Ropelength and Lissajous Diagrams

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Abstract

Investigate ropelengths of the family of knots known as French Sinnet knots. It is shown that the ropelength of this family of knots is linearly bounded by the crossing number. This result is expanded to include all Lissajous knots.

Keywords: Lissajous knots; French Sinnet knots; Fourier Knots; crossing number; ropelength; cubic lattice;

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1 Introduction

First, it is necessary to define a few key terms to establish an adequate background. A *knot* is a simple, closed curve in Euclidean 3-space. A *knot diagram* or *projection* is an image that represents a particular knot. An example of this can be seen with the figure-eight knot in figure 1. The *crossing number*,

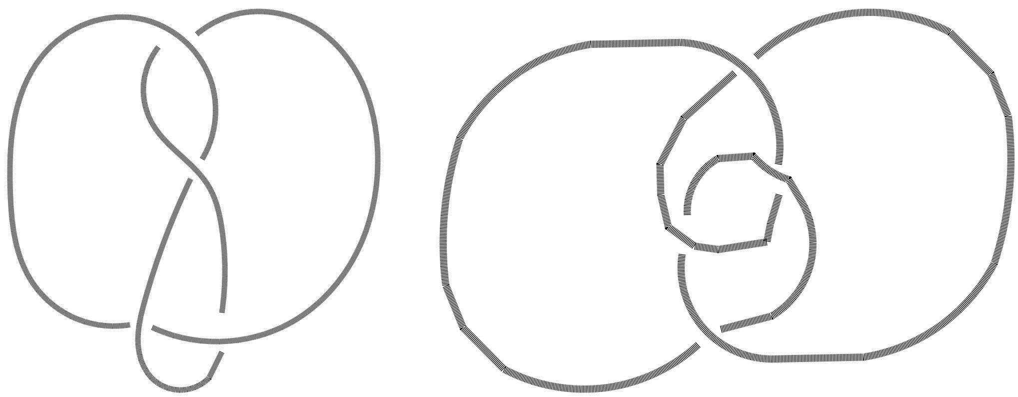

Figure 1: two projections of the figure-eight knots

denoted $Cr(K)$, of a knot type is the minimum number of crossings over all projections. An *alternating diagram* alternates between over and undercrossings as you traverse the diagram. It is worth noting that an alternating diagram has the minimum number of crossings over all projections.

A *polygonal knot* is a knot made from line segments. An example of a polygonal knot can be seen in figure 2. A *cubic lattice knot* is a polygonal knot

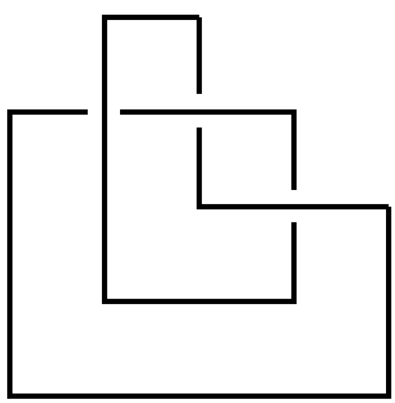

Figure 2: Polygonal knot

with all vertices on $\mathbb{Z}^3$ and all edges length 1. An example of a cubic lattice knot can be seen in figure 3. The *ropelength* of a knot K, denoted $Rop(K)$, where $Rop(K) = \frac{\ell(K)}{r(K)}$. $\ell(K)$ is the *arclength* of K, and $r(K)$ is the *radius* of K. Since we can assume without loss of generality that $r(K) = 1$, we see that $Rop(K) = \ell(K)$. The cubic lattice is useful for finding upperbounds on ropelength, because the length of a knot on a cubic lattice is always greater than or equal to the original length. Then when we find an upperbound for the length of a knot of the cubic lattice, we have found an upperbound for $Rop(K)$.

