\bar{R_2}(\bar{w_2}, \bar{\eta_1}, \bar{\eta_2}, \bar{\eta_3}) \neq 0$. We also know that $\bar{R_2}(\bar{w_1} - \bar{w_2}, \bar{\eta_1}, \bar{\eta_2}, \bar{\eta_3}) = 0$ since $w_1 - w_2 \in V_1$.

Therefore, $0 \neq \bar{R}_2(\bar{w_1}, \bar{\eta_1}, \bar{\eta_2}, \bar{\eta_3}) = \bar{R}_2(\bar{w_2}, \bar{\eta_1}, \bar{\eta_2}, \bar{\eta_3})$. $\bar{\eta_i} \in \bar{V_2}$ means $\bar{\eta_i} = \pi\eta_i$ where $\eta_i \in V$ and more specifically $\eta_i \in V_2$.We know, $\bar{R}_2(\bar{w_1}, \bar{\eta_1}, \bar{\eta_2}, \bar{\eta_3}) = R(w_1, \eta_1, \eta_2, \eta_3)$ and $\bar{R}_2(\bar{w_2}, \bar{\eta_1}, \bar{\eta_2}, \bar{\eta_3}) = R(w_2, \eta_1, \eta_2, \eta_3)$

Then we can say, $0 \neq R(w_1, \eta_1, \eta_2, \eta_3) = R(w_2, \eta_1, \eta_2, \eta_3)$ We would like to split up each η_i into its components in relation to W_1 and W_2 since each $\eta_i \in V$. $\eta_i = \eta_i^1 + \eta_i^2$ where $\eta_i^j \in W_j$.

$0 \neq R(w_1, \eta_1^1+\eta_1^2, \eta_2^1+\eta_2^2, \eta_3^1+\eta_3^2) = R(w_1, \eta_1^1, \eta_2^1, \eta_3^1)$, since $R(w_1, \eta_1^2, *, *) = 0$ because $w_1 \in W_1$ and $\eta_1^2 \in W_2$. $0 \neq R(w_1, \eta_1^1, \eta_2^1, \eta_3^1) = R(w_2, \eta_1^1, \eta_2^1, \eta_3^1) = 0$, since $\bar{R}_2$ is well-defined and $w_2 \in W_2$ and $\eta_i^1 \in W_1$. This is a contradiction, which implies that $\bar{W_1} \cap \bar{W_2} = \{0\}$. This means that $W_1 \oplus W_2$ descends to the decomposition for $\bar{V_2} = \bar{W_1} \oplus \bar{W_2}$.

Now we show that there exists $\bar{A}_1, \bar{A}_2$ on $\bar{W_1}, \bar{W_2}$, respectively, such that, $\bar{R}_2 = \bar{A}_1 \oplus \bar{A}_2$ and $\bar{R}_2(\bar{W_1}, \bar{W_2}, x, y) = 0$ (for any $x, y \in V$). Let $\pi^*\bar{A}_i = A_i$ (to show $\bar{A}_i$ is well defined, see Lemma 2.2). To show the first part, $\bar{R}_2(\bar{x}, \bar{y}, \bar{z}, \bar{w}) = R(x, y, z, w) = A_1(x, y, z, w) + A_2(x, y, z, w) = \bar{A}_1(\bar{x}, \bar{y}, \bar{z}, \bar{w}) + \bar{A}_2(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ Therefore, $\bar{R}_2 = \bar{A}_1 + \bar{A}_2$

To show the second part, Let $w_i \in W_i$, $\bar{A}_1(\bar{w_1}, \bar{w_2}, x, y) + \bar{A}_2(\bar{w_1}, \bar{w_2}, x, y) = \bar{R}_2(\bar{w_1}, \bar{w_2}, x, y) = R(w_1, w_2, x, y) = A_1(w_1, w_2, x, y) + A_2(w_1, w_2, x, y) = 0$ Therefore, $\bar{R}_2$ decomposes into $\bar{A}_1 \oplus \bar{A}_2$.

