Ropelength, Connect Sum, and Cabling of Paired Knots

Jessica Alley
Department of Mathematics
The University of Nebraska
Lincoln, NE 68588

August 19, 2009

Abstract

We define a class of knots called paired knots and show that for any paired conformation, K, with diagram, D, $R(K) \geq 4br(D)$. We then show that the connect sum of two paired knots is paired and that we can cable a paired knot such that the resulting (p,q)-cable knot is paired.

1 Introduction

A **knot** is a simple closed curve in three space. Any given knot has many different diagrams that can represent it. These diagrams are called **projections**. The **crossing number** of a knot, K, is the least number of crossings over all projections of K. An **alternating knot** is a knot with a projection that has crossings that alternate between over and under as we traverse the knot in a fixed direction. An **overpass** is a sub arc of the knot that goes over at least one crossing but never goes under a crossing. A **maximal overpass** is an overpass that could not be made any longer (See Figure 2.1). The **bridge number** of a knot diagram, denoted $br(D)$ is the number of maximal overpasses in the projection. [1]

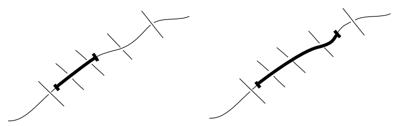

Figure 2.1: On the left we have an overpass. On the right we have a maximal overpass.

- *Example*: One projection of the figure eight knot has a bridge number of 4 (See Figure 2.2)

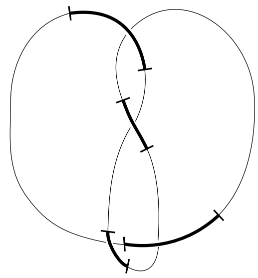

Figure 2.2: The figure eight knot with its maximal overpasses in bold. The bridge number of this diagram is four.

The **ropelength** of a knot, K, denoted $R(K)$, is the ratio of the arc length of K, denoted $l(K)$, and the injectivity radius of K, denoted $r(K)$. That is, $R(K) = \frac{l(K)}{r(K)}$. It is known that ropelength is scale invariant. Therefore, without loss of generality, we can assume $r(K) = 1$. Then $R(K) = l(K)$.

An unsolved problem in knot theory is finding a linear lower bound on the ropelength of a knot. In [3] Sadjadi proves that $R(K) \geq 4cr(K)$ for any alternating conformation. We will define a new class of knots called paired knots and use the methods in [3] to generalize Sadjadi's result and show that $R(K) \geq 4br(D)$ for any paired conformation. We then further investigate paired knots to see how they behave under the operations connect sum and cabling.

2 Finding a Paired Knot

Before we can study paired knots we need to know how to decide whether or not a knot is paired. We will combine knot theory and combinatorics to do so.

In combinatorics, a **graph**, is defined as $G = (V, E)$ where V is a finite set of vertices and E is a set of edges joining different pairs of distinct vertices. A **bipartite graph** is defined as $G = (X, Y, E)$ where G is an undirected graph with two vertex sets X and Y with all edges of the form (x, y) where $x \in X$ and $y \in Y$. A a **matching** in a graph is a set of independent edges with no common end points. A **X-matching** is a matching involving all vertices in X. A **maximal matching** is a matching of the largest possible size (See Figure 2.3). We will define the **maximal matching number**, denoted $m(G)$, of a graph, G, by the number of edges in a maximal matching. [2]

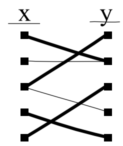

Figure 2.3: A bipartite graph with five vertices and six edges. The edges included in the maximal mathcing are in bold. The maximal matching number of this graph is four. This is not a X-matching since not all of the vertices in X are incident with a bold edge.

We now associate a bipartite graph to a diagram that contains the essential crossing information. Assume we are given a diagram, D, of a knot, K. We begin by assigning an orientation to D. Let $n = br(D)$ Label some maximal overpass 1. Following the direction that D is oriented, we label the adjacent maximal underpass 2. We continue in this manner until all maximal overpasses and underpasses have been labeled, 1 through $2n$. Now, let $G = (X, Y, E)$ where X is the set of all maximal overpasses, Y is the set of all maximal underpasses. Our set of edges, E, join maximal overpasses and maximal underpasses at their corresponing crossings. So E is the set of all crossings in D. Note that the number of maximal overpasses is equal to the number of maximal underpasses, that is, the size of X is equal to the size of Y.

