Second Twist Number of 2-Bridge Knots

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Abstract

In this paper we take a look at the Jones polynomial and the Tutte polynomial for alternating knots as well as the first and second twist number. We then examine the second twist number and extend it to 2-bridge knots and try to compute the second twist number in terms of things we can compute strictly from tangles that correspond to the given knot. The goal at the end of this paper is to be able to get a list of numbers(tangles) and be able to compute its second twist number.

Keywords: Jones Polynomial, Tutte Polynomial, twist number, second twist number, 2-bridge knots

1 Introduction

A *knot* is defined as a simple closed curve in $\mathbb{R}^3$. A *knot diagram* is the projection of a knot onto a plane with over and under crossings. A *link* is made up of one or more components, a knot is a link with one component. In the late 1920's Kurt Reidemeister showed that knots can be equivalent up to isotopy with a combination of moves that are now refered to as Reidemeister Moves. These three moves are illustrated in Figure 1.

In knot theory we want to be able to distinguish between different knots. To make this distinction we have what are called knot invariants. A *knot invariant* is a quantity that is defined for each knot and which is the same if the knots are equivalent up to ambient isotopy. The knot invariant can range from a number up to something like a polynomial. The first knot polynomial discovered was in 1923 by James Waddell Alexander II. Then in 1969, John Conway showed a variation of this polynomial using a skein relation but its significance was not realized until the discovery of the Jones polynomial in 1984.

The paper will be organized as follows. In section 2 the Reidemeister moves will be mentioned as well as the the Bracket polynomial and its rules. We will see how the Bracket polynomial is invariant under Reidemeister moves I, II, and III. In section 3 we introduce the Jones polynomial as well as the Tutte polynomial and their relation. An explanation as to how to get a multigraph and its dual graph from a knot diagram will also be explained. In section 4 we look at the second twist number and the effect certain twists on a knot diagram have on its graph. In section 5 we look at a new way to find the second twist number in terms of the knot diagram.

2 Bracket Polynomial and the Jones Polynomial

I begin by introducing the Kauffman Bracket polynomial which Kauffman defines in [1].

Definition 1 Let L be an unoriented link diagram and let L be the element of the ring $\mathbb{Z}\left[A, A^{-1}, -A^2, -A^{-2}\right]$ defined by:

1. $< O > = 1$
2. $< O \cup L > = (-A^2 - A^{-2}) < L >$

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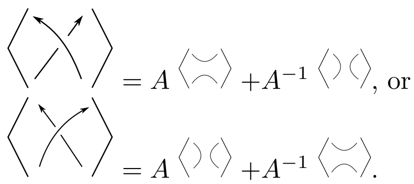
3. 

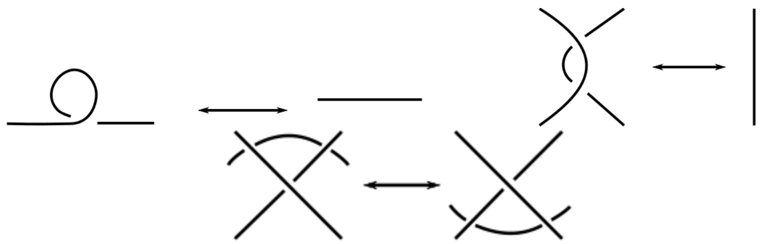
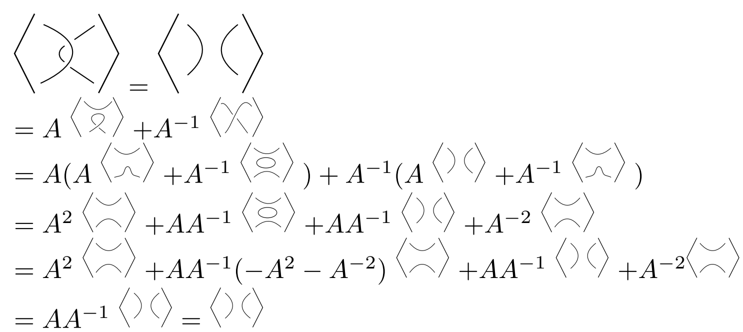


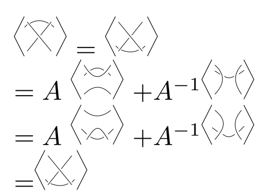
Figure 1: Reidemeister Moves I, II, and III

Kauffman showed that the bracket polynomial is invariant under Reidemeister moves II and III but not I. In order to show that the bracket is invariant under Reidemeister move I the *writhe* of a link L needs to be defined.

