Constructing Algebraic Curvature Tensors Using Symmetric Bilinear Forms

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Abstract

We examine various properties of algebraic curvature tensors in order to construct a basis for all algebraic curvature tensors on an m-dimensional vector space V. We also show that if $\dim(V) = 4$, then we can construct any algebraic curvature tensor using at most 6 symmetric bilinear forms.

1 Introduction and Preliminaries

When seeking to describe a geometric object in space, perhaps the most obvious way to proceed is by describing "how much" the object curves in space. In differential geometry this idea has been formalized as the concept of *curvature*. Algebraically, this can be described using an *algebraic curvature tensor*.

Definition 1.1. *Let V be a vector space, $x, y, z, w \in V$ and $R : V \times V \times V \times V \to \mathbb{R}$, a multi-linear function satisfying*

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$, *and*
3. *(Bianchi Identity)* $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

Then, R is an ***algebraic curvature tensor****.*

Definition 1.2. *If V is a vector space with metric $< \cdot, \cdot >$, orthonormal basis $\{e_1, ..., e_m\}$ and algebraic curvature tensor R, then the* ***Ricci tensor*** *with respect to these objects is defined by*

$$\rho(x, y) = \sum_{i=1}^{m} < e_i, e_i > R(x, e_i, e_i, y)$$

Definition 1.3. *An algebraic curvature tensor R is called an* ***Einstein curvature tensor*** *if there exists $c \in \mathbb{R}$ such that the corresponding Ricci tensor $\rho(x, y) = c < x, y >$ for any $x, y \in V$. Additionally, R is called* ***Ricci-flat*** *if $\rho(x, y) = 0$ for all $x, y \in V$.*

We write $\mathcal{A}(V)$ to denote the set of all algebraic curvature tensors on V. If ϕ is a symmetric bilinear form, denoted $\phi \in S^2(V^*)$, then we can define an algebraic curvature tensor, R_ϕ by

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w).$$

It is known that $\text{span}\{R_\phi \in \mathcal{A}(V) | \phi \in S^2(V^*)\} = \mathcal{A}(V)$[1], however, we will explicitely state a basis when $\dim(V) = 4$ and use that basis to generate a basis when $\dim(V) \geq 5$. If $\{e_1, ..., e_m\}$ is a basis for V, we will write R_{ijkl} for $R(e_i, e_j, e_k, e_l)$ as the components of R with respect to this basis.

Theorem 1.4. *If V is a vector space of dimension m and $\mathcal{A}(V)$ is the set of all algebraic curvature tensors on V, then the dimension of $\mathcal{A}(V)$ is $\frac{m^2(m^2-1)}{12}$.*

Proof. Let $\{e_1, ..., e_m\}$ be an basis for V and let R be an algebraic curvature tensor. It is clear that all of the curvature components of R involve two, three or four of our basis vectors.

Concerning the components that involve two of the basis vectors, consider the number $\binom{m}{2}$. This is the number of all possible pairs of numbers $1, ..., m$, up to the order they appear in. Therefore, $\binom{m}{2}$ corresponds to all of the curvature components of the form R_{ijji}, and this is all of the curvature components involving 2 basis vectors.

Now, we will attempt to determine the number of components involving 3 basis vectors in terms of m. Consider $\binom{m}{3}$, which will give us all of the triples of three distinct numbers i, j and k, up to the order they appear in. Then, each triple corresponds to three independent curvature components, R_{ijki}, R_{ijkj} and R_{kijk}. Thus, there are $3\binom{m}{3}$ independent components involving 3 basis vectors.

Finally, we seek to determine the number of components involving 4 basis vectors. In this case, each component will have a unique basis vector in each slot. We will now consider $\binom{m}{4}$, which will give us all possible quadruples i, j, k, l such that they are all distinct. Because our algebraic curvature tensor must satisfy the Bianchi identity, we know that

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

Hence, R_{ijkl} and R_{jkil} are independent curvature components, but R_{kijl} is not because $R_{ijkl} + R_{jkil} = -R_{kijl}$. Therefore, for each quadruple there are 2 corresponding independent curvature components and the number of components involving 4 basis vectors is $2\binom{m}{4}$.

Summing each of these pieces yields $\binom{m}{2} + 3\binom{m}{3} + 2\binom{m}{4} = \frac{m^2(m^2-1)}{12}$. Therefore, we need $\frac{m^2(m^2-1)}{12}$ independent curvature components to determine an algebraic curvature tensor and thus $\dim(\mathcal{A}(V)) = \frac{m^2(m^2-1)}{12}$ $\square$

Let V be a vector space and $< \cdot, \cdot >$ an inner product on V, then according to [3], [4], we can, in general, construct the **Chern basis** $\{e_1, ..., e_m\}$ such that this basis is orthonormal, and

$$R_{1i1j} = 0 \text{ for } 2 \leq i < j \leq m,\ R_{122j} = 0 \text{ for } 3 \leq j \leq m \text{ and } R_{1323} = 0.$$

From [5], if $\dim(V) = 4$ and R is Einstein, then we can construct the **Singer-Thorpe basis** $\{f_1, ..., f_4\}$ such that

$$\begin{aligned} R_{1221} &= R_{3443} = a, R_{1331} = R_{2442} = b, R_{1331} = R_{2332} = c; \\ R_{1234} &= \alpha, R_{1342} = \beta, R_{1423} = \gamma, and \\ R_{ijki} &= 0, \end{aligned}$$

where $\alpha + \beta + \gamma = 0$ by the Bianchi identity and $a + b + c = \frac{\tau}{4}$, where τ is the scalar curvature.

Definition 1.5. *If V is a vector space of dimension m, $R \in \mathcal{A}(V)$ and $\phi_i \in S^2(V^*)$, then $\nu(R)$ is the least number such that*

$$R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\psi_i}.$$

We then define

$$\nu(m) := \sup_{R \in \mathcal{A}(V)} \nu(R).$$

In other words, $\nu(m)$ is the maximal number of symmetric bilinear forms one could possibly need to construct *any* algebraic curvature tensor in a vector space of dimension m. It was shown in [1] and [2] that $\frac{m}{2} \leq \nu(m) \leq \frac{m(m+1)}{2}$. We will show how to improve on that upper bound if $m = 4$.

2 A Basis for Dimension Four and Higher

Let V be a vector space, $< \cdot, \cdot >$ a positive definite metric and let $\{e_1, e_2, e_3, e_4\}$ be a basis for V. Then, $\dim(\mathcal{A}(V)) = 20$. Consider $\mathcal{B} = \{\phi_i \in S^2(V^*) | i = 1, ..., 20\}$, where

$$\phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \phi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \phi_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \phi_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \phi_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \phi_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \phi_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\phi_{10} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \phi_{11} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \phi_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\phi_{13} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \phi_{14} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \phi_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\phi_{16} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \phi_{17} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \phi_{18} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi_{19} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \phi_{20} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Each ϕ_i represents a symmetric bilinear form written with respect to this inner product, where the i, j entry of each matrix corresponds to $\phi_k(e_i, e_j)$. Each was constructed with a very specific reason in mind. We can see from the proof that $dim(\mathcal{A}(V)) = 4$ that there are 20 independent curvature components that completely determine an algebraic curvature tensor on a 4-dimensional vector space. We can break these independent curvature components into 3 groups. If i, j, k, l are distinct indices and $1 \leq i, j, k, l \leq 4$, then the first group is those curvature components of the form R_{ijji}. The second group is those curvature components of the form R_{ijki}. And the last group is those curvature components of the form R_{ijkl}. We can correspondingly group our symmetric bilinear forms into 3 groups. The first group contains $\phi_1, ..., \phi_6$, the second group contains $\phi_7, ..., \phi_{18}$ and the final group consists of ϕ_{19} and ϕ_{20}.