Now we need to find the maximal matching of G. We will highlight the edges of our bipartite graph that we want to include in the matching. We start by selecting any vertex that is incident with only one edge. Call the vertex v_1 and the edge e_1. We highlight this edge and move to the other vertex it is incident with. Call this vertex v_2. If v_2 is only incident with e_1 then we're finished with these vertices. If v_2 is incident with another edge, e_2, then we follow this edge to its other vertex, v_3, but we do not highlight it, since then v_2 would be incident with two highlighted edges, contradicting the defintion of a matching. If v_3 is only incident with e_2 then we are finished with these vertices. If v_3 is incident with another edge, e_3, we highlight this edge and move to the next vertex, v_4. We continue in this manner until we reach a vertex that is incident with only one edge. Each time we reach a vertex that is incident with only one edge we move to a new vertex that is incident with only one edge and we repeat these steps. We are done when we cannot highlight anymore edges without having a vertex incident with more that one highlighted edge. Note that if there are no vertices on the graph incident with only one edge then we can start at any vertex and follow these steps to find our maximal matching.

- *Example*: We can use the minimal crossing diagram for the 8_{20} knot to illustrate this process. We label the maximal overpasses $1, 3, 5, 7, 9$ and the maximal underpasses $2, 4, 6, 8, 10$ (See Figure 2.4). We then construct the corresponding bipartite graph (See Figure 2.5).

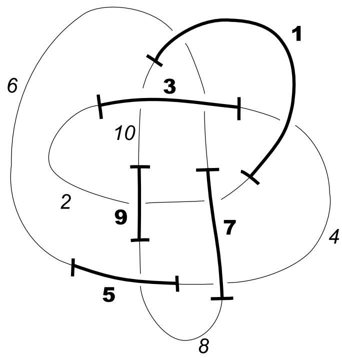

Figure 2.4: The 8_{20} knot with its maximal overpasses and underpasses labeled 1 through $2n$ where $n = br(D)$.

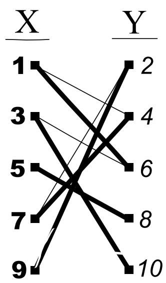

Figure 2.5: The bipartite graph corresponding to the 8_{20} knot where X is the set of all maximal overpasses and Y is the set of all maximal underpasses. Notice that this is an X-matching graph since every vertex in X is incident with one independent edge.

Definition 2.1 *A* ***paired diagram*** *is a diagram, D, that admits a bipartite graph G whose maximal matching is a X-matching.*

Definition 2.2 *A* ***paired knot*** *is any knot that admits a paired diagram.*

Notice that in any X-matching bipartite graph, $m(G)$ is equal to the order of X and, in our graphs, the order of X is equal to the number of maximal overpasses. So, by definition of bridge number, $m(G) = br(D)$ for all paired diagrams.

- *Example*: All alternating knots are paired. Since each maximal overpass and maximal underpass has length one we can simply pair each overpass with the only underpass it passes over.

- *Example*: Looking at the minimum crossing diagrams for the non-alternating 8 and 9 crossing knots we see that 8_{20}, 8_{21}, 9_{42}, 9_{44}, 9_{45}, 9_{46}, 9_{47}, 9_{48}, and 9_{49} are paired diagrams while 8_{19} and 9_{43} are not.

- *Example*: A **torus** is a surface generated by rotating a circle about an axis that is in the same plane as the circle but does not intersect it. A **torus knot** is any knot that lies on the unknotted torus without crossing over or under itself while on the torus. A **meridian curve** is a curve that runs once the short way around the torus. A **longitude curve** is a curve that runs once around the torus the long way. Every torus knot is a (p, q)-torus knot for some relatively prime integers p and q where p is the number of meridian curves and q is the number of longitude curves. [2]

  All torus knots are paired. We can illustrate this with a $(5, 7)$-torus knot. We start at the left most crossing on the first overpass we encounter and label it 1. We label the next crossing to the right 2. We continue in this manner until all the crossings on the overpass are labeled 1 through 4. We label every overpass in this manner. Then we can see that each of our maximal underpasses are also labeled 1 through 4 with no numbers repeating. So we can pair this knot at every crossing labeled 1 since every maximal overpass and maximal underpass is only labeled 1 once (See Figure 2.3)

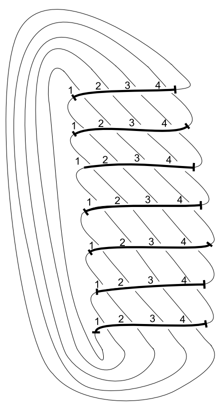

Figure 2.6: The $(5, 7)$-torus knot is paired.