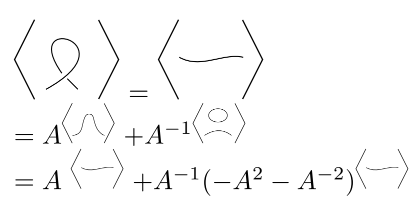
Definition 2 The *writhe* denoted $w(L)$ is the sum of all crossings where each crossing is given a value of either +1 or -1, depending on the orientation.

We start by illustrating invariance of the Bracket polynomial under R-moves II and III:

- 

- 

Now we will see what we get when we compute the bracket polynomial for Reidemeister move I:

- 

$= A \langle \frown \rangle + (-A - A^{-3}) \langle \frown \rangle$
$= (A - A - A^{-3}) \langle \frown \rangle$
$= -A^{-3} \langle \frown \rangle$

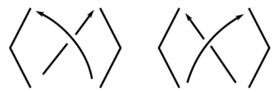


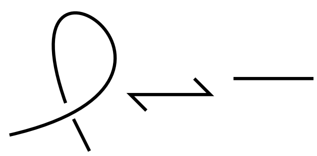
Figure 2: A negative crossing -1, and a positive crossing +1.

Definition 3 The *X polynomial* is a polynomial of oriented links and is defined to be

$$X(L) = (-A^3)^{-w(L)} < L >$$

Since neither $w(L)$ and $< L >$ are affected by moves II and III, $X(L)$ is unaffected by II and III and as we will subsequently see, $X(L)$ is also unaffected by move I.

Let's see what happens to $X(L)$ when we look at Reidemeister move I:



$W \langle \ell \rangle = W - 1$ $\quad W \langle \frown \rangle = W$ $\quad \langle \ell \rangle = -A^{-3} \langle \frown \rangle$
$X \langle \ell \rangle = (-A^3)^{-W\langle \ell \rangle} \langle \ell \rangle$
$= (-A^3)^{-W\langle \frown \rangle + 1} (-A^{-3}) \langle \frown \rangle$
$= (-A^3)^{-W\langle \frown \rangle} (-A^3)(-A^{-3}) \langle \frown \rangle$
$= (-A^3)^{-W\langle \frown \rangle} \langle \frown \rangle$
$= X \langle \frown \rangle$

As we can see, we get what we want to get, showing that in fact it is not affected by R-move I. The *Jones polynomial* is then obtained from the X polynomial by simply replacing A by $t^{-\frac{1}{4}}$.

3 The Jones Polynomial and the Tutte Polynomial

Now, we will look at the Jones polynomial by looking at the simple weighted graphs of twist diagrams. This description can be found in [4], we include it here for completeness.

Notation

1. $G = (V, E)$ is a multi graph with a set V of vertices and a set E of edges.
2. $\tilde{G}$ denotes a spanning subgraph of G in which parallel edges are deleted and instead are given multiplicities, i.e., the number of parallel edges. The subgraph $\tilde{G} = (V, \tilde{E})$ has a set of vertices V and a set $\tilde{E}$ of edges.