Example 1.1 *Here is an example of a knot that has been drawn on the cubic lattice. The under- and overcrossings have been preserved, and the black dots indicate places where a line segment is coming out of the page to lift a piece above another. The length of the side of each square is 2 so that we can place the line segments going up in the middle of them. Thus since the base diagram has 24 line segments, it has a length of 48. Then we also need to take into consideration the 14 line segments coming out of the page, which gives us a total length of 62 for the diagram.*

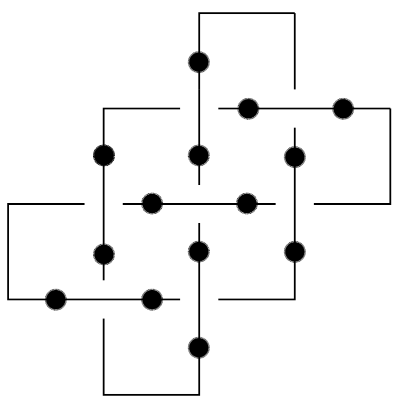

Figure 3: Cubic Lattice

2 French Sinnet Knots

In this section, a class of Lissajous Knots is discussed and named. First we define a Lissajous knot.

Definition 2.1 *Knots which can be parameterized using*

$$x(t) = A_x cos(B_x t + C_x),$$

$$y(t) = A_y cos(B_y t + C_y),$$

$$z(t) = A_z cos(B_z t + C_z).$$

are called Lissajous knots.

We can set $A_x = A_y = A_z = 1$ and $C_z = 0$, because this only changes the amplitudes, which does not change the topology of the knot [2].

Definition 2.2 *A French Sinnet knot is a Lissajous knot K with coprime frequencies $B_y \geq B_x \geq 1$ and $B_z = 2b_xB_y - B_x - B_y$, and phases $C_x = \pi\frac{(2B_x-1)}{(2B_z)}$ and $C_y = \frac{\pi}{2B_z}$.*

This family of Lissajous knots are called French Sinnet knots, because they are a plat closure of French Sinnet braids defined by Kohno [3]. When viewed along the z-axis, K, a French Sinnet knot, has an alternating diagram [2].

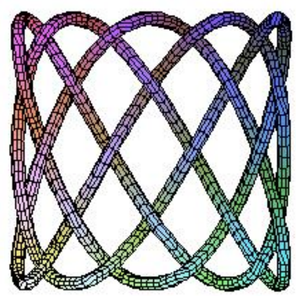

Figure 4: $B_x = 3, B_y = 5$ French Sinnet knot

From Section 2.1 of [1] we see that a Lissajous knot K has $2B_xB_y - B_x - B_y$ crossings. Since French Sinnet knots are a class of Lissajous knots, this also applies to French Sinnet knots. However, since French Sinnet knots are alternating, the number of crossings becomes the crossing number, $Cr(K)$.

Example 2.3 *As an example, we show the crossing number for French Sinnet knot with $B_x = 3, B_y = 4$.*

$$\begin{aligned} Cr(K) &= 4(3-1) + (4-1)(3) \\ &= 4 \cdot 2 + 3 \cdot 3 \\ &= 8 + 9 = 17 \end{aligned}$$

We can see from the cubic lattice in Figure 5 that for a constant B_x it appears that we can calculate how many more segments will need to be added for each additional B_y. We start with the base case when $B_x = 2, B_y = 3$ as in Figure 5, and remembering that each line segment in the diagram is actually length 2 so that we can add segments to raise segments at crossings, we see that the diagram has $24 \cdot 2 = 48$ base segments plus 14 segments coming out of the page. Thus the $B_x = 2, B_y = 3$ case has length 62.

French Sinnet knots and Lissajous knots both require that B_x, B_y be relatively prime. For the following proofs, when induction is used, the case when B_x, B_y are not relatively prime is still acceptable, because this only means that the French Sinnet knots and Lissajous knots become French Sinnet links and Lissajous links. The equations will still work as expected. Thus, in the following diagram of the cubic lattice for the French Sinnet knot where $B_x = 2$ and $B_y = 3$, we then see the next case which is actually a French Sinnet link where $B_x = 2$ and $B_y = 4$.