We know that $(\bar{V_2}, \bar{R_2})$ is indecomposable, which means that either $\bar{W_1}$ or $\bar{W_2}$ is trivial. WLOG assume $\bar{W_1}$ is trivial. Let $\bar{w_1} \in \bar{W_1}$. $\pi w_1 = 0 \Rightarrow w_1 \in \ker(\pi) = V_1$, for all $w_i \in W_1$, which means $W_1 \subseteq V_1$. V_1 is totally isotropic, which arises a contradiction since we assumed that $V = W_1 \oplus W_2$ was a nontrivial decomposition of V.

Example 2.5 *Let $\{e_1, e_2, e_3, e_4\}$ be a basis for V, a four dimensional vector space. Let $V_1 = span\{e_1, e_2\}$, $V_2 = span\{e_3, e_4\}$, so that $V = V_1 \oplus V_2$. Let $v_i \in V_i$, for $i = 1, 2$. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate inner product on V, such that $\langle e_1, e_3 \rangle = \langle e_2, e_4 \rangle = 1$. Set $\phi(e_3, e_3) = \phi(e_4, e_4) = 1$. Set $R_\phi = R_2$.*

With this metric, (V_2, R_2) is indecomposable. Also, V_1 is totally isotropic. By our theorem, $(V, \langle \cdot, \cdot \rangle, R)$ is indecomposable. This is also an interesting example because R_1 is not explicitly given, which means that there is a lot of variation on what R_1 can be.

3 $Ker(R_\phi \pm R_\psi)$

A way to decompose a model space, (V, R) is to look at $\ker(R)$. In this section we look at the weak model space, $(V, R_\phi \pm R_\psi)$, where $\phi, \psi \in S^2(V^*)$. We start with a ϕ that is positive definite in order to be able to diagonalize ψ with respect to ϕ easily. We start looking at $\dim(\ker(R_\phi \pm R_\psi))$ in order to determine how $(V, R_\phi \pm R_\psi)$ can decompose using $\ker(R_\phi \pm R_\psi)$.

Theorem 3.1 *Let V be a vector space with $R = R_\phi \pm R_\psi$, an algebraic curvature tensor, on V. If ϕ is positive definite, and $dim(V) = n$, where $n \geq 3$, then $dim(ker(R_\phi \pm R_\psi)) = 0$, 1, or n.*

Proof Let $\{e_1, e_2, \dots, e_n\}$ be a basis for V. Diagonalize ψ with respect to the positive definite form, ϕ, so that the only nonzero entries of ψ on this basis are $\psi(e_i, e_i) = \lambda_i$.

Let $v \in ker(R_\phi \pm R_\psi)$, where $v \neq 0$. Let $v = \sum c_i e_i$, where $c_i \in \mathbb{R}$. Since v is nonzero, there exists l, such that $c_l \neq 0$.

$$R_\phi(v, e_j, e_j, e_l) \pm R_\psi(v, e_j, e_j, e_l) = 0$$

for some $j \neq l$. Therefore,

$$(\phi(v, e_l)\phi(e_j, e_j) - \phi(v, e_j)\phi(e_j, e_l)) \pm (\psi(v, e_l)\psi(e_j, e_j) - \psi(v, e_j)\psi(e_j, e_l)) = 0$$

$$c_l \pm \lambda_j \lambda_l c_l = 0$$

$$c_l(1 \pm \lambda_j \lambda_l) = 0$$

For $i \neq j$, $i \neq l$, since $\dim(V) \geq 3$, there exists i such that i, j are distinct, and so, $R_\phi(v, e_j, e_j, e_l) \pm R_\psi(v, e_j, e_j, e_l) = 0$, and for the same reasoning,

$$c_l(1 \pm \lambda_i \lambda_l) = 0$$

Since $c_l \neq 0$,

$$1 \pm \lambda_j \lambda_l = 0 = 1 \pm \lambda_i \lambda_l$$

$$\lambda_l(\lambda_j - \lambda_i) = 0$$

So either $\lambda_l = 0$ or $\lambda_i = \lambda_j \ \forall \ i \neq l, j \neq l, i \neq j$.
If $\lambda_l = 0$ then we know that $c_l = 0$ (since $c_l(1 \pm \lambda_j \lambda_l) = 0$), but $c_l \neq 0$, which means $\lambda_l \neq 0$. Therefore, $\lambda_i = \lambda_j$, for all $i, j \neq l$