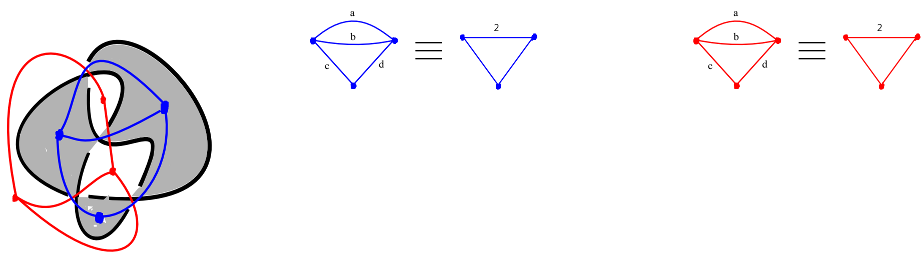


Figure 3: Fig. 8 knot with positive and negative checkerboard graph; multigraph and its weighted simple graph; dual graph

3. Define $n(j)$ to be the number of edges $e \in \tilde{E}$ with a multiplicity $\mu(e)$ and $\mu(j) \geq j$.
4. $k(G)$ is the number of components of a graph G.
5. The Tutte polynomial of a multi graph G is

$$T_G(x, y) := \sum_{F \subseteq E} (x-1)^{k(F)-k(E)} (y-1)^{|F|-|V|+k(F)}$$

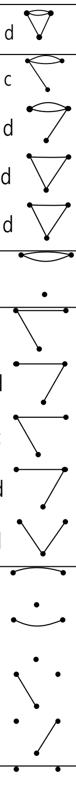
Let K be an alternating link with an alternating plane projection $P(K)$ the projection region can be colored with 2 colors such that adjacent faces have different colors. Each color is assigned a graph and every region gives rise to a vertex. Two adjacent vertices of the same color are connected by an edge if the corresponding regions are adjacent to a common crossing. These graphs are then called checkerboard graphs as explained in [4].

Since for an alternating link K all edges are either positive of negative, we have a positive and negative checkerboard graph. Letting G be the positive checkerboard graph and a be the number of vertices in G then b can be the number of vertices in the negative checkerboard graph. Where w is the writhe number.

From [4], we know that these checkerboard graphs are dual to each other and the Jones polynomial of an alternating link K with positive checkerboard graph G satisfies:

$$V_K(t) = (-1)^w t^{(b-a+3w)/4} T_G(-t, -1/t).$$

Example 1



F	$k(F)$	$\lvert F\rvert$	$(x-1)^{k(F)-k(E)}(y-1)^{\lvert F\rvert-\lvert v\rvert+k(F)}$
a, b, c, d	1	4	$(x-1)^0(y-1)^2$
a, b, c	1	3	
a, b, d	1	3	$4(y-1)^1$
a, c, d	1	3	
b, c, d	1	3	
a, b	2	2	$(x-1)(y-1)$
a, c	1	2	
a, d	1	2	
b, c	1	2	5
b, d	1	2	
c, d	1	2	
a	2	1	
b	2	1	$4(x-1)$
c	2	1	
d	2	1	
∅	3	0	$(x-1)^2$

Now, we have that the Jones polynomial satisfies:

$$V_K(t) = (-1)^w t^{(b-a+3w)/4} T_G(-t, -1/t)$$

Where w is the writhe and in our example when given an orientation, the Fig. 8 knot has $w = 0$ and from looking at both of our graphs we see that $a = 3$ and $b = 3$ which makes the first half just 1. What we are left with is now $T_G(-t, -1/t)$ which is simply adding all the terms from our table and doing the correct substitution,
$T_G(-t, -1/t) = (y-1)^2 + 4(y-1) + 4(x-1) + (x-1)(y-1) + 5 + (x-1)^2$
$= y^2 - 2y + 1 + 4y - 4 + 4x - 4 + xy - x - y + 1 + 5 + x^2 - 2x + 1$
$= t^2 - t + 1 - t^{-1} + t^{-2}$

Which is indeed the correct Jones polynomial for the Fig. 8 knot.

4 2-Bridge Knots

From [4], we have:

Theorem 1 Let $V_K(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_m t^m$ be the Jones polynomial of an alternating knot K and let $G = (V, E)$ be a checkerboard graph of a reduced alternating projection of K. Then,

1. $|a_n| = |a_m| = 1$
2. $|a_{n+1}| + |a_{m-1}| = T(K)$
3. $|a_{n+2}| + |a_{m-2}| + |a_{m-1}||a_{n+1}| = \frac{T(K)+T(K)^2}{2} + n(2) + n^*(2) - tri - tri^*$

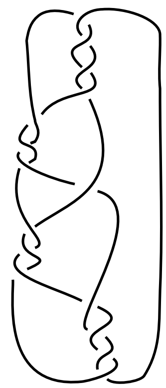


Figure 4: 2-bridge knot with 5 tangles

where $T(K)$ is the twist number, $n(2)$ is the number of edges with multiplicity greater than 2 in $\tilde{E}$ and $n^*(2)$ the corresponding number in the dual checkerboard graph.

