Structure Groups of Pseudo-Riemannian Algebraic Curvature Tensors

Joseph Palmer

August 20, 2010

Abstract

We study the group of endomorphisms which preserve given model spaces under precomposition, known as the structure group of the given space. We are able to greatly characterize the elements of these groups for many different cases and we propose a conjecture which would define their properties in nearly every case. The decomposition of the model spaces and the decomposition of the structure groups are also explored and we show how to decompose the kernel of the algebraic curvature tensor apart from the rest of the model space. The majority of this work is done on a weak model space, with no inner product, but these results could be expanded to spaces with more structure and we always work in the pseudo-Riemannian setting. This work is motivated by a desire to gain a greater understanding of model spaces in order to formulate scalar invariants which are not of Weyl type in the pseudo-Riemannian case.

1 Introduction

Given some pseudo-Riemannian manifold paired with a metric given by (M, g) the Levi-Civita connection can be used to the form the Riemann curvature tensor. This is actually a tensor field because it assigns a tensor to each point on the manifold, so restricting this to some point $p \in M$ will yield a tensor. This restriction is known as an *Algebraic Curvature Tensor*, or ACT, and studying all possible ACTs can provide insight into how the larger tensor field on a manifold will behave. If V is a real vector space of finite dimension n then $R \in \otimes^4 V^*$ is an Algebraic Curvature Tensor if:

$$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y)$$

$$\text{and } R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0$$

for all $x, y, z, w \in V$. Define $\mathcal{A}(V)$ to be the vector space of all such objects. Now, let $\phi \in S^2(V^*)$ and define the *canonical ACT* formed from ϕ as

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$$

for any $x, y, z, w \in V$. It is known [3] that

$$\mathcal{A}(V) = span\{R_\phi | \phi \in S^2(V^*)\}.$$

1.1 Structure Groups

The orthogonal group of matrices of dimension n on some positive definite inner product space $(V, \langle \cdot, \cdot \rangle)$ is denoted $\mathcal{O}(n)$ and is defined as

$$\mathcal{O}(n) = \{A \in Gl(n) | \langle Ax, Ay \rangle = \langle x, y \rangle \, \forall x, y \in V\}.$$

That is, the elements of $\mathcal{O}(n)$ are exactly the endomorphisms of V which preserve the inner product. This idea can be generalized into spaces with more structure then just an inner product. First, note that for $A \in Gl(n)$ we use A^* to denote the *precomposition* of A. This means that when A^* is applied to some covariant tensor T the operator A should act on the arguments of T before sending them to T itself, for example $A^* \langle x, y \rangle = \langle Ax, Ay \rangle$. Thus for $A \in \mathcal{O}(n)$ one could say that $A^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$.

Define a model space as an ordered triple $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ of a vector space, inner product, and an ACT on that space. Each point on a manifold given by (M, g) can be described by a model space where $dim(V)$ is equal to $dim(M)$, the inner product is just the restriction of the metric, and the ACT is the full Riemannian curvature tensor restricted to the point in question. We wish to consider operators that preserve not just the inner product, but the entire model space. Define, for some model space $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$, the *structure group of* $\mathcal{M}$, denoted $G_{\mathcal{M}}$, by

$$G_{\mathcal{M}} = \{A \in Gl(n) | A^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \text{ and } A^* R = R\}.$$

In general define notation such that G_α is the set of endomorphisms which preserve the mathematical object α. It is possible to define a structure group which preserves only some bilinear form, or one that preserves a weak model space $\mathcal{M} = (V, R)$. It is also fruitful to consider the idea of structure groups on several different types of ACTs, including a sum or difference of two canonical curvature tensors. It is the interaction of these structure groups and the form of their elements which is the primary topic of this paper.

1.2 Decomposabilty

It is well known that many manifolds can actually be realized as the products of smaller dimension manifolds and it is we study this through the algebraic methods of a model space and structure groups. This is considered in [1]. If the vector space decomposed as $V = V_1 \oplus V_2$ and the metric satisfied $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_2$ while $R = R_1 \oplus R_2$ with $\langle \cdot, \cdot \rangle_i$ and R_i acting on V_i one could say that the entire model space decomposes. This would be written as $\mathcal{V} = (V_1, \langle \cdot, \cdot \rangle_1, R_1) \oplus (V_2, \langle \cdot, \cdot \rangle_2, R_2)$. The interaction between this decomposition and structure groups is another topic of this paper. The connection between a decomposing model space and the decomposition of the relevant structure groups is also explored. Also, the decomposition of the model space $\mathcal{M}$ can provide information about the elements of $G_{\mathcal{M}}$.