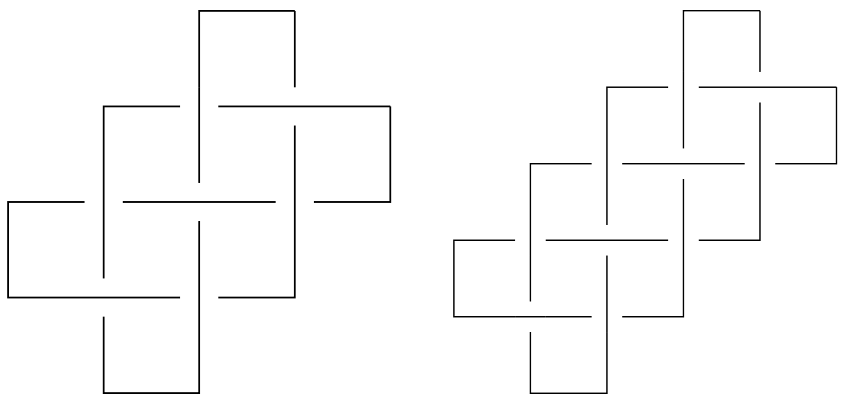

Figure 5: $B_x = 2, B_y = 3$ and $B_x = 2, B_y = 4$ Cubic Lattice

We know from earlier that for $B_x = 2, B_y = 3$ the length of the lattice will be 62. Then we can look at the diagram for $B_x = 2, B_y = 4$ and see that is has $32 \cdot 2 = 64$ base segments and 20 segments coming out of the page for a total length of 84. We see that this requires 22 additional segments when B_y is increased by 1.

Lemma 2.4 *For each additional B_y, the cubic lattice length of the French Sinnet knot becomes $12B_x - 2$ units longer.*

Proof. As seen in figure 5, the only addition to the cubic lattice for $B_y = 4$ is an additional row of B_x squares. Thus there are $2 \cdot 4B_x$ additional base pieces, since each piece is of length 2, and there are $B_x + (B_x - 1)$ additional crossings. This means that for each additional B_y, the Lissajous knot is

$$\begin{aligned} & 2 \cdot 4B_x + 2[B_x + (B_x - 1)] \\ = \quad & 8B_x + 4B_x - 2 \\ = \quad & 12B_x - 2 \end{aligned}$$

units longer. The number of crossings is multiplied by 2, because for each overcrossing, the line segment must be raised which requires two additional raising segments. ∎

Lemma 2.5 *A French Sinnet knot K where $B_x = 2$, can be constructed on the cubic lattice using $\ell(K) = 22B_y - 4$ edges.*

Proof. This is proved by induction.
$P(3) : 22(3) - 4 = 62$ which is the length of the cubic lattice of a Lissajous knot with $B_y = 3$. We start with 3, because it is a requirement of French Sinnet knots that $B_y > B_x > 1$.
Next we want to show that assuming the $P(n)$ case works, the $P(n+1)$ case will work.

$22(n+1)-4 = 22n+22-4 = (22n-4)+22 = \ell(K_n)+22$ by the inductive hypothesis

Therefore by Lemma 2.4 we see that this is equal to $\ell(K_{n+1})$. ■

Lemma 2.6 *For each additional B_x, the cubic lattice length of the French Sinnet knot becomes $12B_y - 2$ units longer.*

Proof. As seen in figure , the only addition to the cubic lattice for $B_x = 4$ is an additional row of B_y squares. Thus there are $2 \cdot 4B_y$ additional base pieces, since each piece is of length 2, and there are $B_y + (B_y - 1)$ additional crossings. This means that for each additional B_x, the Lissajous knot is

$$\begin{aligned} & 2 \cdot 4B_y + 2[B_y + (B_y - 1)] \\ = \;& 8B_y + 4B_y - 2 \\ = \;& 12B_y - 2 \end{aligned}$$

units longer. The number of crossings is multiplied by 2, because for each overcrossing, the line segment must be raised which requires two additional raising segments. ■