The number tri is the number of triangles in the graph $\tilde{G} = (V, \tilde{E})$ and tri^* corresponding to tri in the dual graph.

The proof can be found in [4].

Definition 4 *Given $b_1, b_2, \cdots, b_{2i+1}$, a 2-bridge knot can be constructed by building a knot with a diagram and b_i crossings in each tangle.*

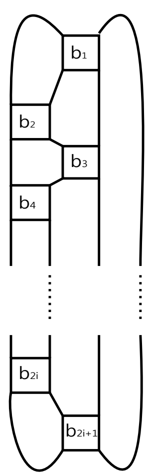


Figure 5: 2-bridge knot with a_{2i+1} tangles

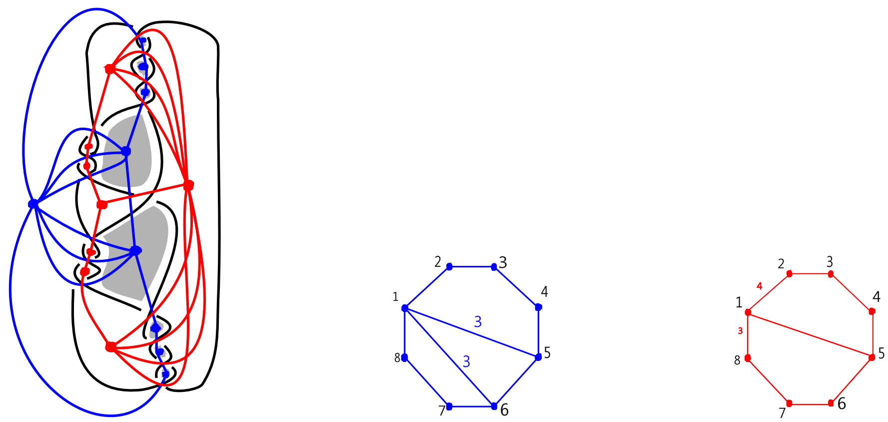
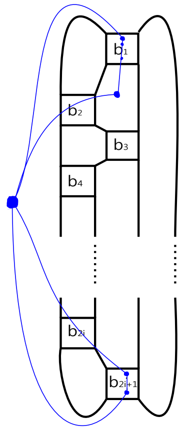


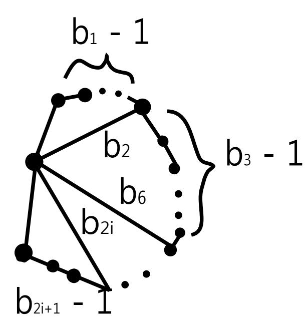
Figure 6: 5 twist knot with the checkerboard graph; weighted simple graph; dual graph

5 SecondTwist Number

Theorem 2 Given $b_1, b_2, \cdots, b_{2i+1}$ with $b_i, b_{2i+1} > 2$ then $tri + tri^* = \{\#b_i = 1\}$

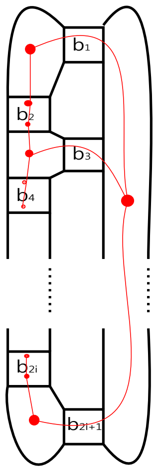
Proof. Looking at the two different cases we being with the positive checkerboard graph:

