Composing Two Non-Tricolorable Knots

Kelly Harlan

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Math REU at CSUSB

Abstract

In this paper we will be using modp-coloring, determinants of coloring matrices and knots, and techniques from linear algebra to prove that the knot sum of two non-tricolorable knots is non-tricolorable.

0.1 Introduction

The goal of this paper is to prove that when you compose two knots together as a knot sum, if neither of the knots used in the knot sum is tricolorable, then the knot sum will also not be tricolorable. First, let's introduce some terms. Some good sources for this and more background material include *Knot Theory* by C. Livingston and *The Knot Book* by C. Adams, A **knot** can be thought of as a rope in 3-space, where different crossings are made and the ends of the rope are connected together. Imagine tying your shoelaces and then connecting the two ends so you cannot undo the knot.
More formally, a knot is an embedding of a circle in 3-dimensional space. [6]

Figure 1: Example of a Knot
[7]

A **link** is a set of knotted loops all tangled up together. [2] A well-known example of a link is the Olympic Rings. It is a link with 5 components, specifically 5 unknots. Many people denote knots as links, because knots are links of one component.

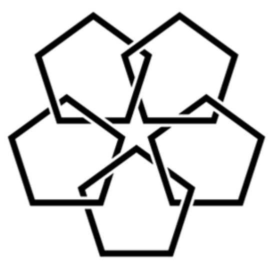


Figure 2: Example of a Link
[8]

When P. G. Tait started working with knot theory in the late 1800's one of the main problems that he posed was developing the means of proving that knots are distinct. [1]
How can we tell that two knots are distinct or equivalent, though?
Two knots K and J are called equivalent if K can be changed into J withouth breaking the any arcs of the knot. The notion of equivalence also satisfies the definition of an equivalence relation in that it is symmetric, transitive, and

reflexive. So proving that it is impossible to deform one knot into another is the same as proving that those knots are in different equivalence classes.
We must also consider knot invariants, which are values or functions that are the same across the entire equivalence class of the knot which we are concerned with. Knot invariants include knot complements, Kauffman and Jones Polynomials [9], minimal crossing number, and p-colorings of knots. Using these knot invariants helps in the proofs of unanswered or open questions in knot theory.
Tricoloring is the simplest invariant which distinguishes the trefoil knot from the unkot, and it was introduced by R. Fox around 1960. [11]
A link diagram L is **tricolored** if every arc is colored red (r), blue (b), or yellow (y), and at any given crossing either all three colors appear or only one color appears. A knot is called tricolorable if its diagrams are tricolorable, and when a knot diagram has every arc colored the same color, it is called a trivial tricoloring.
Also, if a knot diagram must use n colors, where n>3, the knot is not tricolorable, it is n-colorable. [See modp coloring in Section 3]

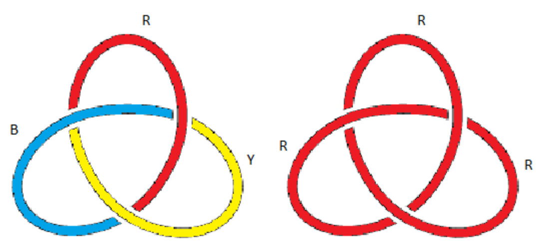


Figure 3: Examples of Tricolorings (Nontrivial and Trivial)

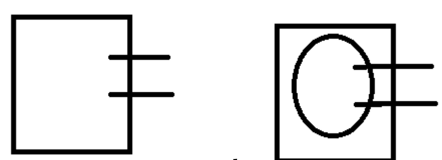
Two oriented knots (or links) can be summed by placing them side by side and joining them by straight bars such that orientation is preserved in the sum. We call this a **knot (or connect) sum**.

The knot sum of knots K_1 and K_2 is denoted $K_1 \# K_2 = K_2 \# K_1$ [See Fig. 8 as an example of the knot sum of the two knots in Fig 7.]

At the Knot Theory Workshop at Wake Forest University in 2002, Colin Adams suggested some open questions about knots, including one on colorability. Adams' suggestion was to find a pair of non-tricolorable knots whose composition (or knot sum) is tricolorable or show that this is not possible. [10] Our goal is to prove that this situation is not possible.