3 Ropelength of Paired Knots

In [1] Sadjadi proves that for an alternating conformation of a knot K, $R(K) \geq 4Cr(K)$. Now we seek to use paired knots to generalize her argument and show $R(K) \geq 4br(D)$ for any paired knot, K, with diagram, D.

Definition 3.1 *A* ***paired conformation*** *is a knot, K, which admits a paired diagram, D, in the z-direction and has $r(K) = 1$.*

Let K be a paired conformation with diagram, D, and G be its corresponding bipartite graph. On G label each pair of vertices that are connected by an edge in the maximal matching (p_i, q_i) where $p_i \in X$ and $q_i \in Y$. Then, in D, p_i and q_i will share the same x and y coordinates. Let o_i and u_i denote the height of p_i and q_i, respectively.

- *Example*: On our bipartite graph for the 8_{20} graph we can label our pairs of vertices (p_1, q_1) through (p_5, q_5). (See Figure 3.1). Back on the knot diagram, we label the overpasses and underpasses accordingly (See Figure 3.2).

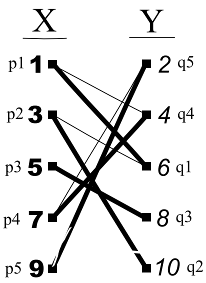

Figure 3.1: The bipartite graph corresponding to the 8_{20} knot with its pairs of vertices labeled (p_1, q_1) through (p_5, q_5).

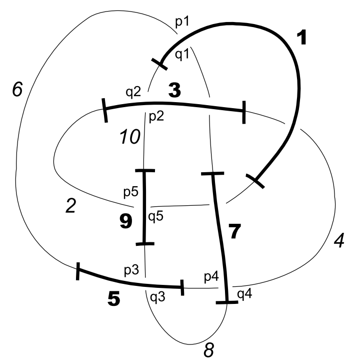

Figure 3.2: The 8_{20} knot with its maximal overpasses and underpasses labeled according to the bipartite graph in Figure 3.1.

Definition 3.2 *Let K be a paired conformation with diagram, D. Define the height function $h : K \to \mathbb{R}$ by $h(x, y, z) = z$. That is, h sends every point on K to its z-coordinate.*

Notice that under this definition $h(p_i) = o_i$ and $h(q_i) = u_i$. Since we previously defined the radius of each of our strands to be one we know that for each pairing, $o_i > u_i$, o_i and u_i must be at least two units apart. That is, $(o_i - u_i) \geq 2$.

- *Example*: We can represent the height function of the 8_{20} knot by placing each overpass above its corresponding underpass on the z-axis and connecting the strands accordingly (See Figure 3.3).

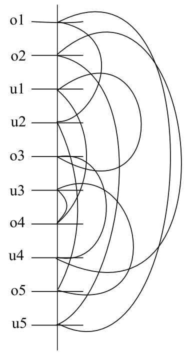

Figure 3.3: The 8_{20} knot projected onto the z-axis.

Since the projection h does not change the distance between any points in K we know that the arc length of K must be at least the length of $h(K)$. Thus, $R(K) \geq l(h(K))$.

Because the length of $h(K)$ is often hard to measure we need find a better way to measure the length of our intervals. In [1] Sadjadi replaces the arc lengths of $h(K)$ with straight line segments connecting overcrossing and undercrossings. She calls this the **taut** image of K, denoted $t(K)$. We will use this taut image as well. However, in this case line segments will connect overpasses and underpasses. In $t(K)$, an **edge** is the line segment connecting succesive overpasses and underpasses. Notice that each o_i and u_i is incident with exactly two edges. Since the edges in $t(K)$ are straight lines it must be true that the length of $h(K)$ is at least the length of $t(K)$. In turn, $R(K) \geq l(t(K))$

- *Example*: In Figure 3.4, we have adjusted our height function into the taut image of K. Note that the edges in the figure have bent slightly in order to be visible.

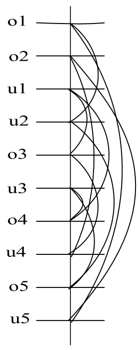

Figure 3.4: The taut image of the 8_{20} knot.

Now that we have established alternating pairs of overpasses and underpasses that are each incident with two edges in the taut image of K our argument follows directly from [1]. We reproduce the arguments here for completeness.