1.3 Invariants

It is often desired to know if a given manifold is locally homogeneous. It is also of interest to determine when manifolds are k-curvature homogeneous, which means that the first k covariant derivatives of R are each locally constant. There exist what are known as *Weyl scalar invariants* built by contractions of R, and in the Riemannian these scalar functions are all constant if and only if the manifold is locally homogenous [4]. This result can not be generalized to the pseudo-Riemannian case. In fact, there exist *vanishing scalar invariant* (VSI) manifolds for which all scalar contractions of R are zero, but not all of these manifolds are locally homogeneous. However, there are classes of pseudo-Riemannian manifolds for which alternate scalar invariants have been found. For example, in [1] an example of a scalar invariant which is not of Weyl type is found on a VSI manifold. Thus, by studying the structure group of a given model space, we hope to inspire methods which could be used to form these new invariants which will also work in the pseudo-Riemannian case.

2 Structure Groups of R_ϕ

First consider the simplest case; for some $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ let $R = R_{\langle\cdot,\cdot\rangle}$ so

$$R(x, y, z, w) = \langle x, w\rangle \langle y, z\rangle - \langle x, z\rangle \langle y, w\rangle \;\; \forall\; x, y, z, w \in V.$$

It is known that this R corresponds exactly to the case of constant sectional curvature [2], even in the pseudo-Riemannian case. Recall, for a non-degenerate plane Π spanned by the vectors u and v, the definition of sectional curvature,

$$\kappa(\Pi) = \frac{R(u, v, v, u)}{\langle u, u\rangle \langle v, v\rangle - \langle u, v\rangle^2}.$$

Now we are prepared to make a connection between this model space and the relevant structure groups. We give our own proof to the following known result.

Lemma 1. *For the model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ we have:*

$$G_{\mathcal{M}} = G_{\langle\cdot,\cdot\rangle} \Leftrightarrow R = cR_{\langle\cdot,\cdot\rangle} \Leftrightarrow R \text{ has constant sectional curvature, } \kappa.$$

Proof. Recall it is already known that $R = \kappa R_{\langle\cdot,\cdot\rangle} \Leftrightarrow R$ has constant sectional curvature, κ.

First show constant κ implies $G_{\mathcal{M}} = G_{\langle\cdot,\cdot\rangle}$. Notice that since $R_{\langle\cdot,\cdot\rangle}$ is completely determined by $\langle\cdot,\cdot\rangle$ any transformation which fixes the inner product will clearly fix $R_{\langle\cdot,\cdot\rangle}$ as well. Thus $G_{\langle\cdot,\cdot\rangle} \subset G_{\mathcal{M}}$. By definition we have the opposite inclusion.

Now assume $G_{\mathcal{M}} = G_{\langle \cdot,\cdot \rangle}$ and show that κ is constant. In dimension less than or equal to 2 we know κ is trivially constant since there do not exist distinct 2-planes, so assume dimension strictly greater than 2. We assume that κ is not constant and proceed looking for a contradiction.

Since κ is not constant choose two distinct planes Π_1 and Π_2 for which $\kappa(\Pi_1) \neq \kappa(\Pi_2)$. Let $\{e_1, e_2\}$ be an orthonormal basis for Π_1, and take $A \in G_{\langle \cdot,\cdot \rangle}$ so $span\{Ae_1, Ae_2\} = \Pi_2$ and label each $Ae_i = f_i$. Notice since A preserves $\langle \cdot, \cdot \rangle$ we know that f_1 and f_2 will also form an orthonormal basis on Π_2. Now consider

$$\begin{aligned}
\kappa(\Pi_1) \neq \kappa(\Pi_2) \quad & \Rightarrow \quad \kappa(e_1, e_2) \neq \kappa(f_1, f_2) \\
& \Rightarrow \quad A^*\kappa(e_1, e_2) \neq A^*\kappa(f_1, f_2) \\
& \Rightarrow \quad \kappa(Ae_1, Ae_2) \neq A^*\kappa(f_1, f_2) \\
& \Rightarrow \quad \kappa(f_1, f_2) \neq A^*\kappa(f_1, f_2).
\end{aligned}$$

Now recall that both basis are orthonormal, so this implies that

$$R(f_1, f_2, f_2, f_1) \neq A^*R(f_1, f_2, f_2, f_1)$$

so $A \notin G_{\mathcal{M}}$. Thus we have a contradiction and the proof is complete.