In dimension 4, the only curvature components that depend on all 4 basis vectors are R_{1234} and R_{1432}. We grouped our forms so that ϕ_{19} and ϕ_{20} correspond to the curvature components of the form R_{ijkl}. Examining $R_{\phi_{19}}$ and $R_{\phi_{20}}$, it is easy to see that $R_{\phi_{19}}(e_1, e_2, e_3, e_4) = 1$ and $R_{\phi_{20}}(e_1, e_4, e_3, e_2) = 1$. There are other non-zero curvature components of $R_{\phi_{19}}$ and $R_{\phi_{20}}$ - notice that $R_{\phi_{19}}(e_1, e_4, e_4, e_1) = 1$ and $R_{\phi_{20}}(e_3, e_4, e_4, e_3) = 1$ - however, for any $i \neq 19, 20$, $R_{\phi_i}(e_1, e_2, e_3, e_4) = R_{\phi_i}(e_1, e_4, e_3, e_2) = 0$.

If we examine our second grouping of $\phi_7, ..., \phi_{18}$, it is easy to see that each of these symmetric bilinear forms corresponds to a unique curvature component such that $R_{\phi_n}(e_i, e_j, e_k, e_i) = 1$, and, $R_{\phi_m}(e_i, e_j, e_k, e_i) = 0$ for any $m \neq n$. For example, $R_{\phi_7}(e_1, e_2, e_3, e_1) = 1$, and, for any $i \neq 7$, $R_{\phi_i}(e_1, e_2, e_3, e_1) = 0$.

In our last group, which contains $\phi_1, ..., \phi_6$, each form was constructed to give only one non-zero entry. It is easy to see from the structure of these forms that $R_{\phi_1}(e_1, e_2, e_2, e_1) = 1$, $R_{\phi_2}(e_1, e_3, e_3, e_1) = 1$, and so on. Because of how we constructed these 20 symmetric bilinear forms, we present the following proposition.

Proposition 2.1. *The set $\mathcal{B}$ is a basis for $\mathcal{A}(V)$.*

Proof. Fix $\alpha_i \in \mathbb{R}$ such that

$$R = \sum_{i=1}^{20} \alpha_i R_{\phi_i} = 0.$$

We will show that $\alpha_i = 0$ for $i = 1, ..., 20$, and thus $\mathcal{B}$ is a linearly independent set. We will do so by observing what happens if we plug in each of the 20 curvature components known to be independent and to completely determine an algebraic curvature tensor in dimension four.

To begin, notice that $R_{\phi_i}(e_1, e_4, e_3, e_2) \neq 0$ if and only if $i = 20$. Therefore, it is clear that $\alpha_{20} = 0$. Similarly $R_{\phi_i}(e_1, e_2, e_3, e_4) \neq 0$ if and only if $i = 19$. Again, it is easy to see that this implies $\alpha_{19} = 0$.

If we consider the curvature components of three distinct indecies, then $R_{\phi_\gamma}(e_i, e_j, e_k, e_i) \neq 0$ if and only if $\gamma = 7, ..., 18$. Each possible combination of three of our basis vectors corresponds to exactly one ϕ_i with $i = 7, ..., 18$. For example, $R_{\phi_i}(e_1, e_2, e_3, e_1) \neq 0$ if and only if $i = 7$. Therefore, $\alpha_i = 0$ for $i = 7, ..., 18$.

Lastly, we must consider what happens when we use only two of our basis vectors. There are six such situations. Observe that $R_{1441} = \alpha_3 - \alpha_{11} - \alpha_{14} - \alpha_{19} = 0$. However, we know that $\alpha_{11} = \alpha_{14} = \alpha_{19} = 0$, and thus $\alpha_3 = 0$. Since we know that if $i \geq 7$, then $\alpha_i = 0$, a similar case will arise for each α_j for $j = 1, ..., 6$. Therefore, if $\sum_{i=1}^{20} \alpha_i R_{\phi_i} = 0$, then $\alpha_i = 0$ for $i = 1, ..., 20$, and, the set $\{R_{\phi_i} \in \mathcal{A}(V) | i = 1, ..., 20\}$ is linearly independent and a basis for $\mathcal{A}(V)$. $\square$

An interesting observation can be made about using this basis to generate a basis when $dim(V) \geq 5$. If V is a 5-dimensional vector space, then $\dim(\mathcal{A}(V)) = 50$. Therefore, there are 50 independent curvature components which completely and uniquely determine $R \in \mathcal{A}(V)$. We can split these curvature components into groups as we did previously, a group of the curvature components of 2 indices, a group of the curvature components of 3 indices and a group of the curvature components of 4 indices. We could proceed as we did in dimension 4 and simply construct a symmetric bilinear form for each of these 50 curvature components. However, this would simply be an excercise in patience. Notice that since $R : V \times V \times V \times V \to \mathbb{R}$, there are no curvature components of 5 indices. Therefore, we can obtain a basis for $\mathcal{A}(V)$ by taking our basis for algebraic curvature tensors in dimension 4 and inserting rows and columns of zeros to make them determine algebraic curvature tensors in dimension n.

Let $1 \leq i, j, k, l \leq n$ and i, j, k, l be distinct. We can easily define ϕ_γ such that $R_{\phi_\gamma}(e_i, e_j, e_k, e_l) = 1$. If $i \leq j \leq k \leq l$, then, so long as $x \neq i, j, k, l$, $e_x \in \ker(\phi_\gamma)$. We define the remaining 16 entries of ϕ_γ so that $\phi_\gamma(e_i, e_l) = \phi_\gamma(e_l, e_i) = 1$ and $\phi_\gamma(e_j, e_k) = \phi_\gamma(e_k, e_j) = 1$ and $\phi_\gamma(e_n, e_m) = 0$ otherwise. Notice that the matrix representation of ϕ_γ looks similar to ϕ_{19}, except that ϕ_γ has extra rows and columns of zeros. The other case is where $i \leq l \leq k \leq j$. We construct ϕ_ξ the same way, except now it will look like ϕ_{20} with extra rows and columns of zeros.

We will now seek to define ϕ_ω such that $R_{\phi_\omega}(e_i, e_j, e_k, e_i) = 1$. To begin, choose one $1 \leq i \leq n$ and let $\phi_\omega(e_i, e_i) = 1$. Then, we get a different symmetric bilinear form for by letting every combination of $\phi_\omega(e_j, e_k) = 1$ where $j, k \neq i$. This will give us a different for for every combination of i, j and k, and, each ϕ_ω will resemble $\phi_7, ..., \phi_{18}$ with extra rows and columns of zeros.