Definition 3.3 *Let z_0 be any height on the z-axis. A pair (o_i, u_i) is **split** if z_0 lies between the pairing. That is, $o_i \geq z_0 \geq u_i$. Otherwise the pair is **unsplit**.*

Lemma 3.1 *Let K be a paired conformation with diagram, D, z_0 a particular height on the z-axis and b the number of pairs (o_i, u_i) split by z_0. In $t(K)$ at least $2b$ edges must cross z_0.*

Proof: Let a be the number of unsplit pairs above z_0, b be the number of pairs split by z_0, and c be the number of pairs below z_0. Note that $a + b + c = n$. The case that results in the least number of edges crossing z_0 occurs when all the overpasses above z_0 connect with underpasses above z_0. There are $a + b$ overpasses above z_0 and a underpasses above z_0. Thus there are $2(a + b)$ edges incident with the overpasses above z_0 and $2a$ edges incident with the underpasses above z_0. So $2(a + b) - 2a = 2b$ edges remain above z_0. Therefore at least $2b$ edges must cross z_0 to connect with the undercrossings below. □

Lemma 3.2 *Let K be a paired conformation with diagram, D, and h_i, $1 \leq i \leq 2n$, a representation of the overpass and underpass heights in descending order. The length of $t(K) \geq \sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1})$.*

Proof: Represent the overpass and underpass heights as an ordered sequence of points $h_1, ..., h_{2n}$, along the z-axis. Define b_m to be the number of pairs split by the point z_0 on the z axis between the point h_m and h_{m+1}. Then on the interval $[h_m, h_{m+1}]$ there are at least $2b_m$ edges spanning the interval. So the length of the edges on this interval must be at least $2b_m(h_m - h_{m+1})$. Since there are $2n-1$ total intervals the total length of the edges is at least $\sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1})$. That is $t(K) \geq \sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1})$. □

- *Example*: Represent the heights $o_1, o_2, u_1, u_2, o_3, u_3, o_4, u_4, o_5$, and u_5 as $h_1, ...h_{2n}$, $1 \leq n \leq 5$, along the z-axis. Let z_0 intersect every interval between heights h_m and h_{m+1} (See Figures 3.5, 3.6, 3.7, 3.8).

Lemma 3.3 *If K is a paired conformation, then $\sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1}) = \sum_{i=1}^{n} 2(o_i - u_i)$*

Proof: Let o_i and u_i be represented by h_x and h_y respectively. The interval $[h_x, h_y]$ can be partitioned into smaller heights $h_x \geq h_{x+1} \geq ... \geq h_{y-1} \geq h_y$. Then the length of the interval $[o_i, u_i]$ is equal to $\sum_{j=x}^{y-1}(h_j - h_{j+1})$. So we have $\sum_{i=1}^{n}(o_i - u_i) = \sum_{i=1}^{n}(\sum_{j=x}^{y-1}(h_j - h_{j+1}))$. Now notice that the height $h_m - h_{m+1}$ occurs in the expansion of $o_i - u_i$ if and only if the interval $[h_m, h_{m+1}]$ splits the pair (o_i, u_i). So the length $h_m - h_{m+1}$ occurs in the double sum exactly b_m times. Therefore $\sum_{i=1}^{n}(o_i - u_i) = \sum_{m=1}^{2n-1} b_m(h_m - h_{m+1})$. Multiplying both sides be two we find $\sum_{i=1}^{n} 2(o_i - u_i) = \sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1})$, as desired. □