Theorem 2.7 *For a French Sinnet knot K, the length of the knot is denoted $\ell(K)$ such that $\ell(K) = 2[4B_xB_y + Cr(K)]$.*

Proof. We will prove this using double induction. For the initial case we will use $P(2, 3)$, since $B_y > B_x > 1$. By lemma 2.5, we know that the equation for length for the initial case is $\ell(K) = 22B_y - 4$. After plugging in our B_y, we see that $\ell(K) = 22(3) - 4 = 62$, which we know is true from our diagram.
Next we want to show that assuming $P(2, B_y)$ is true, $P(2, B_y+1)$ is true. This is proved by Lemma 2.5, since we are keeping $B_x = 2$, and letting $B_y = n + 1$.
Now we want to assume that $P(B_x, B_y)$ is true, and show that $P(B_x + 1, B_y)$ is true. $P(B_x + 1, B_y) = 2[4(B_x + 1)B_y + Cr(K_{B_x+1})]$. Then

$$\begin{aligned} P(B_x + 1, B_y) &= 2[4(B_x + 1)B_y + Cr(K_{B_x+1})] \\ &= 2[4B_xB_y + 4B_y + 2B_y(B_x + 1) - B_y - (B_x + 1)] \text{ by Proposition ??} \\ &= 2[6B_xB_y + 6B_y - B_y - B_x - 1] \\ &= 2[4B_xB_y + Cr(K)] + 12B_y - 2 \\ &= \ell(K) + 12B_y - 2 \text{ by the inductive hypothesis} \\ &= \ell(K_{B_x+1,B_y}) \end{aligned}$$

■

This result is improved upon by Corollary 3.5 in the next section.

3 Lissajous Knots

For alternating knot diagrams, we know that the number of crossings is the minimum number over all projections, and thus is the crossing number of the knot. In this aspect, it was useful to use French Sinnet knots to develop the equations from the previous sections. However, some Lissajous knots are nonalternating, and the equations for additional B_x and B_y still work for nonalternating knots. Thus we define a new variable c as follows:

Definition 3.1 *Let K be a Lissajous knot. Then $c(K)$ is the number of crossings in the Lissajous diagram of K.*

The following two lemmas are used to prove the theorem at the end of the section.

Lemma 3.2 *For $B_y > B_x > 1, B_y B_x > B_y + B_x$.*

Proof. Since $B_y > B_x > 1$ and $B_x - 1 \geq$
$B_y(B_x - 1) \geq B_y$, then
$B_y(B_x - 1) - B_x \geq B_y \geq B_y - B_x > 0$.
Thus, $B_y B_x - B_y - B_x > 0$, and $B_y B_x > B_y + B_x$. ■

Lemma 3.3 $B_x B_y \leq c$

Proof.

$$\begin{aligned} c &= 2B_xB_y - B_x - B_y \text{ by Section 2.1 of [1]} \\ &= 2B_xB_y - (B_x + B_y) \\ &> 2B_xB_y - B_xB_y \text{ by lemma 3.2} \\ &= B_xB_y \end{aligned}$$

■

Theorem 3.4 *If K has a Lissajous diagram with c crossings, then the ropelength of K is $O(c)$.*

Proof.

$$\begin{aligned} \ell(K) &\leq 2[B_xB_y + c] \\ &= 2B_xB_y + 2c \\ &\leq 2c + 2c \text{ by lemma 3.3} \\ &= 4c \end{aligned}$$

■

Corollary 3.5 *If K is a French Sinnet knot with crossing number $Cr(K)$, then the ropelength of K is $O(Cr(K))$.*

4 Crossing Number

It is known that not all knots are Lissajous knots, for example, the trefoil knot. However, it is known that all knots are $Fourier - (1,1,k)$ knots by Theorem 3.3 in [4]. A $Fourier - (1,1,k)$ knot is one which can be represented by the parametric equations

$$\begin{aligned} x(t) &= A_x cos(B_x t + C_x), \\ y(t) &= A_y cos(B_y t + C_y), \\ z(t) &= A_{z_1} cos(B_{z_1} t + C_{z_1}) + \ldots + A_{z_k} cos(B_{z_k} t + C_{z_k}). \end{aligned}$$

Since a Lissajous diagram is determined by the projection in the xy plane, we see that this shows that all knots have a Lissajous diagram. In this section we will investigate the relationship between the crossing number of a knot K and the number of crossings in its Lissajous diagram while discussing Lamm's proof. First we consider the case for two-bridge knots.

