Algebraic Curvature Tensors and Antisymmetric Forms

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ABSTRACT

Previous studies have been made on algebraic curvature tensors generated by symmetric two-forms. Two topics in the paper discuss similar results in the case of algebraic curvature tensors built from antisymmetric two-forms. More specifically, two such tensors will never be linearly dependent given enough rank. Also, for any algebraic curvature tensor R on a three dimensional vector space, there are at most three antisymmetric algebraic curvature tensors needed to sum to R. One other result states that no curvature tensor built from a anitsymmetric form will ever equal a curvature tensor built from a symmetric form.

1 Introduction

An algebraic curvature tensor gives information about the curvature at a given point on a manifold. The set of all algebraic curvature tensors is denoted as $\mathcal{A}(V)$. Algebraic curvature tensors can be constructed by symmetric or antisymmetric two-forms. Two of the following results have been studied in the symmetric case. The purpose of this research is to draw similar conclusions about these kinds of algebraic curvature tensors on antisymmetric forms. Surprisingly, one of the results differs.

Suppose that V is a k-dimensional, real vector space. An algebraic curvature tensor $R \in \otimes^4(V^*)$ satisfies the following properties:

$$R(x,y,z,w) = -R(y,x,z,w) = R(z,w,x,y), \text{ and}$$

$$R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.$$

The second is called the Bianchi Identity.

The vector spaces $S^2(V^*)$ and $\Lambda^2(V^*)$ are the set of all symmetric bilinear forms and antisymmetric bilinear two-forms respectively. If $\varphi \in S^2(V^*)$ and $\psi \in \Lambda^2(V^*)$, then we define

$$\begin{aligned} R_\varphi(x,y,z,w) &= \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w) \\ R_\psi(x,y,z,w) &= \psi(x,w)\psi(y,z) - \psi(x,z)\psi(y,w) - 2\psi(x,y)\psi(z,w). \end{aligned}$$

Note that $R_\varphi, R_\psi \in \mathcal{A}(V)$. In fact, [3]

$$span_{\psi\in\Lambda^2(V^*)}\{R_\psi\} = span_{\varphi\in S^2(V^*)}\{R_\varphi\} = \mathcal{A}(V).$$

We define

$$\eta(k) = max\{min\{n | R = \sum_{i=1}^{n} \kappa_i R_{\psi_i}, \text{ where } \psi_i \in \Lambda^2(V^*) \text{ and } \kappa_i \in \mathbb{R}\}\}.$$

It has been shown in [2] that the analogous symmetric case $\nu(3) = 2$, i.e. the maximum number of symmetric two-forms needed to make any algebraic curvature tensor in the third dimension is two.

2 Results

Previous results show some basic linear independence facts about algebraic curvature tensors built from symmetric forms. Similar questions can be asked about the antisymmetric case. For example, it is shown in [1] that two symmetric algebraic curvature tensors never sum to zero given the rank is high enough. Here is an analogous result:

Theorem 1. *If $\psi \in \Lambda^2(V^*)$ and $rank(\psi) = 2k \geq 4$, then there does not exist $\tau \in \Lambda^2(V^*)$ such that $R_\psi + R_\tau = 0$.*

Proof: Suppose such a $\tau \in \Lambda^2(V^*)$ exists. Since $rank(\psi) = 2k$, choose an orthonormal basis $\{e_1, f_1, \ldots, e_k, f_k, n_1, \ldots, n_s\}$ so that $\psi(e_i, f_j) = \delta_{ij}$ and $\tau(e_i, f_j) = \lambda_{ij}$. Thus

$$R_\psi(e_i, f_i, f_i, e_i) + R_\tau(e_i, f_i, f_i, e_i) = 3 + 3\lambda_{ii}^2 = 0.$$

This implies that $\lambda_{ii}^2 = -1$, a contradiction. $\square$

With the fact that $\mathcal{A}(V)$ is independently spanned by the curvature tensors built from symmetric and antisymmetric forms, it is interesting to ask if an element from the first set will ever equal an element from the second.