□

We also consider more general cases. It is almost always true that $G_{R_\phi} = G_\phi$ on the weak model space with $\phi \in S^2(V^*)$, but in the balanced signature case this does not necessarily hold.

Theorem 1. *Let $\phi \in S^2(V^*)$ with rank greater than or equal to 3. Then the following are true:*

1. *If $A \in G_{R_\phi}$ then $A^*\phi = \pm\phi$.*
2. *If the signature is unbalanced then $G_{R_\phi} = G_\phi$.*

Proof. Take $A \in G_\phi$ and define $\psi(x, y) = \phi(Ax, Ay)$ for $x, y \in V$ and notice that $\psi \in S^2(V^*)$. By assumption $R_\phi = A^*R_\phi$ but also be the definition of ψ we know $A^*R_\phi = R_\psi$. Thus we have

$$R_\phi = R_\psi \Rightarrow \phi = \pm\psi,$$

because rank of ϕ is greater than or equal to 3 by [3]. Now we have shown assertion 1.

To show 1 we first notice that clearly $G_\phi \subseteq G_{R_\phi}$ and so we take an arbitrary $A \in G_{R_\phi}$ and must show it is in G_ϕ. We have two cases. If $\phi = \psi$ then $\phi = A^*\phi$ so $A \in G_\phi$ and we are done. We will now consider the other case and show that it creates a contradiction.

We have that $\phi = -\psi$ so

$$\phi(x, y) = -\phi(Ax, Ay)$$

for all $x, y \in V$. Now take $\{e_1, \ldots, e_n\}$ to be an orthonormal basis for V with respect to ϕ. Thus

$$\phi(e_i, e_j) = \epsilon_i \delta_{ij}$$

where $\epsilon_i = \pm 1$ depending on the signature. So

$$-\phi(Ae_i, Ae_j) = \phi(e_i, e_j) \Rightarrow \phi(Ae_i, Ae_j) = -\epsilon_i \delta_{ij}$$

and we see that the set $\{Ae_1, \ldots, Ae_n\}$ also forms an orthonormal basis for V with respect to ϕ but now it has signature (q, p). All orthonormal bases have the same signature so this produces a contradiction since we have assumed that $q \neq p$. Thus we have shown assertion 2 as well.

$\square$

Examples of endomorphisms which preserve R_ϕ but not ϕ are known to exist in the balanced signature case, but the full characterization of these operators is still unknown. As an example consider

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

in the four dimensional balanced signature case written in an ordered basis. Then for $x, y, z, w \in V$ we have $A^*\phi(x, y) = -\phi(x, y)$ but

$$A^* R_\phi(x, y, z, w) = (-\phi(x, w))(-\phi(y, z)) - (-\phi(x, z))(-\phi(y, w)) = R_\phi(x, y, z, w).$$

3 The Decomposabilty of Structure Groups

A structure group is said to decompose if it can be written as the group direct product of two other groups, so $G_{\mathcal{M}} \simeq G_1 \oplus G_2$, and a model space $\mathcal{M} = (V, R)$ is said to decompose if $(V, R) = (V_1, R_1) \oplus (V_2, R_2)$. The goal of this section is to explore the relation between the decomposition of the structure group and the decomposition of the model space. We will assume that the model space decomposes and consider when $G_{\mathcal{M}}$ decomposes. Also note that we will use the bar notation to suggest when the object in question is not associated with the entire vector space V, but instead with some lower dimensional subspace.

Lemma 2. *Assume $\mathcal{M} = (V, R) = (V_1, R_1) \oplus (V_2, R_2)$ with $\overline{G}_i$ as the structure group for (V_i, R_i). Then having V_1, V_2 as invariant subspaces for all $A \in G$ is equivalent to $G \simeq \overline{G}_1 \oplus \overline{G}_2$.*

Proof. Define $G_1 = \{A \in G \mid A = A_1 \oplus I \text{ for } A_1 \in \overline{G}_1\}$ and similarly define G_2. Assume each V_i is invariant and we must show

(a) $G_1 \cap G_2 = id$

(b) $G = G_1 G_2$

(c) $G_1, G_2 \trianglelefteq G$.