0.2 Przytycki's Proof

In 1994,Józef Przytycki published a proof which proves this question of the tricolorability of knots sums. His approach is slightly different from the approach in this paper.
For instance, when Przytycki talks of tricolorability, he denotes the number of different tricolorings by $tri(\mathrm{L})$. Note that the unknot only has trivial colorings, so $tri(\text{unknot}) = 3$.
Przytycki proves many facts about tricolorings, including:
Lemma 1.4: $tri(\mathrm{L})$ is always a power of 3.
For this situation, lemma 1.5 is more important. This answers Adams' question on tricolorability. We repeat Przytycki's argument for completeness
Lemma 1.5a: $tri(\mathrm{L}_1)tri(\mathrm{L}_2) = 3tri(\mathrm{L}_1\#\mathrm{L}_2)$ [4]
Proof: An n-tangle is a part of a link diagram placed in a 2-disk, with $2n$ points on the disk boundary (n inputs and n outputs). We show first that for any 3-coloring of a 1-tangle (a tangle with one input and one output; See Figure 1), the input arc has the same color as the output arc. Consider a trivial component, C, such as an unknot. Let T be our 3-colored tangle, and let the 1-tangle T' be obtained from T by adding C close to the boundary of the tangle, where it only cuts T near the input and output. The 3-coloring of T can be extended to a 3-coloring of T' (in three different ways) because of ambient isotopy. But if we try to color C, we immediately see that it is possible if and only if the input and out put arcs of T have the same color. Thus if we consider a knot sum $L_1\#L_2$, we see from above that the arcs joining L_1 and L_2 have the same color. Therefore the formula $tri(\mathrm{L}_1)tri(\mathrm{L}_2) = 3tri(\mathrm{L}_1\#\mathrm{L}_2)$ follows. We can see from this lemma that if you take 2 links



and

Figure 4: 1-tangles

which are non-tricolorable, or in otherwords only have trivial tricolorings ($tri(\mathrm{L})$=3), then the number of tricolorings of $L_1\#L_2$ has to be 3. [4]

0.3 Determinants and Coloring Matrices

To algebraically prove the knot sum of two nontricolorable knots is nontricolorable, we will use the concept of determinants of knots.

A knot diagram can be labeled with a **mod$p$ coloring** if each edge can be labeled with an integer from 0 to p-1 such that:

1. at each crossing the relation $2x - y - z \equiv 0$ modp holds, where x is the label on the overcrossing and y,z are the labels on the undercrossings, and
2. at least two labels are distinct

Note: p is restricted to odd prime numbers. [1] We will see later that if the determinant of a knot is not prime it can be reduced down to a prime number. Also, observe when $p = 3$, this is a tricoloring.

The **coloring system of equations** (CSE) of a knot diagram is the system of equations assigned to the knot diagram by reading the equation 2x-y-z≡0 with respect to the specified modulus. [3]

The **coloring matrix** of a knot is the matrix of the coefficients of the CSE, where the rows are the crossings and the columns are the arcs. [3]
EXAMPLE:

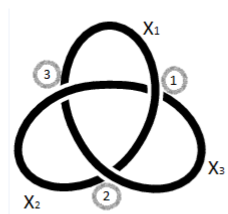


Figure 5: Trefoil

CM:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

CSE:
2(x_1)-x_2-x_3≡0
2(x_3)-x_1-x_2≡0
2(x_2)-x_1-x_3≡0

The above example labels each crossing and arc of the trefoil graph. The coloring matrix looks at the crossings and labels the overcrossing with a 2 and undercrossings with -1. If there is an arc that is not in the crossing we are looking at, then we label it with 0 in the coloring matrix. [See Figure 3]
But how do we find p for a mod p coloring?
When we look at the coloring matrix of a knot diagram, the determinant of that matrix is zero because it is linearly dependent. Deleting any one column and any one row of the coloring matrix of a knot diagram yields a new matrix.

This new matrix will give you a new determinant. The determinant of a knot is the absolute value of the determinant of the associated $(n-1)$ x $(n-1)$ matrix. [1]

Theorem 1: The knot can be labeled mod p if and only if the corresponding set of equations has a nontrivial mod p solution. [1] In other words, so long as the determinant of the smaller matrix is p or a multiple of p, then the knot can be labeled mod p.

Also, the number of solutions of the matrix is determined by the modp nullity of the matrix. This is to say how many p-colorings are possible on a knot diagram. An example of this is the trefoil. The trefoil is tricolorable, but there are 9 different tricolorings (3 of which are trivial).

Theorem 2: The determinant of a knot and its mod p rank are independent of the choice of diagram and labeling. [1] This tells us that no matter what projection of the knot we are looking at, every projection has the same determinant and mod p rank

Lemma: A knot is p-colorable if and only if p divides d, where d is the determinant of the knot.

Notice from this lemma that if the determinant of a knot is not divisible by three, then the knot is not tricolorable.