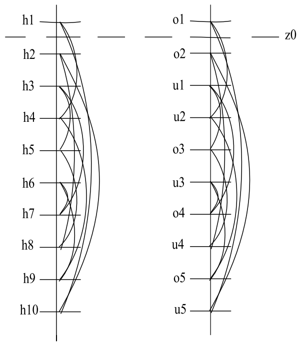

Figure 3.5: $b_1 = 1$

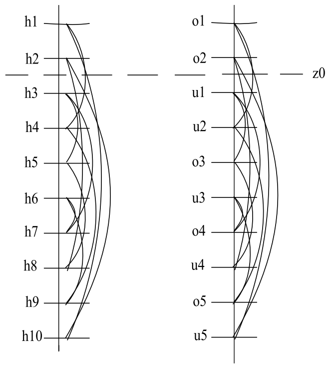

Figure 3.6: $b_2 = 2$

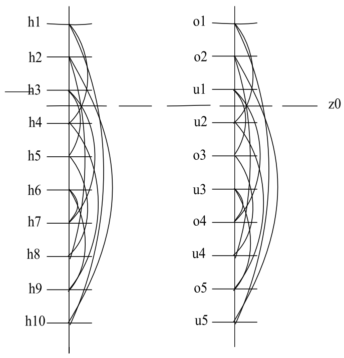

Figure 3.7: $b_3 = 1$

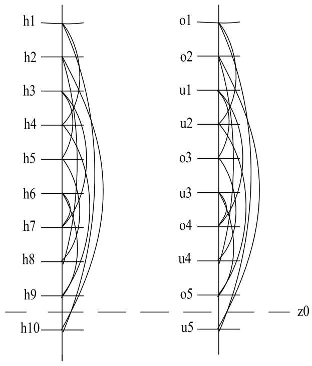

Figure 3.8: $b_9 = 1$

Theorem 3.1 *If K is a paired conformation with diagram, D, then $R(K) \geq 4br(D)$.*

Proof: Since $(o_i - u_i) \geq 2$ we have $\sum_{i=1}^{n} 2(o_i - u_i) \geq \sum_{i=1}^{n} 2(2) = 4n = 4br(D)$. So, from our Lemmas this gives us $4br(D) = \sum_{m=1}^{2n-1} 2b_m(h_m - h_{m+1}) \leq l(t(K)) \leq l(h(K)) \leq R(K)$. □

Notice that for any alternating conformation, K, with diagram, D, $Cr(D) = br(D)$. Therefore Theorem 3.1 is a generalization of Sadjadi's result in [1].

4 Connect Sum of Paired Knots

In this section we investigate what happens when we connect sum two paired knots.

Given two projections of knots, K_1 and K_2 we can define a new knot obtained by removing a small arc from each knot projection and then connecting the four endpoints by two new arcs. We will call this operation the **connect sum**, denoted $K_1 \# K_2$. The resulting knot is called the **composition** of the two knots. Then a **composite knot** is a knot that can be expressed as the composition of two or more knots, neither of which is the trivial knot. **Factor knots** are the knots that make up the composite knot.[2]

- *Example*: We can connect sum the trefoil and the figure eight knot to produce a composite knot with the trefoil and figure eight as its factor knots (See Figure 4.1).

Definition 4.1 *Given a paired diagram, D, of a knot, K, an* ***arc*** *is a strand in D that runs between two adjacent crossings.*

Definition 4.2 *Given a paired diagram, D, of a knot, K, a* ***switched arc*** *is any arc that runs between one undercrossing and one overcrossing(See Figure 4.2).*

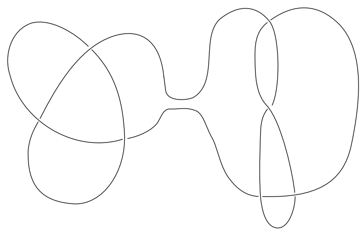

Figure 4.1: The connect sum of the trefoil and the figure eight knot.

Definition 4.3 *Given a paired diagram, D, of a knot, K, a **full overarc** is any arc that runs between two overcrossings (See Figure 4.3).*

Definition 4.4 *Given a paired diagram, D, of a knot, K, a **full underarc** is any arc that runs between two undercrossings.*

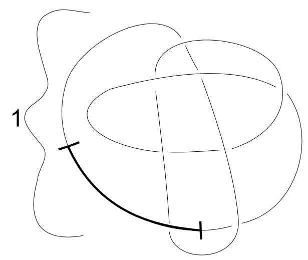

Figure 4.2: Arc 1 of the 8_{20} knot is a switched arc since it lies between an undercrossing and an overcrossing.

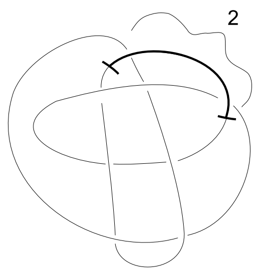

Figure 4.3: Arc 2 of the 8_{20} knot is a full overarc since it lies between two overcrossings.

Theorem 4.1 *Given two paired diagrams, D_1 and D_2, of knots, K_1 and K_2, respectively, there is always a way to connect sum D_1 and D_2 such that $D_1 \# D_2$ is a paired diagram.*

Proof: We will break up the proof into cases.