Let $\lambda_i = \lambda$, for all $i \neq l$. Then we know that

$$c_l(1 \pm \lambda \lambda_l) = 0$$

$$1 = \mp \lambda \lambda_l \Rightarrow \lambda \lambda_l \neq 0,$$

$$\mp \lambda_l = \frac{1}{\lambda}$$

We arrange the basis so that this exceptional index $l = 1$. Then $\psi(e_1, e_1) = \mp \frac{1}{\lambda}$ and $\psi(e_i, e_i) = \lambda$ for all $i \neq 1$.

$$\begin{aligned} R_\phi(c_1 e_1, e_j, e_j, e_1) \pm R_\psi(c_1 e_1, e_j, e_j, e_1) &= c_1 \pm c_1 \lambda_1 \lambda \\ &= c_1(1 \pm (\mp \frac{1}{\lambda})\lambda \\ &= c_1(1 \pm (\mp 1)) \\ &= 0 \end{aligned}$$

We know that $c_1 e_1 \in ker(R_\phi \pm R_\psi)$, which means that $dim(ker(R_\phi \pm R_\psi)) \geq 1$
For $i, j \neq l$ and $i \neq j$,

$$\begin{aligned} &R_\phi(e_i, e_j, e_j, e_i) \pm R_\psi(e_i, e_j, e_j, e_i) \\ &\qquad = \phi(e_i, e_i)\phi(e_j, e_j) \pm \psi(e_i, e_i)\psi(e_j, e_j) \\ &\qquad = 1 \pm \lambda^2 \end{aligned}$$

In order for $1 \pm \lambda^2 = 0$, $\lambda = \pm 1$ only in considering $R_\phi - R_\psi$. The next case will deal with what happens when $\lambda = \pm 1$. As long as ψ has the form from above, and $\lambda \neq \pm 1$, then $dim(ker(R_\phi \pm R_\psi)) = 1$.

Let $v = \sum c_i e_i \in \ker(R_\phi \pm R_\psi)$, and there exists $c_l, c_p \neq 0$ $(v \neq 0)$, where $l \neq p$. For $j \neq l$, $j \neq p$,

$$R_\phi(v, e_j, e_j, e_l) \pm R_\psi(v, e_j, e_j, e_l) = 0 = R_\phi(v, e_j, e_j, e_p) \pm R_\psi(v, e_j, e_j, e_p)$$

$$c_l \pm c_l \lambda_l \lambda_j = 0 = c_p \pm c_p \lambda_p \lambda_j$$

Since $c_l \neq 0$, $c_p \neq 0$ then,

$$1 \pm \lambda_l \lambda_j = 1 \pm \lambda_p \lambda_j$$
$$\pm \lambda_l \lambda_j = \pm \lambda_p \lambda_j$$
$$\lambda_l = \lambda_p$$

For $i \neq j$, $i, j \neq l$,

$$R_\phi(v, e_j, e_j, e_l) \pm R_\psi(v, e_j, e_j, e_l) = 0 = R_\phi(v, e_i, e_i, e_l) \pm R_\psi(v, e_i, e_i, e_l)$$

$$c_l \pm c_l \lambda_l \lambda_j = 0 = c_l \pm c_l \lambda_l \lambda_i$$
$$\lambda_i = \lambda_j$$

Let $\lambda_i = \lambda$. Since $p \neq i, l$, then by the same argument, $\lambda_p = \lambda_i = \lambda$. Since $\lambda_l = \lambda_p$, then $\lambda_l = \lambda$. All $\lambda_j = \lambda$, which means $\psi = \lambda\phi$.