Take $\beta = \beta_1 \oplus \beta_2$ as a basis for V where β_i is a basis for V_i. We will use this basis for the remainder of the proof. First notice that

$$\overline{A}_1 \in \overline{G}_1 \Leftrightarrow A_1 = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & I \end{bmatrix} \in G$$

and

$$\overline{A}_2 \in \overline{G}_2 \Leftrightarrow A_2 = \begin{bmatrix} I & 0 \\ 0 & \overline{A}_2 \end{bmatrix} \in G$$

so

$$A_1 A_2 = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{bmatrix} \in G. \tag{1}$$

Next notice that each V_i as an invariant subspace implies that the form of any $A \in G$ will be as given in (1), so we have shown b. This also shows that $G \simeq \overline{G}_1 \oplus \overline{G}_2$ implies that each V_i is invariant.

Now consider an A such that $A \in G_1$ and $A \in G_2$. Then

$$A = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \overline{A}_2 \end{bmatrix} \Rightarrow \overline{A}_1 = \overline{A}_2 = I \Rightarrow A = I.$$

Thus we have shown condition a.

Now we must only show condition c. Take arbitrary $g \in G$ and take $A \in G_1$ without loss of generality. We must show $gAg^{-1} \in G_1$, which means that gAg^{-1} would preserve R_1. So consider

$$\begin{aligned}
(gAg^{-1})^* R_1(x, y, z, w) &= R_1(gAg^{-1}x_1 + gAg^{-1}x_2, gAg^{-1}y_1 + gAg^{-1}y_2, \\
&\quad gAg^{-1}z_1 + gAg^{-1}z_2, gAg^{-1}w_1 + gAg^{-1}w_2) \\
&= R_1(gAg^{-1}x_1, gAg^{-1}y_1, gAg^{-1}z_1, gAg^{-1}w_1) \\
&= R(gAg^{-1}x_1, gAg^{-1}y_1, gAg^{-1}z_1, gAg^{-1}w_1) \\
&= (gAg^{-1})^* R(x_1, y_1, z_1, w_1) \\
&= R(x_1, y_1, z_1, w_1) \\
&= R_1(x, y, z, w)
\end{aligned}$$

and thus we have condition c as well. □

Notice that in this proof we have characterized the matrix form of the elements of $G_1 \oplus G_2$ in a specific basis.

4 Separating the Kernel by Decomposition

For $R \in \mathcal{A}(V)$ take $A \in G_R$ and let $\beta = \{e_1, \ldots, e_k, n_1, \ldots, n_l\}$ be an basis in which $span\{n_1, \ldots, n_l\} = kerR$. For this section we will consider A to be written as a matrix in this basis.

Define the *null rows* of A as the last l rows, that is, the rows which correspond to the n_i vectors, and define the *null columns* similarly. We will obtain results about the elements of these rows and columns. It is obvious that A will map the kernel of R to the kernel of R, and thus we obtain the following lemma.

Lemma 3. *The elements of $A \in G_R$ which are in null columns but not in the null rows are all zero.*

Next, we show that once we know $A \in Gl(k+l)$ there are no further constraints on the null rows of A to have $A \in G_R$.

Lemma 4. *If $A, B \in Gl(k+l)$ and these matrices written in the basis β agree on all terms of non-null rows, then $A \in G_R \Leftrightarrow B \in G_R$.*

Proof. Let $A, B \in Gl(k+l)$, with $A \in G_R$, agree on every element except for the elements in the null rows. Let arbitrary $x \in V$ and note that Ax and Bx agree on all but the last r components. Thus write $Ax = \tilde{x} + x_{nA}$ and $Bx = \tilde{x} + x_{nB}$ where $\tilde{x}$ is a linear combination of the non-null basis vectors and x_{nA} and x_{nB} are linear combinations of the null basis vectors. Similarly define arbitrary $y, z, w \in V$. Now notice

$$A^* R(x, y, z, w) = R(\tilde{x} + x_{nA}, \tilde{y} + y_{nA}, \ldots) = R(\tilde{x}, \tilde{y}, \ldots)$$

and also

$$B^* R(x, y, z, w) = R(\tilde{x} + x_{nB}, \tilde{y} + y_{nB}, \ldots) = R(\tilde{x}, \tilde{y}, \ldots).$$