- Case 1: D_1 and D_2 each have a switched arc on the outside of the knot that are mirror images of each other. That is, when these arcs are broken, creating an overpass strand and an underpass strand on both D_1 and D_2, the overpass strand on D_1 can be connected to the underpass strand on D_2 and the underpass strand on D_1 can be connected to the overpass strand on D_2 without creating any new crossings.
  - Since we are only changing one arc on each knot we can assume that each maximal overpass and maximal underpass not touching the broken arc stays paired as it was before we broke the arc. So we can simply look at the arc we break. Notice that breaking a switced arc will result in two new strands on each knot, an overpass and an underpass strand. We connect the overpass on D_1 to the underpass on D_2 and the underpass on D_1 to the overpass on D_2. Then since we haven't added any overcrossings to an overpass or any undercrossings to an underpass we have not altered any of our original passes. Therefore, each of the passes can stay paired as they were before we broke the arc. Then, since D_1 and D_2 were paired originally, $D_1 \# D_2$ must also be paired.
- *Example*: We can illustrate case 1 with the 8_{20} and 8_{21} knots (See Figure 4.4). Connecting these knots at two switched arcs produces a paired diagram, $8_{20} \# 8_{21}$ (See Figure 4.5).

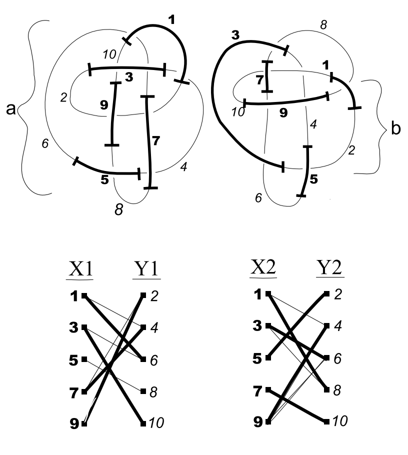

Figure 4.4: The 8_{20} and 8_{21} knots, which are both paired, have switched arcs, a and b respectively.

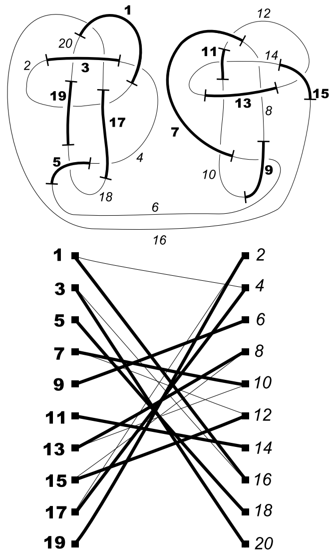

Figure 4.5: The connect sum of 8_{20} and 8_{21}, $8_{20}\#8_{21}$, connected at their switched arcs. From the bipartite graph we see that $8_{20}\#8_{21}$ is paired.

- Case 2: D_1 and D_2 each have a full overarc on the outside of the knot. The maximal overpasses that the arcs are contained in are paired on opposite sides. Note that the case where D_1 and D_2 each have a full underarc on the outside of the knot is similar.
  - We can again assume that any maximal overpass or maximal underpass untouched by the broken arc stay paired as they were before we broke the arc. So we can simply look at the arcs we break. Each full overarc that we are breaking is contained in a maximal overpass. Any given maximal overpass can only be paired at only one of its crossing. In D_1 call the paired crossing p_1 and the unpaired crossing u_1. In D_2 call the paired crossing p_2 and the unpaired crossing u_2. Since our original overpasses were paired on opposite sides when we connect our broken strands without creating any new crossings we get two new maximal overpasses where one overpass contains p_1 and u_2 and the other contains p_2 and u_2. Since our new maximal overpasses still only contain one paired crossing, either p_1 or p_2, we can leave these crossings paired as they were in D_1 and D_2. So $D_1\#D_2$ is paired.
- *Example*: We illustrate this case with the 8_{20} and 8_{21} knots (See Figure 4.6). Connecting these knots at two full overpasses results in a paired diagram, $8_{20}\#8_{21}$ (See Figure 4.7)

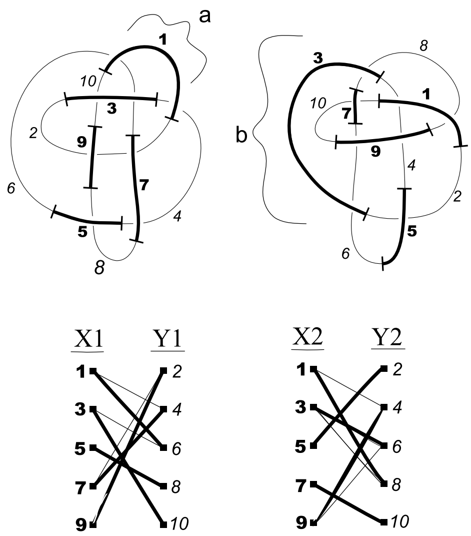

Figure 4.6: The 8_{20} and 8_{21} knots, which are both paired, have full overpasses, a and b respectively. Notice that overpass 1 on 8_{20} is paired with underpass 6 and overpass 3 on 8_{21} is paired with underpass 6. So when we break these arcs the resulting paired and unpaired strands will be on opposite sides, allowing us to connect a paired strand to an unpaired strand without creating any new crossings, as desired.