EXAMPLE

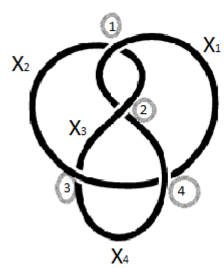


Figure 6: Figure 8 knot

$$\begin{pmatrix} -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 2 & 0 & -1 & -1 \end{pmatrix}$$

This is the coloring matrix for this figure eight knot diagram. The determinant of this matrix is 0. A row and column must still be deleted to find the determinant of the knot. If the last row and column are eliminated the new matrix looks like:

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

The determinant of this 3 x 3 coloring matrix is 5, which implies the determinant of the knot is 5. So by the above lemma, this knot diagram is 5-colorable. If the determinant of a knot is 15, then it is 3-colorable and 5-colorable.
In this paper, when we see det(K_1) for some knot we are looking at the absolute value of the determinant of the coloring matrix for that knot. These determinants of knots are the central part of the algebraic proof of Adams' question.
Conjecture: det(K_1#K_2)= $\pm det(K_1)$*det(K_2)
This is to say that the determinant of the knot sum K_1#K_2 is equal to the multiplication of the determinants of the knots K_1 and K_2
We will prove this conjecture in section 5.

0.4 Knot sums and determinants

Let's look at what happens with the coloring matrices of knot sums. Notice the x_i's are the arc labels, and the circled numbers are the crossing labels.

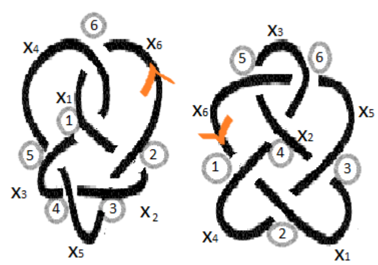


Figure 7: Two 6-Crossing Knots

Here we have two 6 crossing knots and their respective coloring matrices. Denote the knot on the left side as K_1 and knot on the right side as K_2. The coloring matrix for K_1= M_1 =

$$\begin{pmatrix} 2 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2 & 0 & -1 \end{pmatrix}$$

and M_2=

$$\begin{pmatrix} -1 & 0 & 0 & 2 & 0 & -1 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & -1 & -1 \end{pmatrix}$$

det(K_1)= 9 and det(K_2)= 11. The knot sum of these two knots looks like:

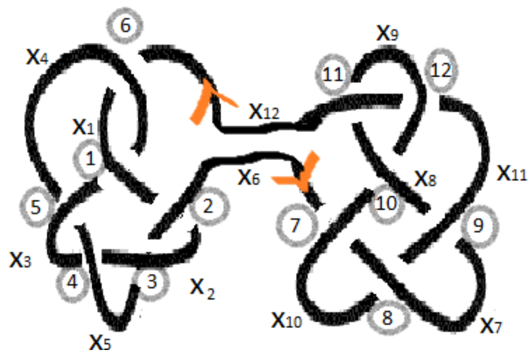


Figure 8: $K_1\#K_2$

$$\left(\begin{array}{cccccc|cccccc} 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & \boxed{0} & 0 & 0 & 0 & 0 & 0 & \boxed{-1} \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1 \end{array}\right)$$

Figure 9: Coloring Matrix of $K_1\#K_2$

det($K_1\#K_2$)= 99 (after we delete a row and column from the CM)
Notice this 12 x 12 coloring matrix. The lines indicate a unique sectioning that occurs. The top left 6 x 6 matrix looks almost exactly like the coloring matrix M_1. The only change is the squared 0 where the squared -1 should be. Similarly, the bottom right 6 x 6 matrix looks like the coloring matrix of M_2 except for the circled 0 where the circled -1 should be. We will call these similar matrices M_1* and M_2*, respectively.

Lemma: Let K_1 be the knot diagram on the left side of the knot sum with m crossings and the coloring matrix M_1, and K_2 be the knot diagram on the right side with n crossings and the coloring matrix M_2. There is a way to always get a knot sum coloring matrix $M_1 \# M_2$ to have the structure where we have M_1* (the similar matrix to M_1) in the m x m block in the top left corner, M_2* in the n x n block in the bottom right corner, with the rest of the matrix $M_1 \# M_2$ filled with zeros with the exception of a -1 in the (m+1, m) spot and (m,n) spot.

Proof: Assume K_1 be the knot diagram on the left side of the knot sum with m crossings and the coloring matrix M_1, and K_2 be the knot diagram on the right side with n crossings and the coloring matrix M_2.