Lastly, and most easily, we get construct forms that represent the remaining curvature components by simple choosing all possible combinations of i and j such that $\phi(e_i, e_i) = \phi(e_j, e_j) = 1$ and $\phi(e_n, e_m) = 0$ otherwise. Each of these will resemble $\phi_1, ..., \phi_6$ with extra rows and columns of zeros. Proceeding as such will yield a basis for $\mathcal{A}(V)$, regardless of the dimension of V.

3 An Upper Bound for $\nu(4)$.

As we stated in the introduction, $\nu(m) \leq \frac{m(m+1)}{2}$. Therefore, it is known that $\nu(4) \leq 10$. We will improve on this upper bound.

Theorem 3.1. $\nu(4) \leq 6$.

Proof. Let V be a four-dimensional vector space with a positive definite metric g and an algebraic curvature tensor R. We can define a Chern basis for V, $\{e_1, e_2, e_3, e_4\}$, with respect to g and R such that

$$R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1314} = R_{1323} = 0.$$

Therefore, since we know that in dimension 4 there are 20 independent curvature components, with respect to a Chern basis, there are only 14 independent curvature components that could possibly be non-zero. Thus, $R = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{12} R_{\phi_{12}} + \sum_{j=14}^{20} \alpha_j R_{\phi_j}$. We will show that R can be written as $\sum_{i=1}^{6} \beta_i R_{\psi_i}$ for six symmetric bilinear forms ψ_i. To do this, we will start by viewing R in terms of our basis and proceed to build a new algebraic curvature tensor $\hat{R}$ by building new symmetric bilinear forms $\psi_1, ..., \psi_6$ and substituting R_{ψ_i} for some combination of the basis vectors. We will then check to see where $\hat{R}$ and R differ and proceed to change some of our constants, namely $\alpha_1, ..., \alpha_6$, so that $\hat{R} = R$. We will examine 4 possible cases, which depend on the values of α_{19} and α_{20}.

We will first examine the case where $\alpha_{19} = \alpha_{20} = 0$. In this case, note that $R = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{12} R_{\phi_{12}} + \sum_{j=14}^{18} \alpha_j R_{\phi_j}$. To begin to reduce the number of symmetric bilinear forms we use to construct R, we will construct a new form that will contribute the same information to $\hat{R}$ as $\alpha_{12} R_{\phi_{12}} + \alpha_{15} R_{\phi_{15}} + \alpha_{18} R_{\phi_{18}}$. We will call this form $\phi_{12,15,18}$ and will define it by

$$\phi_{12,15,18} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha_{12}\alpha_{15} & 0 & -\alpha_{15}\alpha_{18} \\ 0 & 0 & -\alpha_{12}\alpha_{15} & \alpha_{12}\alpha_{18} \\ 0 & -\alpha_{15}\alpha_{18} & \alpha_{12}\alpha_{18} & 0 \end{bmatrix}.$$

If we let $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{14} R_{\phi_{14}} + \alpha_{16} R_{\phi_{16}} + \alpha_{17} R_{\phi_{17}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}}$, then notice that $R \neq \hat{R}$. This can be seen since, for example, $R_{2332} = \alpha_4 - \alpha_{18} - \alpha_{19}$, but, $\hat{R}_{2332} = \alpha_4 - \frac{\alpha_{12}\alpha_{15}}{\alpha_{18}} - \alpha_{19}$, where in this case $\alpha_{19} = 0$. However, notice that the unique information that was contributed to R by $R_{\phi_{12}}$, $R_{\phi_{15}}$ and $R_{\phi_{18}}$ is preserved since $R_{2342} = \hat{R}_{2342} = \alpha_{12}$, $R_{3243} = \hat{R}_{3243} = \alpha_{15}$ and $R_{4234} = \hat{R}_{4234} = \alpha_{18}$. Any changes this substitution made to the overall structure of R will be accounted for by altering some of our constants later.

There is the question of what happens if $\alpha_{12} = 0$, $\alpha_{15} = 0$ or $\alpha_{18} = 0$. Let i, j, k be distinct with $i, j, k = 12, 15, 18$, in no particular order.

If $\alpha_i = \alpha_j = 0$, then we would simply let $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{14} R_{\phi_{14}} + \alpha_{16} R_{\phi_{16}} + \alpha_{17} R_{\phi_{17}} + \alpha_k R_{\phi_k}$, and $\hat{R}$ would be built using the same number of symmetric bilinear forms as when α_{12}, α_{15} and α_{18} were non-zero.

If $\alpha_i = 0$ and $\alpha_j, \alpha_k \neq 0$, then define $\phi_{j,k}$ by

$$\begin{aligned}
\phi_{12,15} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_{12} & \alpha_{12}\alpha_{15} \\ 0 & -\alpha_{12} & \alpha_{15} & 0 \\ 0 & \alpha_{12}\alpha_{15} & 0 & 0 \end{bmatrix}, \\
\phi_{12,18} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{12}\alpha_{18} & -\alpha_{12} \\ 0 & \alpha_{12}\alpha_{18} & 0 & 0 \\ 0 & -\alpha_{12} & 0 & \alpha_{18} \end{bmatrix} \text{ and} \\
\phi_{15,18} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{15}\alpha_{18} & 0 \\ 0 & \alpha_{15}\alpha_{18} & 0 & -\alpha_{15} \\ 0 & 0 & -\alpha_{15} & \alpha_{18} \end{bmatrix}.
\end{aligned}$$

We then let $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{14} R_{\phi_{14}} + \alpha_{16} R_{\phi_{16}} + \alpha_{17} R_{\phi_{17}} + \frac{1}{\alpha_j \alpha_k} R_{\phi_{j,k}}$. Notice that this construction of $\hat{R}$ requires the same number of symmetric bilinear forms as in the case where α_{12}, α_{15} and α_{20} are non-zero.

In each of the above cases it is still true that $R_{2342} = \hat{R}_{2342} = \alpha_{12}$, $R_{3243} = \hat{R}_{3243} = \alpha_{15}$ and $R_{4234} = \hat{R}_{4234} = \alpha_{18}$. For sake of notation, we will write $\hat{R}$ as if we are assuming that $\alpha_i \neq 0$ unless it is explicitly stated otherwise. However, we will always show how to accomidate if $\alpha_i = 0$ for some $i = 1, ..., 18$.