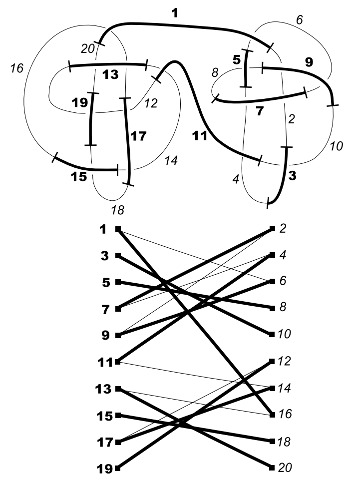

Figure 4.7: The connect sum of 8_{20} and 8_{21}, $8_{20}\#8_{21}$, at their full overpasses. Note that $8_{20}\#8_{21}$ is paired.

- Case 3: D_1 and D_2 don't contain arcs on the outside of the knot that make it possible to connect them as in case 1 or case 2.

– Locate arcs on D_1 and D_2 that match the criteria in either case 1 or case 2. Call the arcs a_1 and a_2 where $a_1 \in D_1$ and $a_2 \in D_2$. One of the knots needs to have its arc on the outside. Without loss of generality, assume a_1 is on the outside of D_1. Connect a_1 to D_2 on the arc closest to a_2. Shrink D_1 until it is small enough to fit inside D_2. Slide D_1 along D_2 until a_1 and a_2 are connected as in case 1 or case 2 (See Figures 4.8 and 4.9). Now the proof follows from case 1 and case 2.

□

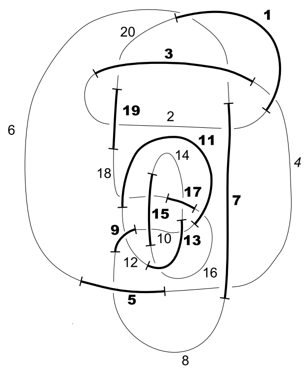

Figure 4.8: The connect sum of 8_{20} and 8_{21}, $8_{20}\#8_{21}$, after we shrink 8_{21} and slide it inside of 8_{20}.

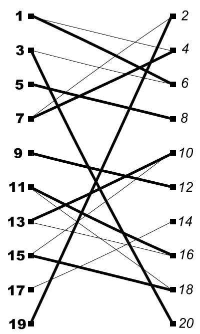

Figure 4.9: The bipartite graph for the knot in Figure 4.8. Notice that this knot is paired.

5 (p, q)-Cables of Paired Knot Diagrams

In this section we will try and determine if we can cable a paired knot, K, in such a way that the resulting (p, q)-cable of K is paired. Recall from our example in section 2 that a torus is a surface generated by rotating a circle about an axis that is in the same plane as the circle but does not intersect it and a torus knot is any knot that lies on the unknotted torus without crossing over or under itself while on the torus. A **solid torus** includes the interior of the torus instead of just the surface. Let K_1 be a knot inside an unknotted solid torus. If we knot the solid torus in the shape of a second knot, K_2 then this will take the knot K_1 that lies inside the original solid torus to a new knot inside the knotted solid torus. This new knot, K_3, is a **satellite knot**. The knot K_2 is called the **companion knot**. If K_1 is a torus knot, then we call the resulting satellite knot with companion K_2 a **cable knot** on K_2. If K_1 is a $(p.q)$-torus knot then K_3 is a (p, q)-cable knot. [2]

- *Example*: We can cable the trefoil using three strands and five twists (See Figure 5.1).

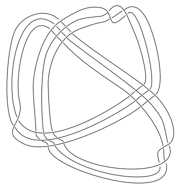

Figure 5.1: The $(3, 5)$ cable of the trefoil.

Definition 5.1 *Let K be a (p, q)-cable knot. A* ***cable arc*** *in K is any section of the knot that contains the p strands that run between two crossings.*

Definition 5.2 *Let K be a (p, q)-cable knot.* ***switched cable arc*** *in K is any cable arc that runs between one overcrossing and one undercrossing.*

Let K_0 be a paired knot with diagram, D. Let K_1 be a (p, q)-cable of K_0. We examine locally what happens when we add a twist to a cable knot. If we add one twist to a switched cable arc we merely extend one of our overpasses (See Figure 5.2). Since no new overpasses or underpasses are created none of our existing pairings are affected. So we can add one twist to every switched cable arc and still have a paired knot.