The knot sum arcs are the two connecting arcs in the diagram. For example, in Figure 5, the knot sum arcs are x_6 and x_{12}.

To clarify any misconception, when it is mentioned that an arc is "going into" a crossing on an oriented knot is shown below in Figure 10:

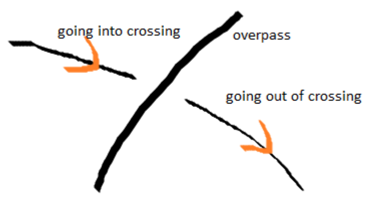


Figure 10: Crossing Example

To get the specific structure of the coloring matrix shown above, then:

1. The orientation of the arcs on K_1 and K_2 where the knot sum arcs will connect the two knots cannot be the same direction. So if we look at Figure 4, x_6 in K_1 and x_6 in K_2 end up having the knot sum arcs in Figure 5. So those two arcs cannot have the same orientation, and they do not (one of the arcs is oriented up, the other is oriented down).

2. At the first crossings that knot sum arcs pass going from one side of the diagram to the other side, the knot sum arc should be going into a crossing as an underarc. In Figure 5, notice the knot sum arc x_{12} goes from the right side to the left side of the diagram, and at the first crossing it comes to on the left side (crossing 6) it is an underarc.

3. After labeling the orientation of each knot, knowing which arcs will turn into knot sum arcs, label the mth crossing of K_1 and the first crossing of K_2 such that both of those crossings will have a knot sum arc going into them.

4. When looking at the individual diagrams (before they are summed together), the arcs going into the mth crossing of K_1 and the first crossing of K_2 should be labeled x_m and x_n respectively.

If we cannot do this at first, we can switch K_1 and K_2 (such that K_2 is now on the left and K_1 on the right), rotate one of the knot diagrams, or switch all the crossings in one of the diagrams to achieve the above requirements.

0.5 Proof

In this section we will prove using these coloring matrices and determinants that when we knot sum two nontricolorable knots together, the knot sum will also be nontricolorable.
First, take a closer look at the coloring matrix $M_1 \# M_2$. We have the m x m coloring matrix M_1* in the top left corner and the n x n coloring matrix M_2* in the bottom right corner.
Proposition 1: det(M_1*) = $\pm det(K_1)$ and det(M_2*)= $\pm det(K_2)$
First, let A=[a_{ij}] be an n x (n+1) matrix whose columns sum to the zero vector. Let $\hat{\mathcal{A}}_k$ be the matrix with the kth column removed.
Lemma: det($\hat{\mathcal{A}}_k$) = (-1)$^{k+1}$det($\hat{\mathcal{A}}_1$)
Proof: Let $\hat{\mathcal{A}}_k$ be the matrix with the kth column removed.
The determinant of this matrix will have the following structure:

$$\begin{aligned}
det &= a_{11}|\hat{A}_{k11}| - a_{21}|\hat{A}_{k21}|...... + a_{(n+1)1}|\hat{A}_{k(n+1)1}| \\
&= -\sum a_{1j}|\hat{A}_{k1j}| - a_{2j}|\hat{A}_{k2j}|..... + a_{(n+1)j}|\hat{A}_{k(n+1)j}| \\
&= -\sum_{j=2}^{n+1} det[A_j, A_2, ..., \hat{A}_k, ...A_{n+1}] \\
&= -det[A_k, A_2, ..., \hat{A}_k, ..., A_{n+1}] \\
&= (-1)^{k+1} det[A_2, ..., A_k, ..., A_{n+1}] \\
&= (-1)^{k+1} det\hat{A}_1
\end{aligned}$$

Corollary: det($\hat{\mathcal{A}}_k$) = (-1)$^{j+k}$det($\hat{\mathcal{A}}_j$)
Proof:

$$\begin{aligned}
det(\hat{\mathcal{A}}_k) &= (-1)^{k+1} det(\hat{\mathcal{A}}_1) \\
&= (-1)^{k+1}((-1)^{j+1} det(\hat{\mathcal{A}}_j)) \\
&= (-1)^{k+j} det(\hat{\mathcal{A}}_j)
\end{aligned}$$

If we look at the row with just one -1 and a 2 in either M_1 or M_2, we will say the -1 is in the jth column and the 2 is in the kth column. From this we get:

$$\begin{aligned}
detM &= (-1)^{1+j}(-1)det(\hat{\mathcal{A}}_j) + (-1)^{1+k} * 2 * det(\hat{\mathcal{A}}_k) \\
&= (-1)^j det(\hat{\mathcal{A}}_j) + (-1)^{1+k} * 2 * (-1)^{j+k} * det(\hat{\mathcal{A}}_j) \\
&= det(\hat{\mathcal{A}}_j)[(-1)^j + 2(-1)^{j+1}] \\
&= (-1)^j det(\hat{\mathcal{A}}_j)[1-2] \\
&= (-1)^{j+1} det(\hat{\mathcal{A}}_j)
\end{aligned}$$