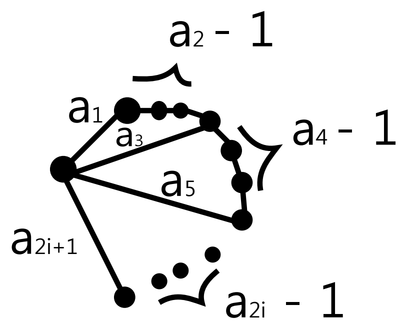




Given a list of tangles with corresponding crossings the weighted simple graph will have tri triangles when b_{2i+1} has only 1 crossing. Looking at figure 6, it can be seen that the third tangle has 1 crossing which corresponds to a triangle in the weighted simple graph. There is only a triangle for each k such that $a_{2k+1} - 1 = 0$. The number of vertices corresponds to one less than the number of crossings in each of the right tangles while the weighted edges correspond to the number of crossings in each tangle on the left.
Since we know that adding crossings to the right side tangles will add more vertices and edges, it is clear that the only way to get triangles is by having only one crossing.

For the dual graph we have:





Looking at the diagram and its corresponding graph it can be seen that the triangles will correspond to tangles that are on the right side of the diagram with one crossing. The weighted edges will also correspond to the right tangles and will have a weighted number equal to the number of crossings. The number of vertices will now correspond to the left tangles and will be one vertex less than the number of crossings. □

From [4] we have that
$a_{m-1} = (-1)^{|v|-1}(|v| - 1 - |\tilde{E}|)$
$|a_{m-1}| = |\tilde{E}| + 1 - |v|$
$|a_{n+1}| = |\tilde{E}^*| - |v^*| + 1$

Looking at the general forms of the graphs we get that:
$|\tilde{E}^*| = (i+1) + \sum_{j=1}^{i} a_{2j}$
$|v^*| = \sum_{j=1}^{i} a_{2j} + 2$
We can then conclude that
$|a_{n+1}| = (i+1) + \sum_{j=1}^{i} a_{2j} - (\sum_{j=1}^{i} a_{2j} + 2) + 1 = i$

For G, we have that:
$|\tilde{E}| = \sum_{j=0}^{i} a_{2j+1} + i$
$|v| = \sum_{j=0}^{i} a_{2j+1}$
$|a_{m-1}| = |\tilde{E}| - |v| + 1 = i + 1$

Theorem 3 *Given $b_1, b_2, \cdots, b_{2i+1}$ the second twist number of a 2-bridge knot will be*

$$|a_{n+2}| + |a_{m-2}| = \tfrac{(T(K)-1)^2}{4} + 2(n(2) + n^*(2))$$

Proof. We have that $n(2) + n^*(2) + tri + tri^* = 2i + 1 = T(K)$ therefore,
$|a_{n+2}| + |a_{m-2}| + i(i+1) = \frac{T(K)+T(K)^2}{2} + n(2) + n^*(2) - tri - tri^*$
$|a_{n+2}| + |a_{m-2}| + \frac{T(K)-1}{2}(\frac{T(K)-1}{2} + 1) = \frac{T(K)+T(K)^2}{2} + n(2) + n^*(2) - tri - tri^*$
$|a_{n+2}| + |a_{m-2}| + \frac{T(K)^2-1}{4} = \frac{T(K)+T(K)^2}{2} + n(2) + n^*(2) - tri - tri^*$
$|a_{n+2}| + |a_{m-2}| = \frac{2T(K)+T(K)^2+1}{4} + n(2) + n^*(2) - tri - tri^*$

With $tri + tri^* = T(K) - n(2) - n^*(2)$:
$|a_{n+2}| + |a_{m-2}| = \frac{2T(K)T(K)^2+1}{4} + n(2) + n^*(2) - T(K) + n(2) + n^*(2)$
$|a_{n+2}| + |a_{m-2}| = \frac{(T(K)-1)^2}{4} + 2(n(2) + n^*(2))$ $\square$

Now that we have been able to find the second twist number and can compute the second twist number for any 2-bridge knot what it means diagrammatically remains an open question.

6 Acknowledgements

It has been my pleasure to have taken part in the CSUSB REU and be able to work alongside Dr. Trapp. I would like to thank Dr. Trapp for his guidance throughout this REU and for his willingness to help out whenever possible. I would also like to thank Dr. Dunn for lending a helping hand whenever needed and for the many interesting stories told. This paper was made possible by NSF Grant # DMS-0850959.

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