Let b be the number of switched cable arcs on K_1. Notice that a switched cable arc occurs each time a cable arc strand switches from overpass to underpass. This occurs once for each bridge in D. So $b = br(D)$. Define a **full twist** to be a twist in which the bridge number of the twist is equal to q, or the number of strands. As noted by Kauffman in <u>On Knots</u>, a full twist can be expressed as a writhe [4] The advantage of writhes is that they are easily paired. (See Figure 5.3). We can add writhes in such a way that each writhe adds a switched cable arc. Let $x = q \mod p$. Then $q = mp + x$ for some integer $|m|$ where $|m|$ is the number of full twists or writhes in K. So can have up to $br(D) + |m|$ switched arcs on our cable and x twists that cannot be exchanged for writhes.

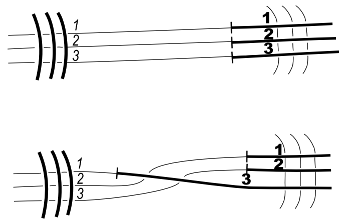

Figure 5.2: Adding one twist to the switched arc above extends overpass 3 but does not affect our pairing.

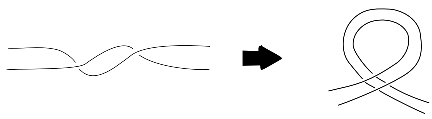

Figure 5.3: Exchanging a twist for a writhe.

Theorem 5.1 *Given a paired diagram, D, of a knot, K, the (p,q)-cable of K, is paired if $x \leq |m| + br(K)$.*

Proof: Let K be a (p,q) cable of D where D is a paired knot diagram. Begin by pairing K in the same manner as D. That is, if overpass a is paired with underpass b in D then in K we can pair the p overpasses that correspond to overpass a arbitrarily to the p underpasses that correspond to underpass b.

Now we can replace our $|m|$ full twists with $|m|$ writhes. Note that when exchanging a twist for a writhe we must make sure that the sign of the twist is the same as the sign of the resulting writhe. Since we can slide a twist anywhere we want along the knot, we can also slide our writhes wherever we would like. We slide all of our writhes to a switched cable arc such each writhe adds a new switched cable arc(See Figure 5.4). Adding a writhe in this manner creates p new overpasses that pass over p new underpasses. So we can simply pair the new overpasses with the new underpasses and all of our other pairings stay the same.

We can add one twist to every switched arc without affecting our pairing. Since each writhe adds one switched arc to our diagram we have $|m| + br(D)$ switched arcs. After we replace every full twist with a writhe we are left with x twists. So, K is paired if $x \leq |m| + br(D)$. □

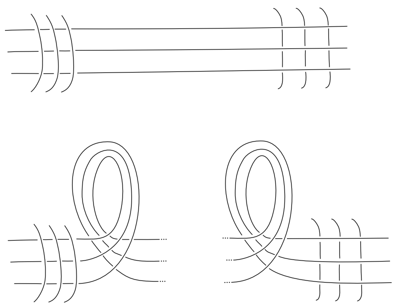

Figure 5.4: We can slide our $|m|$ writhes onto the switched arc above to create one new switched cable arc for every writhe.

6 Conclusion

We began by defining a class of knots called paired knots. We then imitated the process in [3] to generalize Sadjadi's argument and find a linear lower bound on ropelength for all paired conformations. We then investigated how paired knots act under the knot operations connect sum and cabling. We showed that the connect sum of two paired knots is paired. Then we showed that when cabling a paired knot, given that the number of twists is less than or equal to the number of bridges in the paired knot plus the number of writhes in the cable, the resulting (p, q)-cable is paired.

7 Acknowledgements

I would like to thank Dr. Rolland Trapp and Dr. Corey Dunn at CSUSB. I would also like to thank Charley Mathes who I worked together with on defining paired knots and proving Theorem 3.1. This project was made possible by the NSF grant DMS-0850959.

References

[1] Adams, Colin C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* W.H. Freeman and Company, 2001. Print.

[2] Tucker, Alan. *Applied Combinatorics.* 4th ed. John Wiley & Sons, 2002. Print

[3] Sadjadi, Shala. "On Ropelength of Alternating Links". California State University, San Bernardino, 2007.

[4] Kauffman, Louis H. *On Knots.* Princeton University Press, 1987. Print.