Next, we will build a new symmetric bilinear form that will contribute the same information to $\hat{R}$ as $\alpha_{14} R_{\phi_{14}} + \alpha_{17} R_{\phi_{17}}$. Thus, we will substitute $R_{\phi_{14,17}}$ for $\alpha_{14} R_{\phi_{14}} + \alpha_{17} R_{\phi_{17}}$, where

$$\phi_{14,17} = \begin{bmatrix} 0 & 0 & 0 & \alpha_{14}\alpha_{17} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{14} & -\alpha_{17} \\ \alpha_{14}\alpha_{17} & 0 & -\alpha_{17} & 0 \end{bmatrix}.$$

So now,

$$\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{16} R_{\phi_{16}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}}.$$

Notice that if $\alpha_{14} = 0$ or $\alpha_{17} = 0$, then making this substitution is not necessary since writing $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{16} R_{\phi_{16}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}} + \alpha_i R_{\phi_i}$, where $i = 14$ or 17 and $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{16} R_{\phi_{16}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}}$

requires the same number of symmetric bilinear forms. After this substitution, we still have that $R \neq \hat{R}$ and this substitution leads to further differences between R and $\hat{R}$, similar to the example we cited after the first substitution. We will begin to rectify these problems by changing our coefficients and forcing $R = \hat{R}$.

For any curvature component with 4 distinct indices $R_{ijkl} = \hat{R}_{ijkl} = 0$, since $\alpha_{19} = \alpha_{20} = 0$. For any curvature component with 3 distinct indices, $R_{ijki} = \hat{R}_{ijki}$, even after the substitutions. Therefore, we will check to make sure that $R_{ijji} = \hat{R}_{ijji}$ for any 2 distinct indices i and j. Since $R_{1331} = \alpha_2 - \alpha_{17}$ and $\hat{R}_{1331} = \alpha_2$, and, since $R_{1441} = \alpha_3 - \alpha_{14}$ and $\hat{R}_{1441} = \alpha_3 - \alpha_{14}\alpha_{17}$, let

$$\tilde{\alpha}_2 = \alpha_2 - \alpha_{17}, \text{and}$$

$$\tilde{\alpha}_3 = \alpha_3 - \alpha_{14} + \alpha_{14}\alpha_{17}.$$

Define $\phi_{1,2,3}$ by

$$\phi_{1,2,3} = \begin{bmatrix} \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3 & 0 & 0 & 0 \\ 0 & \tilde{\alpha}_1 & 0 & 0 \\ 0 & 0 & \tilde{\alpha}_2 & 0 \\ 0 & 0 & 0 & \tilde{\alpha}_3 \end{bmatrix}.$$

Where, because we have not specified otherwise, it can be assumed that $\tilde{\alpha}_1 = \alpha_1$. In $\hat{R}$, we then replace $\alpha_1 R_{\phi_1} + \alpha_2 R_{\phi_2} + \alpha_3 R_{\phi_3}$ with $R_{\phi_{1,2,3}}$ so that

$$\hat{R} = \sum_{i=4}^{6} \alpha_i R_{\phi_i} + \alpha_{16} R_{\phi_{16}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}} + \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}}.$$

Now, $\hat{R}_{1221} = \alpha_1 - \alpha_{16} - \alpha_{20} = R_{1221}$, $\hat{R}_{1331} = \alpha_2 - \alpha_{17} = R_{1331}$ and $\hat{R}_{1441} = \tilde{\alpha}_3 - \alpha_{14}\alpha_{17} = \alpha_3 - \alpha_{14} = R_{1441}$, as needed. Notice that if $\alpha_i = 0$ for $i = 1, 2, 3$, then by making $\phi_{1,2,3}(e_i, e_i) = 0$ and removing α_i from the other entries of $\phi_{1,2,3}$, we have a symmetric bilinear form that still yields the information we desire.

We will now define $\tilde{\alpha}_4$, $\tilde{\alpha}_5$ and $\tilde{\alpha}_6$, to ensure $\hat{R}_{2332} = R_{2332}$, $\hat{R}_{2442} = R_{2442}$ and $\hat{R}_{3443} = R_{3443}$. Since $R_{2332} = \alpha_4 - \alpha_{18}$ and $\hat{R}_{2332} = \alpha_4 - \frac{\alpha_{12}\alpha_{15}}{\alpha_{18}} + \frac{1}{\tilde{\alpha}_3}$, $R_{2442} = \alpha_5 - \alpha_{15}$ and $\hat{R}_{2442} = \alpha_5 - \frac{\alpha_{15}\alpha_{18}}{\alpha_{12}} + \frac{1}{\tilde{\alpha}_2}$ and $R_{3443} = \alpha_6 - \alpha_{12}$ and $\hat{R}_{3443} = \alpha_6 - \frac{\alpha_{12}\alpha_{18}}{\alpha_{15}} - \frac{\alpha_{17}}{\alpha_{14}} + \frac{1}{\tilde{\alpha}_1}$, let

$$\begin{aligned} \tilde{\alpha}_4 &= \alpha_4 - \alpha_{18} + \frac{\alpha_{12}\alpha_{15}}{\alpha_{18}} - \frac{1}{\tilde{\alpha}_3} \\ \tilde{\alpha}_5 &= \alpha_5 - \alpha_{15} + \frac{\alpha_{15}\alpha_{18}}{\alpha_{12}} - \frac{1}{\tilde{\alpha}_2} \text{and} \\ \tilde{\alpha}_6 &= \alpha_6 - \alpha_{12} + \frac{\alpha_{12}\alpha_{18}}{\alpha_{15}} + \frac{\alpha_{17}}{\alpha_{14}} - \frac{1}{\tilde{\alpha}_1}. \end{aligned}$$

If $\alpha_4 = \alpha_5 = \alpha_6 = 0$, then $\hat{R} = \alpha_{16} R_{\phi_{16}} + R_{\phi_{12,15,18}} + R_{\phi_{14,17}} + R_{\phi_{1,2,3}} = R$, and we have therefore expressed R using less than or equal to 6 symmetric bilinear forms.

If $\tilde{\alpha}_i = 0$ and $\tilde{\alpha}_j = 0$ for $4 \leq i < j \leq 6$, then we can replace $\alpha_4 R_{\phi_4} + \alpha_5 R_{\phi_5} + \alpha_6 R_{\phi_6}$ with $\tilde{\alpha}_k R_{\phi_k}$ in $\hat{R}$, where $4 \leq k \leq 6$ and $k \neq i, j$. Therefore, $\hat{R} = \tilde{\alpha}_k R_{\phi_k} + \alpha_{16} R_{\phi_{16}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}} + \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}} = R$, and we have expressed R using less than or equal to 6 symmetric bilinear forms, as we set out to do.