From this we can see that the determinant of M_1 and M_2 will always be equal to the positive or negative determinant of its respective knot
Now consider the general coloring matrix $M_1\#M_2$ again [See Figure 8]. Recall that M_1 is an m x m matrix and M_2 is an n x n, making the coloring matrix $M_1\#M_2$ an $(m+n)$ x $(m+n)$ matrix. The determinant of this $(m+n)$ x $(m+n)$ matrix is zero. To find the determinant of this knot we must delete a row and column.
Notice in Figure 8 the placement of the circled (and squared) -1's. One is in the (m,n) spot, and the other is in the $(m+1,m)$ spot.
Since we can choose any row and column to delete we will delete the mth row and mth column such that both of those -1's are also deleted. Figure 9 shows the structure of the coloring matrix after the deletion of the row and column.

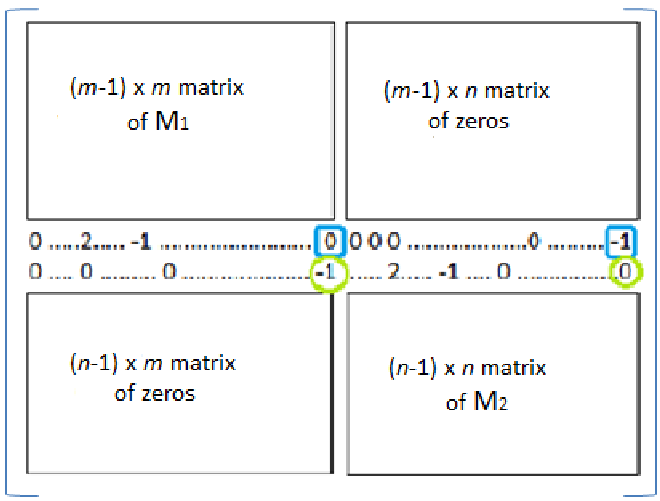


Figure 11: General $(m+n)$ x $(m+n)$Coloring Matrix

A well-known fact and a property of square matrices is that if you have:

M=

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A,B are smaller square matrices, the det(M)=(detA)(detB)
We have already shown that det(M_1*)= $\pm K_1$ and det(M_2*)=$\pm K_2$, so we know the determinant of the m x m or n x n matrix that is left after deleting a row or column.
The smaller $(m$-1) x $(m$-1) or $(n$-1) x $(n$-1) matrix will have the same determinant as its respective knot up to sign. (det((m-1) x (m-1))= $\pm det(K_1)$, similar for K_2) We have no doubt of this because we deleted the only row that changed in that respective coloring matrix.

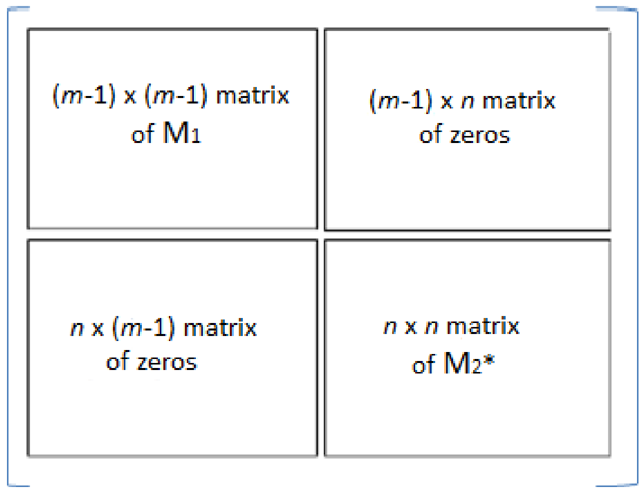


Figure 12: General (m+n-1) x (m+n-1) Coloring Matrix

So taking into consideration both cases, in case 1, det(A)= $\pm det(K_1)$ and det(B)=det(M_2*)= $\pm det(K_2)$. It clearly follows that det(K_1#K_2) = $\pm det(K_1)$det(K_2).
Thus, if neither of the knots K_1 nor K_2 are tricolorable, meaning neither of their determinants are divisible by 3, then det(K_1#K_2) will also not be divisible by 3, and therefore not tricolorable.

0.6 Acknowledgements

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