Theorem 2. *If $\psi \in \Lambda^2(V^*)$, $rank(\psi) \geq 4$ there does not exist $\phi \in S^2(V^*)$ such that $R_\psi = R_\phi$.*

Proof. Suppose to the contrary that there is such a solution. Consider the following cases:

Case 1: The solution ϕ is definite when restricted to a particular basis. If necessary, replace ϕ with $-\phi$. Since $rank(\psi)$ is even, there exists an orthonormal basis $\{e_1, f_1, \ldots, e_k, f_k, n_1, \ldots, n_s\}$ so that $\psi(e_i, f_j) = \delta_{ij}$. Since $\{e_1, e_2\}$ are orthonormal, we get

$$0 = R_\psi(e_1, e_2, e_2, e_1) = R_\phi(e_1, e_2, e_2, e_1) = 1.$$

Case 2: The proposed solution ϕ has signature (p, q). There exists an orthonomal basis $\{e_1^-, \ldots, e_p^-, e_1^+, \ldots, e_q^+, n_1, \ldots, n_s\}$ with respect to ϕ. Then

$$-1 = R_\phi(e_1^-, e_1^+, e_1^+, e_1^-) = R_\psi(e_1^-, e_1^+, e_1^+, e_1^-) = 3\psi(e_1^-, e_1^+)^2.$$

This implies that $-1/3 \geq 0$. Either case arrives at a contradiction. $\square$

3 Example

It is a known fact that in a three dimensional vector space, a given algebraic curvature tensor is completely characterized by its Ricci tensor [2]. With this, we can prove the following result. Firstly, we will show that any algebraic curvature tensor can expressed as the sum of three algebraic curvature tensors on antisymmetric forms. Subsequently, we will show that there exist some algebraic curvature tensors that can not be expressed as the sum of two.

Theorem 3. $\eta(3) = 3$.

Proof. Let $R \in \mathcal{A}(V)$ with $dim(V) = 3$ and let ρ be the Ricci tensor for R. Put a positive definite metric on V. Choose an orthonormal basis $\{e_1, e_2, e_3\}$ that diagonalizes ρ so that $\rho = diag\{\lambda_1, \lambda_2, \lambda_3\}$. Define

$$\beta_1 = \frac{\lambda_1 + \lambda_2 - \lambda_3}{6}, \qquad \tau_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\beta_2 = \frac{\lambda_2 + \lambda_3 - \lambda_1}{6}, \qquad \tau_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\beta_3 = \frac{\lambda_3 + \lambda_1 - \lambda_2}{6}, \qquad \tau_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then

$$R = \sum_{i=1}^{3} \beta_i R_{\tau_i}.$$

Hence $\eta(3) \leq 3$. Note that if an eigen value of ρ is the sum of the other two, then one of $\{\beta_1, \beta_2, \beta_3\}$ is zero. This means that only two of $\{\tau_1, \tau_2, \tau_3\}$ are needed.

Now assume that there exists a solution $\{\psi_1, \psi_2\}$ such that $R = \sum R_{\psi_i}$. Then set

$$\psi_1 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \text{ and}$$

$$\psi_2 = \begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix}.$$

By the definition of the Ricci tensor,

$$\rho(e_j, e_k) = \sum_{i=1}^{3} R(e_i, e_j, e_k, e_i).$$

From this we get the following system of equations:

$$ab + de = 0$$

$$ac + df = 0$$

$$bc + ef = 0$$

$$3a^2 + 3d^2 = \frac{\lambda_1 + \lambda_2 - \lambda_3}{2}$$

$$3b^2 + 3e^2 = \frac{\lambda_1 + \lambda_3 - \lambda_2}{2}$$

$$3c^2 + 3f^2 = \frac{\lambda_3 + \lambda_2 - \lambda_1}{2}$$

Note that if $[\psi_i]_{jk} = 0$ for $j \neq k$ then one of the eigenvalues of ρ is the sum of the other two, which can be expressed as the sum of two antisymmetric curvature tensors with the above method. Thus we can assume that $[\psi_i]_{jk} \neq 0$ for $j \neq k$. By multiplying the first equation by c and the second by b, we get

$$ce = bf \Rightarrow cef = bf^2 \Rightarrow -c^2 = f^2.$$

Since c and f are nonzero, this is a contradiction. Thus $\eta(3) \geq 3$. $\square$

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