If $\tilde{\alpha}_i = 0$ and $\tilde{\alpha}_j, \tilde{\alpha}_k \neq 0$ for $4 \leq i, j, k \leq 6$, then we can define a new symmetric bilinear form $\phi_{j,k}$, where,

$$\begin{aligned}
\phi_{4,5} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{\alpha}_4\tilde{\alpha}_5 & 0 & 0 \\ 0 & 0 & \tilde{\alpha}_4 & 0 \\ 0 & 0 & 0 & \tilde{\alpha}_5 \end{bmatrix}, \\
\phi_{4,6} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{\alpha}_4 & 0 & 0 \\ 0 & 0 & \tilde{\alpha}_4\tilde{\alpha}_6 & 0 \\ 0 & 0 & 0 & \tilde{\alpha}_6 \end{bmatrix} \text{ and} \\
\phi_{5,6} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{\alpha}_5 & 0 & 0 \\ 0 & 0 & \tilde{\alpha}_6 & 0 \\ 0 & 0 & 0 & \tilde{\alpha}_5\tilde{\alpha}_6 \end{bmatrix}.
\end{aligned}$$

We then replace $\alpha_4 R_{\phi_4} + \alpha_5 R_{\phi_5} + \alpha_6 R_{\phi_6}$ with $\frac{1}{\tilde{\alpha}_i\tilde{\alpha}_j} R_{\phi_{j,k}} - R_{\phi_i}$ in $\hat{R}$. Thus,

$$\begin{aligned}
\hat{R} = \alpha_{16} R_{\phi_{16}} \quad & + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}} \\
& + \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}} + \frac{1}{\tilde{\alpha}_i\tilde{\alpha}_j} R_{\phi_{j,k}} - R_{\phi_i} \quad = R,
\end{aligned}$$

and we have expressed R using less than or equal to 6 symmetric bilinear forms, as needed.

If $\tilde{\alpha}_4$, $\tilde{\alpha}_5$ and $\tilde{\alpha}_6$ are all non-zero, then we can then write

$$\begin{aligned}
\hat{R} = \alpha_{16} R_{\phi_{16}} \quad & + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}} R_{\phi_{12,15,18}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}} \\
& + \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}} + \frac{1}{\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\alpha}_6} R_{\phi_{4,5,6}} \quad = R,
\end{aligned}$$

where

$$\phi_{4,5,6} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{\alpha}_4\tilde{\alpha}_5 & 0 & 0 \\ 0 & 0 & \tilde{\alpha}_4\tilde{\alpha}_6 & 0 \\ 0 & 0 & 0 & \tilde{\alpha}_5\tilde{\alpha}_6 \end{bmatrix}.$$

Then we have again expressed R using less than or equal to 6 symmetric bilinear forms.

Therefore, since this exhausts all of the cases , we have shown that if $\alpha_{19} = \alpha_{20} = 0$, then we can find 6 symmetric bilinear forms such that $R = \sum_{i=1}^{6} \beta_i R_{\psi_i}$, as was our goal.

Next, we will consider the case where $\alpha_{19} \neq 0$ and $\alpha_{20} = 0$. The process will be similar to the previous case, except that our substitutions and how we redefine our constants at the end will be different.

To begin, we will define the symmetric bilinear form $\phi_{16,17}$ by

$$\phi_{16,17} = \begin{bmatrix} 0 & 0 & 0 & -\alpha_{16}\alpha_{17} \\ 0 & 0 & 0 & \alpha_{16} \\ 0 & 0 & 0 & \alpha_{17} \\ -\alpha_{16}\alpha_{17} & \alpha_{16} & \alpha_{17} & 0 \end{bmatrix}.$$

Then, let

$$\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{12} R_{\phi_{12}} + \alpha_{14} R_{\phi_{14}} + \alpha_{15} R_{\phi_{15}} + \alpha_{18} R_{\phi_{18}} + \alpha_{19} R_{\phi_{19}} + \frac{1}{\alpha_{16}\alpha_{17}} R_{\phi_{16,17}}.$$

Again, $\hat{R} \neq R$, but by changing some of our constants in our last step, as we did in the previous case, we will fix their differences. Notice, however, that $R_{4234} = \alpha_{18}$ and $\hat{R}_{4234} = \alpha_{18} - 1$, so let $\tilde{\alpha}_{18} = \alpha_{18} + 1$.

Next, we will remove $\alpha_{12} R_{\phi_{12}} + \alpha_{15} R_{\phi_{15}}$ from $\hat{R}$ and repace them with $R_{\phi_{12,15}}$, which we defined in case 1. Now,

$$\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{14} R_{\phi_{14}} + \alpha_{18} R_{\phi_{18}} + \alpha_{19} R_{\phi_{19}} + \frac{1}{\alpha_{16}\alpha_{17}} R_{\phi_{16,17}} + \frac{1}{\alpha_{12}\alpha_{15}} R_{\phi_{12,15}}.$$

Our next substitution is to remove $\alpha_{14} R_{\phi_{14}} + \alpha_{18} R_{\phi_{18}} + \alpha_{19} R_{\phi_{19}}$ from $\hat{R}$ and substitute $R_{\phi_{14,18,19}}$ in their place, where

$$\phi_{14,18,19} = \begin{bmatrix} 0 & 0 & 0 & \alpha_{14}\alpha_{19} \\ 0 & 0 & \tilde{\alpha}_{18}\alpha_{19} & 0 \\ 0 & \tilde{\alpha}_{18}\alpha_{19} & \alpha_{14}\tilde{\alpha}_{18} & 0 \\ \alpha_{14}\alpha_{19} & 0 & 0 & \alpha_{14}\tilde{\alpha}_{18} \end{bmatrix}.$$

After this substitution,

$$\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \frac{1}{\alpha_{16}\alpha_{17}} R_{\phi_{16,17}} + \frac{1}{\alpha_{12}\alpha_{15}} R_{\phi_{12,15}} + \frac{1}{\alpha_{14}\tilde{\alpha}_{18}\alpha_{19}} R_{\phi_{14,18,19}}.$$

We have assumed that $\alpha_{19} \neq 0$. If $\alpha_{14} = 0$ and $\alpha_{18} = 0$, then there is no need to make this substitution. If either $\alpha_{14} = 0$ or $\alpha_{18} = 0$, we can change our form to accomidate this fact. For example, if $\alpha_{14} = 0$, then define

$$\phi_{18,19} = \begin{bmatrix} 0 & 0 & 0 & \alpha_{19} \\ 0 & 0 & \tilde{\alpha}_{18}\alpha_{19} & 0 \\ 0 & \tilde{\alpha}_{18}\alpha_{19} & 0 & 0 \\ \alpha_{19} & 0 & 0 & \tilde{\alpha}_{18} \end{bmatrix}.$$

Then, $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \frac{1}{\alpha_{16}\alpha_{17}} R_{\phi_{16,17}} + \frac{1}{\alpha_{12}\alpha_{15}} R_{\phi_{12,15}} + \frac{1}{\tilde{\alpha}_{18}\alpha_{19}} R_{\phi_{18,19}}$. We will now evaluate the differences between R and $\hat{R}$ and change some of our coefficients to force $R = \hat{R}$.