Recall $A^* R = R$ and thus we have

$$B^* R(x, y, z, w) = A^* R(x, y, z, w) = R(x, y, z, w) \ \forall x, y, z, w \in V.$$

So $B \in G_R$ and finally we see that the null rows do not matter. $\square$

Now we have results about both the null rows and null columns of A. Define $\overline{A}$ to be the submatrix which is formed by removing both the null rows and the null columns of A. Notice that $\overline{A}$ will have dimension $k \times k$. We will show how the structure groups of any k-form α and the corresponding $\overline{\alpha}$ are related.

Lemma 5. *Take α as some k-form on V and take $A \in G_\alpha$. Define $\overline{\alpha}$ as the k-form on $\overline{V} = V/ker\alpha$ so that $\pi^* \overline{\alpha} = \alpha$ where $\pi : V \to \overline{V}$ is a projection. Then $A \in G_\alpha \Leftrightarrow \overline{A} \in G_{\overline{\alpha}}$.*

Proof. First we must show that $\overline{\alpha} : \overline{V} \to \mathbb{R}$ is well-defined. We have defined

$$\overline{\alpha}(x_1 + kerR, \ldots, x_k + kerR) = \overline{\alpha}(x_1, \ldots, x_k),$$

with each $x_i \in V$, and we must show that the result does not depend on which representative of the inputted cosets is chosen. Take $\tilde{x}_1 \in V$ such that $x_1 + kerR = \tilde{x}_1 + kerR$ and recall this implies that $x_1 - \tilde{x}_1 \in kerR$ and thus we see, for $a_1, \ldots, a_{k-1} \in V$, that

$$\overline{\alpha}(x_1 + kerR, a_1, \ldots, a_{k-1}) - \overline{\alpha}(\tilde{x}_1 + kerR, a_1, \ldots, a_{k-1}) = \overline{\alpha}(x_1 - \tilde{x}_1, a_1, \ldots, a_{k-1}) = 0$$

so

$$\overline{\alpha}(x_1 + kerR, a_1, \ldots, a_{k-1}) = \overline{\alpha}(\tilde{x}_1 + kerR, a_1, \ldots, a_{k-1})$$

and we see that the representative chosen does not matter in the first slot. This can be repeated in each slot to see that $\overline{\alpha}$ is well defined.

Notice that $\overline{A}(\pi x) = \pi(Ax)\ \forall x \in V$. Now, keeping in mind that $\pi^*\alpha = \alpha$,

$$\begin{aligned}
& A^*\alpha = \alpha \\
\Leftrightarrow\ & \alpha(Ax_1, \ldots, Ax_k) = \alpha(x_1, \ldots, x_k)\ \forall x_1, \ldots, x_k \in V \\
\Leftrightarrow\ & \pi^*\overline{\alpha}(Ax_1, \ldots, Ax_k) = \pi^*\overline{\alpha}(x_1, \ldots, x_k)\ \forall x_1, \ldots, x_k \in V \\
\Leftrightarrow\ & \overline{\alpha}(\pi(Ax_1), \ldots, \pi(Ax_k)) = \overline{\alpha}(\pi x_1, \ldots, \pi x_k)\ \forall x_1, \ldots, x_k \in V \\
\Leftrightarrow\ & \overline{\alpha}(\overline{A}(\pi x_1), \ldots, \overline{A}(\pi x_k)) = \overline{\alpha}(\pi x_1, \ldots, \pi x_k)\ \forall x_1, \ldots, x_k \in V \\
\Leftrightarrow\ & \overline{A}^*\overline{\alpha} = \overline{\alpha} \\
\Leftrightarrow\ & \overline{A} \in G_{\overline{\alpha}}
\end{aligned}$$

□

Now we will see that $kerR$ can always be decomposed apart from the ACT $\overline{R}$ which has a trivial kernel. This will allow us to apply results which require $kerR = \{0\}$ to the model space $(\overline{V}, \overline{R})$.

Lemma 6. *Define $\overline{V} = V/kerR$ and also define $\pi : V \to \overline{V}$ to be a projection. If $\overline{R}$ is defined by $\pi^*\overline{R} = R$ as an algebraic curvature tensor on $\overline{V}$, then $(V, R) = (\overline{V}, \overline{R}) \oplus (kerR, 0)$.*

Proof. Clearly $V = V/kerR \oplus kerR = \overline{V} \oplus kerR$.