Lamm begins with a theorem that if $\alpha \in B_s$ is a braid, with a closure a knot, then α is conjugate to a rosette braid, and he follows this with Corollary 1.4 to say that every knot K is the plat closure of a rosette braid with $br(K)$ strings. He goes on to define a **checkerboard diagram**, and says a knot diagram is called a checkerboard diagram of type $(2b, n)$, if it is the plat closure of a braid $\sigma_2^{\epsilon_2} \ldots \sigma_{2b-2}^{\epsilon_{2b-2}} \cdot \alpha$ with $\alpha \in R(2b, n)$ and $\epsilon_2, \ldots, \epsilon_{2b-2} \in \pm 1$

The group of pure braid group generators for a two-bridge knot is $R(4,4) = \{A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}\}$. The braid images of these generators can be seen in figure 6.

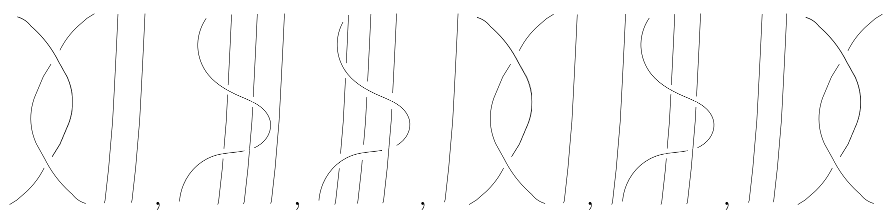

Figure 6: The pure braid group generators

We can make each of the pure braid group generators into a checkerboard diagram. Once this is done, we see that each of them has 12 crossings in this form.

Let K be a two-bridge knot where K is the plat closure of a braid α. For two-bridge knots $\pi_0 = (2,3)$, and by Lemma 2.2 in [4], we see that there is a sequence of operations which transforms $\bar{\alpha}$ to $\bar{\beta}$ such that $\pi(\beta) = \pi_0$. Then in the proof of Theorem 2.3 in [4], we see that for two-bridge knots we can create a pure braid $\beta = \sigma_2 \alpha$ where $\pi(\alpha) = \pi_0$. Since β is a pure braid, by definition it can be written in terms of the pure braid generators.

Conjecture 4.1 *Let the number of pure braid generators needed to generate β be denoted $g(\beta)$. Then $g(\beta) \leq Cr(K)$.*

Conjecture 4.2 *Let the number of crossings in the checkerboard diagram for a knot K be $cc(K)$. Then $cc(K) \leq 12Cr(K) + 1$.*

Example 4.1

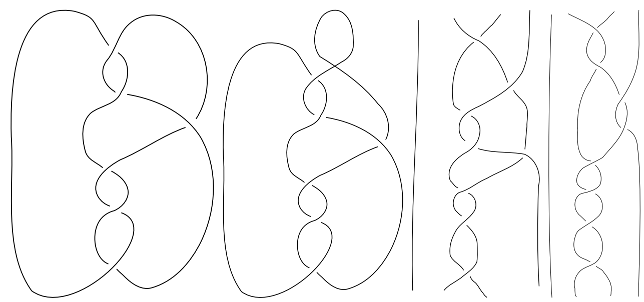

Figure 7: Two-bridge example

We see that the knot above can be generated by 4 of the pure braid generators using Lamm's method. Since each generator has 12 crossings in the checkerboard diagram and there is an additional σ^{-1}, we see that this knot will have $4 \cdot 12 + 1 =$ 49 *crossings in the checkerboard diagram.*

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