This last step will be almost identical to the last step in the first case. The only difference will be how we define out constants. Again, if i, j, k, l are distinct indices, then $R_{ijkl} = \hat{R}_{ijkl}$ and $R_{ijki} = \hat{R}_{ijki}$. We will let

$$\begin{aligned} \tilde{\alpha}_1 &= \alpha_1 - \alpha_{16}, \\ \tilde{\alpha}_2 &= \alpha_2 - \alpha_{17} \text{and} \\ \tilde{\alpha}_3 &= \alpha_3 - \alpha_{14} - \alpha_{19} + \alpha_{16}\alpha_{17} + \frac{\alpha_{14}\alpha_{19}}{\alpha_{18}}. \end{aligned}$$

Again, we will replace $\alpha_1 R_{\phi_1} + \alpha_2 R_{\phi_2} + \alpha_3 R_{\phi_3}$ with $R_{\phi_{1,2,3}}$ in $\hat{R}$. Now,

$$\hat{R} = \sum_{i=4}^{6} \alpha_i R_{\phi_i} + \frac{1}{\alpha_{16}\alpha_{17}} R_{\phi_{16,17}} + \frac{1}{\alpha_{12}\alpha_{15}} R_{\phi_{12,15}} + \frac{1}{\tilde{\alpha}_{18}\alpha_{19}} R_{\phi_{18,19}} + \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}}.$$

Notice that now $\hat{R}_{1221} = \alpha_1 - \alpha_{16} - \alpha_{20} = R_{1221}$, $\hat{R}_{1331} = \alpha_2 - \alpha_{17} = R_{1331}$ and $\hat{R}_{1441} = \tilde{\alpha}_3 - \alpha_{16}\alpha_{17} - \frac{\alpha_{14}\alpha_{19}}{\alpha_{18}} = \alpha_3 - \alpha_{14} - \alpha_{19} = R_{1441}$.

Now, let

$$\begin{aligned} \tilde{\alpha}_4 &= \alpha_4 - \alpha_{18} - \alpha_{19} - \frac{\alpha_{12}}{\alpha_{15}} + \frac{\tilde{\alpha}_{18}\alpha_{19}}{\alpha_{14}} - \frac{1}{\tilde{\alpha}_3}, \\ \tilde{\alpha}_5 &= \alpha_5 - \alpha_{15} + \frac{\alpha_{16}}{\alpha_{17}} + \alpha_{12}\alpha_{15} - \frac{1}{\tilde{\alpha}_2} \text{and} \\ \tilde{\alpha}_6 &= \alpha_6 - \alpha_{12} + \frac{\alpha_{17}}{\alpha_{16}} - \frac{\alpha_{14}\tilde{\alpha}_{18}}{\alpha_{19}} - \frac{1}{\tilde{\alpha}_1} \end{aligned}$$

and write

$$\hat{R} = \frac{1}{\alpha_{16}\alpha_{17}} R_{\phi_{16,17}} + \frac{1}{\alpha_{12}\alpha_{15}} R_{\phi_{12,15}} + \frac{1}{\alpha_{14}\tilde{\alpha}_{18}\alpha_{19}} R_{\phi_{14,18,19}} + \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}} + \frac{1}{\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\alpha}_6} R_{\phi_{4,5,6}}.$$

It is easy to check that $\hat{R} = R$, and so we have found less than or equal to 6 symmetric bilinear forms $\psi_1, ..., \psi_6$ such that $R = \sum_{i=1}^{6} \beta_i \psi_i$, as needed.

Our next case is the case in which $\alpha_{19} = 0$ and $\alpha_{20} \neq 0$. To begin this case, let $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{12} R_{\phi_{12}} + \alpha_{14} R_{\phi_{14}} + \alpha_{16} R_{\phi_{16}} + \alpha_{17} R_{\phi_{17}} + \alpha_{20} R_{\phi_{20}} + \frac{1}{\alpha_{15}\alpha_{18}} R_{\phi_{15,18}}$, where $\phi_{15,18}$ is as it was defined in the first case.

Next, define

$$\phi_{12,16,20} = \begin{bmatrix} 0 & \alpha_{16}\alpha_{20} & 0 & 0 \\ \alpha_{16}\alpha_{20} & \alpha_{12}\alpha_{16} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{12}\alpha_{20} \\ 0 & 0 & \alpha_{12}\alpha_{20} & \alpha_{12}\alpha_{16} \end{bmatrix}.$$

We will replace $\alpha_{12} R_{\phi_{12}} + \alpha_{16} R_{\phi_{16}} + \alpha_{20} R_{\phi_{20}}$ in $\hat{R}$ with $R_{\phi_{12,16,20}}$ so that $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{14} R_{\phi_{14}} + \alpha_{17} R_{\phi_{17}} + \frac{1}{\alpha_{15}\alpha_{18}} R_{\phi_{15,18}} + \frac{1}{\alpha_{12}\alpha_{16}\alpha_{20}} R_{\phi_{12,16,20}}$. Similarly to when we constructed $\phi_{14,18,19}$, if $\alpha_{12} = 0$ or $\alpha_{16} = 0$, we can construct $\phi_{16,20}$ or $\phi_{14,20}$ to accomodate this fact.

Our next step is to replace $\alpha_{14} R_{\phi_{14}} + \alpha_{17} R_{\phi_{17}}$ with $R_{\phi_{14,17}}$, which we defined in case 1. This leaves us with

$$\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \frac{1}{\alpha_{15}\alpha_{18}} R_{\phi_{15,18}} + \frac{1}{\alpha_{12}\alpha_{16}\alpha_{20}} R_{\phi_{12,16,20}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}}.$$

The only remaining step to show that we can write $R = \sum_{i=1}^{6} \beta_i R_{\psi_i}$ is essential the same to the last step in the first two cases. The only difference, once again, is how we define our constants. First, let

$$\begin{aligned}
\tilde{\alpha}_1 &= \alpha_1 - \alpha_{16} - \alpha_{20} + \frac{\alpha_{16}\alpha_{20}}{\tilde{\alpha}_{12}}, \\
\tilde{\alpha}_2 &= \alpha_2 - \alpha_{17} \text{and} \\
\tilde{\alpha}_3 &= \alpha_3 - \alpha_{14} + \alpha_{14}\alpha_{17}.
\end{aligned}$$

Then, let

$$\begin{aligned}
\tilde{\alpha}_4 &= \alpha_4 - \alpha_{18} + \alpha_{15}\alpha_{18} - \frac{1}{\tilde{\alpha}_3}, \\
\tilde{\alpha}_5 &= \alpha_5 - -\alpha_{15} - \frac{\tilde{\alpha}_{12}\alpha_{16}}{\alpha_{20}} - \frac{1}{\tilde{\alpha}_2} \text{and} \\
\tilde{\alpha}_6 &= \alpha_6 - \alpha_{12} - \alpha_{20} + \frac{\alpha_{15}}{\alpha_{18}} + \frac{\tilde{\alpha}_{12}\alpha_{20}}{\alpha_{16}} + \frac{\alpha_{17}}{\alpha_{14}} - \frac{1}{\tilde{\alpha}_1}.
\end{aligned}$$

If we define

$$\hat{R} = \frac{1}{\alpha_{15}\alpha_{18}} R_{\phi_{15,18}} + \frac{1}{\alpha_{12}\alpha_{16}\alpha_{20}} R_{\phi_{12,16,20}} + \frac{1}{\alpha_{14}\alpha_{17}} R_{\phi_{14,17}} + \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}} + \frac{1}{\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\alpha}_6} R_{\phi_{4,5,6}},$$

then it is easy to check that $R = \hat{R}$ and so we have found less than or equal to 6 symmetric bilinear forms such that, as needed.

Our last case is that in which $\alpha_{19} \neq 0$ and $\alpha_{20} \neq 0$. This case then splits into 2 new cases.