Now, clearly for $R = 0$ we have $R(v, \cdot, \cdot, \cdot) = 0$ for any $v \in \overline{V}$. Also for $\overline{R}$ we have that $\overline{R}(v, \cdot, \cdot, \cdot) = 0$ for $v \in kerR$ by definition because $\overline{R}$ is only defined to be nonzero on elements of $\overline{V}$.

Next take arbitrary $x, y, z, w \in V$ and write $x = x_1 + x_2$ with $x_1 \in \overline{V}$ and $x_2 \in kerR$. Define decompositions of the other vectors similarly. Keeping in mind that $\pi^*x = \pi^*(x_1 + x_2) = \pi^*x_1$, and that $\overline{R}$ is only defined to be nonzero on elements of $\overline{V}$, notice

$$\begin{aligned}
R(x, y, z, w) &= R(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) = R(x_1, y_1, z_1, w_1) \\
&= \pi^*\overline{R}(x_1, y_1, z_1, w_1) = \overline{R}(x, y, z, w) + 0(x, y, z, w) \\
&= (\overline{R} + 0)(x, y, z, w).
\end{aligned}$$

Thus we see that $R = \overline{R} \oplus 0$. □

Now we can combine the result of Lemmas 3, 4, 5, and 6 to obtain a characterization of each $A \in G_R$ in the ordered basis β.

Theorem 2. *On some V of dimension n with $R \in \mathcal{A}(V)$ with $dim(kerR) = n - k$ the model space decomposes so*

$$\mathcal{M} = (V, R) = (\overline{V} = V/kerR, \overline{R}) \oplus (kerR, 0)$$

with $\overline{R}$ defined by $\pi^*\overline{R} = R$ where $\pi : V \to \overline{V}$ is a projection. Furthermore, for any $A \in G_{\mathcal{M}}$ written as a matrix in the ordered basis β in which the last $n-k$ vectors span the kernel of R, we have

$$A = \left[\begin{array}{c|c} \overline{A} & 0 \\ \hline B & C \end{array}\right]$$

where $\overline{A} \in G_{\overline{V}}$ is $k \times k$, B can be any $(n-k) \times k$ matrix, and $C \in Gl(n-k)$.

We have shown how to remove the kernel of R from our concern when considering the elements of $G_{\mathcal{W}}$. Now we will work to characterize $\overline{A} \in G_{\mathcal{W}}$ when $kerR = \{0\}$ so that we may form an entire description of the elements of $G_{\mathcal{W}}$ in any case.

5 The Properties of $\overline{A} \in G_{\mathcal{W}}$

Next we would like to characterize the properties of $\overline{A} \in G_{(\overline{V},\overline{R})}$ where $(\overline{V}, \overline{R})$ has a trivial kernel. We are working to prove or disprove the following conjecture.

Conjecture 1. *If a model space $(V, R) = \bigoplus_{i=1}^{n} (V_i, R_i)$ with each (V_i, R_i) indecomposable and $kerR = \{0\}$, then, for $\overline{A} \in G_{(V,R)}$ and some permutation σ we know $\overline{A} : V_i \to V_{\sigma(i)}$.*

This would allow us to greatly characterize the elements of the structure group for any ACT. Unfortunately, at this point we require that each R_i be a R_ϕ for some $\phi \in S^2(V^*)$ instead of just being indecomposable. Thus we have the following result.

Theorem 3. *If a model space $(V, R) = \bigoplus_{i=1}^{n} (V_i, R_{\phi_i})$ with each $\phi_i \in S^2(V^*)$ and $kerR = \{0\}$, then, for $\overline{A} \in G_{(V,R)}$ and some permutation σ we know $\overline{A} : V_i \to V_{\sigma(i)}$.*

Proof. Let $\overline{A} \in G_{(\overline{V},\overline{R})}$. Since $\overline{V} = \oplus_{i=1}^n V_i$ we can use $\beta = \{e_1, \ldots, e_n\} = \oplus_{i=1}^n \beta_i$ as a basis for $\overline{V}$ where each β_i is a basis for V_i. Using this basis we will write A as a matrix and show that its form implies the desired result. Define $Ae_i = f_i$ and write the element of $\overline{A}$ from the i^{th} row and j^{th} column as a_{ji}.