We also know that for all j, l, $1 \pm \lambda_j \lambda_l = 0 \Rightarrow 1 \pm \lambda^2 = 0 \Rightarrow \mp\lambda^2 = 1$. We cannot have a nonzero kernel for $ker(R_\phi + R_\psi)$ in this case because $-\lambda^2 \neq 1$, for all $\lambda \in \mathbb{R}$. So if we are in the case of $R_\phi + R_\psi$, then $dim(ker(R_\phi + R_\psi)) = 0$. Otherwise, $\lambda = \pm 1$, which means $\psi = \pm\phi$.

When $\psi = \pm\phi$, then $R_\phi - R_\psi = R_\phi - R_{\pm\phi} = 0 = R_\phi - (\pm 1)^2 R_\phi =$ $R_\phi - R_\phi = 0$ which is the zero tensor, so $\dim(\ker(R_\phi - R_\psi)) = \dim(V) = n$.

Theorem 3.2 *Let V be a vector space and let ϕ, $\psi \in S^2(V^*)$, and $R = R_\phi \pm R_\psi$ be an algebraic curvature tensor on V. If $\dim(V) = n$, then* $\dim(\ker(R_\phi \pm R_\psi)) \neq n - 1$.

Proof (by contradiction)

Let V be a vector space of dimension n and let $R = R_\phi \pm R_\psi$ be an algebraic curvature tensor on V. Suppose $dim(ker(R_\phi \pm R_\psi)) = n-1$. Let $\bar{V} = V/ker(R)$, which means $dim(\bar{V}) = 1$. Define $\bar{R}$ as $\pi^*\bar{R} = R$, this can be shown to be well-defined (see Lemma 2.1). Since $\dim(\bar{V}) = 1$, $\bar{R} = 0$ It will now be shown that R is the zero tensor, thus contradicting the assumption. Let $x, y, z, w \in V$. Show $R(x, y, z, w) = 0$.

$$\begin{aligned} R(x, y, z, w) &= \pi^*\bar{R}(x, y, z, w) \\ &= \bar{R}(\pi x, \pi y, \pi z, \pi w) \\ &= 0 \end{aligned}$$

Which means R is the zero tensor, which means $dim(ker(R)) = n$, thus contradicting the statement that $dim(ker(R)) = n - 1$.

4 Future Work

The next step on the first project is to figure out how to weaken the hypothesis that V_1 is totally isotropic. The assumption could instead involve the rank of V_1, and somehow come up with a contradiction.

One needs to go back to the beginning of the theorem 3.1 and see where this can go. If one knows that ϕ is positive definite and the dimensions of the kernel, what does that get in relation to the decomposition of $(V, R_\psi \pm R_\psi)$?

Is there any relation between, $\ker(R_\phi \pm R_\psi)$, $\ker(R_\phi) \cap \ker(R_\psi)$, $\ker(\phi) \cap \ker(\psi)$? If ϕ is positive definite, then is $\ker(\psi) \supseteq$ or $\subseteq \ker(R_\phi \pm R_\psi)$?

One should look for semi-positive definite forms of ϕ and using projection maps to create a decomposition on V and figure out what I can do from there. This also requires seeing if you can extend a basis from one subspace to the whole thing.

Look at the Lorentzian case, where $\phi(e_1, e_1) = -1, \phi(e_i, e_i) = 1$, for $i \neq 1$. Determine how this affects $\dim(\ker(R_\phi \pm R_\psi))$. Partial results have been found in dimension 3.

Look at nondegenerate forms of ϕ, such as having ϕ have signature (p, q).

Another thing to continue with is working with Jordan blocks. Using the paper that deals with different cases, I need to see what I would be able to use from there in order to figure out what happens to $\ker(R_\phi \pm R_\psi)$ as well as a decompostion of $(V, R_\phi \pm R_\psi)$.

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