If $\alpha_{19} \neq 1$, then let

$$\phi_{15,17} = \begin{bmatrix} 0 & 0 & 0 & \alpha_{17} \\ 0 & 0 & \alpha_{15} & 0 \\ 0 & \alpha_{15} & 0 & -\alpha_{15}\alpha_{17} \\ \alpha_{17} & 0 & -\alpha_{15}\alpha_{17} & 0 \end{bmatrix}.$$

Then, define $\hat{R} = \sum_{i=1}^{6} \alpha_i R_{\phi_i} + \alpha_{12} R_{\phi_{12}} + \alpha_{14} R_{\phi_{14}} + \alpha_{16} R_{\phi_{14}} + \alpha_{18} R_{\phi_{18}} + \alpha_{19} R_{\phi_{19}} + \alpha_{20} R_{\phi_{20}} + \frac{1}{\alpha_{15}\alpha_{17}} R_{\phi_{15,17}}$. Notice that $R_{3243} = \hat{R}_{3243}$ and $R_{4134} = \hat{R}_{4134}$. However, $R_{1234} = \alpha_{19}$ and $\hat{R}_{1234} =$

$\alpha_{19}+1$. Thus, let $\tilde{\alpha}_{19}=\alpha_{19}-1$. Since we assumed that $\alpha_{19}\neq 1$, we know that $\tilde{\alpha}_{19}\neq 0$.

If we continue to redefine $\hat{R}$ so that we have

$$\hat{R}=\sum_{i=1}^{6}\alpha_i R_{\phi_i}+\frac{1}{\alpha_{15}\alpha_{17}}R_{\phi_{15,17}}+\frac{1}{\alpha_{12}\alpha_{16}\alpha_{20}}R_{\phi_{12,16,20}}+\frac{1}{\alpha_{14}\alpha_{18}\tilde{\alpha}_{19}}R_{\phi_{14,18,19}},$$

where $\phi_{12,16,20}$ and $\phi_{14,18,19}$ are defined as in previous cases, the only difference being that α_{19} is replaced with $\tilde{\alpha}_{19}$, then $R_{ijkl}=\hat{R}_{ijkl}$ and $R_{ijki}=\hat{R}_{ijki}$ for any distinct indices $1\leq i,j,k,l\leq 4$. We will proceed to define our constants. In this case, let

$$\begin{aligned}\tilde{\alpha}_1 &= \alpha_1-\alpha_{16}-\alpha_{20}+\frac{\alpha_{16}\alpha_{20}}{\alpha_{12}},\\ \tilde{\alpha}_2 &= \alpha_2-\alpha_{17}\text{and}\\ \tilde{\alpha}_3 &= \alpha_3-\alpha_{14}-\tilde{\alpha}_{19}+\frac{\alpha_{17}}{\alpha_{15}}+\frac{\alpha_{14}\tilde{\alpha}_{19}}{\alpha_{18}}.\end{aligned}$$

Then, let

$$\begin{aligned}\tilde{\alpha}_4 &= \alpha_4-\alpha_{18}-\alpha_{19}+\frac{\alpha_{15}}{\alpha_{17}}+\frac{\alpha_{18}\tilde{\alpha}_{19}}{\alpha_{14}}-\frac{1}{\tilde{\alpha}_3},\\ \tilde{\alpha}_5 &= \alpha_5-\alpha_{15}-\frac{\alpha_{12}\alpha_{16}}{\alpha_{20}}-\frac{1}{\tilde{\alpha}_2}\text{and}\\ \tilde{\alpha}_6 &= \alpha_6-\alpha_{12}-\alpha_{20}+\alpha_{15}\alpha_{17}+\frac{\alpha_{12}\alpha_{20}}{\alpha_{16}}-\frac{\alpha_{14}\alpha_{18}}{\tilde{\alpha}_{19}}-\frac{1}{\tilde{\alpha}_1}.\end{aligned}$$

By letting

$$\begin{aligned}\hat{R}=\tfrac{1}{\alpha_{15}\alpha_{17}}R_{\phi_{15,17}} &+\tfrac{1}{\alpha_{12}\alpha_{16}\alpha_{20}}R_{\phi_{12,16,20}}+\tfrac{1}{\alpha_{14}\alpha_{18}\tilde{\alpha}_{19}}R_{\phi_{14,18,19}}\\ &+\tfrac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3}R_{\phi_{1,2,3}}+\tfrac{1}{\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\alpha}_6}R_{\phi_{4,5,6}},\end{aligned}$$

it is easy to check that $R=\hat{R}$ and thus we have written R using less than or equal to 6 symmetric bilinear forms, as needed.

The other case is where $\alpha_{19}=1$. In this case, let

$$\hat{\phi}_{15,17}=\begin{bmatrix}0&0&0&\alpha_{17}\\0&0&\alpha_{15}&0\\0&\alpha_{15}&0&-2\alpha_{15}\alpha_{17}\\\alpha_{17}&0&-2\alpha_{15}\alpha_{17}&0\end{bmatrix}.$$

We will then let $\tilde{\alpha}_{19}=\frac{\alpha_{19}}{2}$. Since $\alpha_{19}\neq 0$ by assumption, $\tilde{\alpha}_{19}\neq 0$. We then define $\hat{R}=\sum_{i=1}^{6}\alpha_i R_{\phi_i}+\alpha_{12}R_{\phi_{12}}+\alpha_{14}R_{\phi_{14}}+\alpha_{16}R_{\phi_{14}}+\alpha_{18}R_{\phi_{18}}+\alpha_{19}R_{\phi_{19}}+\alpha_{20}R_{\phi_{20}}+\frac{1}{2\alpha_{15}\alpha_{17}}R_{\phi_{15,17}}$, and so, $R_{1234}=\hat{R}_{1234}$.

If we continue to redefine $\hat{R}$ so that we have

$$\hat{R}=\sum_{i=1}^{6}\alpha_i R_{\phi_i}+\frac{1}{2\alpha_{15}\alpha_{17}}R_{\phi_{15,17}}+\frac{1}{\alpha_{12}\alpha_{16}\alpha_{20}}R_{\phi_{12,16,20}}+\frac{1}{\alpha_{14}\alpha_{18}\tilde{\alpha}_{19}}R_{\phi_{14,18,19}},$$

where $\phi_{12,16,20}$ and $\phi_{14,18,19}$ are defined as in previous cases, the only difference being that α_{19} is replaced with $\tilde{\alpha}_{19}$, then $R_{ijkl} = \hat{R}_{ijkl}$ and $R_{ijki} = \hat{R}_{ijki}$ for any distinct indices i, j, k, l. We will proceed to define our constants. In this case, let

$$\begin{aligned}
\tilde{\alpha}_1 &= \alpha_1 - \alpha_{16} - \alpha_{20} + \frac{\alpha_{16}\alpha_{20}}{\alpha_{12}}, \\
\tilde{\alpha}_2 &= \alpha_2 - \alpha_{17} \text{and} \\
\tilde{\alpha}_3 &= \alpha_3 - \alpha_{14} - \tilde{\alpha}_{19} + \frac{\alpha_{17}}{2\alpha_{15}} + \frac{\alpha_{14}\tilde{\alpha}_{19}}{\alpha_{18}}.
\end{aligned}$$