Now, since $\overline{A}$ has full rank choose some nonzero term in row i where $e_i \in \beta_m$. Label the column of this element as j, so we are considering a_{ij}. Now, choose some l so that f_l and f_j are not in the same V_p. Because they do not come from the same subspace, we know, for e_k coming from the same V_p as e_i but $i \neq j$, that

$$R(e_k, f_j, e_i, f_l) = 0$$

but also notice that

$$R(e_k, f_j, e_i, f_l) = a_{ij}a_{kl}$$

and recall that $a_{ij} \neq 0$. So now we have that $a_{kl} = 0$ for any $k \neq i$ where e_k is in the same subspace as e_i and where f_l and f_j are not in the same subspace. Now we know that the only nonzero elements in the rows associated with V_m but not in the same block as a_{ij} are in the same row as a_{ij}.

Now take some element not in the same block as a_{ij} which is nonzero and we will look for a contradiction. We can repeat the above process on this element causing all of the elements in some row adjacent to row i to be zero, and since $\overline{A}$ has full rank this is our contradiction.

Thus we have that each set of rows associated with a given V_i has nonzero entries only in the columns associated with another specific V_j. Now, since $\overline{A}$ has full rank we know there are no rows or columns of all zeros, and thus we have that $\overline{A} : V_i \to V_{\sigma(i)}$ for some permutation σ. $\square$

Also notice that there exists some $p \in \mathbb{Z}$ such that $\overline{A}^p$ has each V_i as an invariant subspace. Furthermore, since $\overline{A} : V_i \to V_{\sigma(i)}$ and $\overline{A} \in G_{(V,R)}$ we know for $x, y, z, w \in V_i$ we must have that $R(x, y, z, w) = R_{\phi_i}(x, y, z, w)$ and also

$$
\begin{aligned}
R(x, y, z, w) &= \overline{A}^* R(x, y, z, w) \\
&= R(\overline{A}x, \overline{A}y, \overline{A}z, \overline{A}w) \\
&= R_{\phi_{\sigma(i)}}(\overline{A}x, \overline{A}y, \overline{A}z, \overline{A}w) \\
&= \overline{A}^* R_{\phi_{\sigma(i)}}(x, y, z, w)
\end{aligned}
$$

and thus the block in $\overline{A}$ which sends vectors from V_i to $V_{\sigma(i)}$ must be some $D_i \in Gl(dim V_i)$ such that $R_{\phi_i} = D_i^* R_{\phi_{\sigma(i)}} \Rightarrow \phi_i = \pm D_i^* \phi_{\sigma(i)}$. So we now see that the corresponding restricted ACTs must be highly similar in order for $\overline{A}$ to permute between them, because the D_i required will not always exist. For example, V_i and $V_{\sigma(i)}$ must have the same dimension. Also, it is very important to note that all of these results following Theorem 3 do not at all depend on the summed curvature tensors being canonical. Thus, if Conjecture 1 can be proven, all of these relations will still hold in that more general case.

6 Conclusions and Further Questions

Between Theorem 3 and Theorem 2 we have a strong characterization of the elements of $A \in G_{(V,R)}$ for many different weak model spaces. If we could prove Conjecture 1 then many results would follow and the properties of the elements of structure groups of weak model spaces would be very well characterized. An obvious next step would be to consider adding an inner product to the model space for any of these results, as we have done very little with full model spaces. Also, now that we have learned about the elements of $G_{\mathcal{M}}$ we may be able to learn more about potential scalar invariants.

If we wanted to find a counterexample for Conjecture 1 we would have to look in dimension 6. This is because in dimension 2 all ACTs are canonical and thus Theorem 3 would apply. So the next step in finding a counterexample would be to check $R = R_1 \oplus R_2$ where $dimR_1 = dimR_2 = 3$ and each R_i is chosen specifically to not be equal to some R_ϕ for a $\phi \in S^2(V^*)$. Perhaps we should work in the field of representation theory in order to create further progress on this conjecture, because the groups with which we are working lend themselves to this treatment.

7 Acknowledgments

I would like to thank Dr. Corey Dunn for all of his fantastic guidance and help over the course of this program and also Dr. Rolland Trapp for being so helpful. I would also like to thank California State University at San Bernardino and NSF grant DMS-0850959 for funding.

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