Then, let

$$\begin{aligned}
\tilde{\alpha}_4 &= \alpha_4 - \alpha_{18} - \alpha_{19} + \frac{\alpha_{15}}{2\alpha_{17}} + \frac{\alpha_{18}\tilde{\alpha}_{19}}{\alpha_{14}} - \frac{1}{\tilde{\alpha}_3}, \\
\tilde{\alpha}_5 &= \alpha_5 - \alpha_{15} - \frac{\alpha_{12}\alpha_{16}}{\alpha_{20}} - \frac{1}{\tilde{\alpha}_2} \text{and} \\
\tilde{\alpha}_6 &= \alpha_6 - \alpha_{12} - \alpha_{20} + 2\alpha_{15}\alpha_{17} + \frac{\alpha_{12}\alpha_{20}}{\alpha_{16}} - \frac{\alpha_{14}\alpha_{18}}{\tilde{\alpha}_{19}} - \frac{1}{\tilde{\alpha}_1}.
\end{aligned}$$

By letting

$$\begin{aligned}
\hat{R} = \tfrac{1}{2\alpha_{15}\alpha_{17}} R_{\phi_{15,17}} &+ \tfrac{1}{\alpha_{12}\alpha_{16}\alpha_{20}} R_{\phi_{12,16,20}} + \tfrac{1}{\alpha_{14}\alpha_{18}\tilde{\alpha}_{19}} R_{\phi_{14,18,19}} \\
&+ \tfrac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3} R_{\phi_{1,2,3}} + \tfrac{1}{\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\alpha}_6} R_{\phi_{4,5,6}},
\end{aligned}$$

it is easy to check that $R = \hat{R}$ and thus we have written R using less than or equal to 6 symmetric bilinear forms.

This then covers all of our cases. Since in each case we have shown that, given R, we can find six symmetric bilinear forms $\psi_1, \cdots, \psi_6$ such that $R = \sum_{i=1}^{6} \beta_i R_{\psi_i}$, we have shown that $\nu(4) \leq 6$, as we set out to. □

Remark 3.2. *It is interesting to note that the only time we actually needed 6 symmetric bilinear forms to construct R was when $\tilde{\alpha}_i = 0$ and $\tilde{\alpha}_j, \tilde{\alpha}_k \neq 0$ for $4 \leq i, j, k \leq 6$. If R is defined so that this does not happen, then $\nu(R) \leq 5$.*

4 Summary

We found a basis for algebraic curvature tensors on a 4-dimensional vector space and were able to generalize this for a vector space of dimension m. In constructing the basis for dimension 4, the most important observation was that $\dim(\mathcal{A}(V))$ is derived from how many independent curvature components determine an algebraic curvature tensor in V. Building a different symmetric bilinear form to represent each of these curvature components yields the basis. In dimension m, it is easy to find the forms because they look similar to those we used in dimension 4, especially the forms that correspond to R_{ijkl}.

We were also successful in reducing the upper bound on $\nu(4)$ from 10 to 6. Ignoring some of the details - the details were very explicitly shown above - the process was as follows.

If $\alpha_{19} = 0$ and $\alpha_{20} = 0$, we define $\phi_{14,17}$ and $\phi_{12,15,18}$ to contribute the same information as $\alpha_{12}R_{\phi_{12}} + \alpha_{14}R_{\phi_{14}} + \alpha_{15}R_{\phi_{15}} + \alpha_{17}R_{\phi_{17}} + \alpha_{18}R_{\phi_{18}}$. By letting $\hat{R} = \frac{1}{\alpha_{14}\alpha_{17}}R_{\phi_{14,17}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}}R_{\phi_{12,15,18}} + \alpha_{16}R_{\phi_{16}}$, we have constructed a curvature tensors such that $\hat{R}_{ijkl} = R_{ijkl}$ and $\hat{R}_{ijki} = R_{ijki}$, but $\hat{R}_{ijji} \neq R_{ijji}$. To fix this, we define $\tilde{\alpha}_1, ..., \tilde{\alpha}_6$ such that

$$\hat{R} = \sum_{i=1}^{6} \tilde{\alpha}_i R_{\phi_i} + \frac{1}{\alpha_{14}\alpha_{17}}R_{\phi_{14,17}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}}R_{\phi_{12,15,18}} + \alpha_{16}R_{\phi_{16}} = R$$

. Then, define $\phi_{1,2,3}$ and $\phi_{4,5}$ such that

$$\hat{R} = \frac{1}{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3}R_{\phi_{1,2,3}} + \frac{1}{\tilde{\alpha}_4\tilde{\alpha}_5}R_{\phi_{4,5}} + \tilde{\alpha}_6 R_{\phi_6} + \frac{1}{\alpha_{14}\alpha_{17}}R_{\phi_{14,17}} + \frac{1}{\alpha_{12}\alpha_{15}\alpha_{18}}R_{\phi_{12,15,18}} + \alpha_{16}R_{\phi_{16}} = R$$

There are many nuances to the proof, such as when certain $\alpha_i = 0$. There are also other cases with other values of α_{19} and α_{20}, but the process is similar.

5 Open Questions

We end with some open questions that arose from this research.

1. Is $\nu(4) = 6$? Can we decrease the upper bound of $\nu(m)$ by a similar amount as m grows? Can we find a new upper bound for $\nu(m)$ in general?

2. Can we find a general way to determine $\tilde{\nu}(m)$, which denotes $sup(\nu(R))$, where R is a specific kind of algebraic curvature tensor (i.e. Einstein, IP, etc.)? We know that in dimension 4, the Singer-Thorpe basis exists for Einstein tensors. This basis send all but a few curvature components to zero and even forces some equalities among the remaining components.

   Also, because the Ricci tensor depends on an orthonormal basis, and the Ricci tensor is not affected by the value of R_{ijkl}, it seem the dimension of all Einstein tensors on a vector space should be however many components are needed to determine if the Ricci tensor is diagonal with only 1 eigenvalue plus $2\binom{m}{4}$ for the components of the form R_{ijkl}.

3. Can we find a basis-free way to determine when $\nu(R) = 1, 2, 3, 4, 5$ or 6 in dimension 4? In [2], they show that in dimension 3, $\nu(R)$ dependes on the Eigenvalues of the Ricci tensor. Can we find a similar result in dimension 4?

4. It is obvious that if we can write

$$R = \sum_{i=1}^{\nu(R)} \beta_i R_{\psi_i}$$

and $Rank(\psi_j) = 2$ for some $1 \leq j \leq \nu(R)$, then the linear combination of $\nu(R)$ symmetric bilinear forms which we used to construct R is not unique.

Can we always write

$$R = \sum_{i=1}^{\nu(R)} \beta_i R_{\psi_i}$$

such that $Rank(\psi_i) \geq 3$ for all i? If we can, then is this construction of R unique?

6 References

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