Skew-Tsankov algebraic curvature tensors in the Lorentzian setting

Kaetlin Taylor
Research Experience for Undergraduates
California State University, San Bernardino
San Bernardino, California 92407

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Abstract

We begin by surveying already known results about model spaces with a metric of signature (p, q) whose skew-symmetric curvature operator commutes. We then determine possible Jordan normal forms for a skew-adjoint operator and study the decomposability of skew-Tsankov models in the Lorentzian ($p = 1$) setting.

1 Introduction

In the study of differential geometry, an object is measured by how much it curves in space. To measure this curvature, we will utilize a portrait of the Riemann curvature tensor known as the *algebraic curvature tensor*, defined below.

Definition 1.1 Let V be a vector space with $e_1, e_2, e_3, e_4 \in V$ and $R \in \otimes^4 V^*$, a multilinear function satisfying the following:

$$R(e_1, e_2, e_3, e_4) = R(e_3, e_4, e_1, e_2) = -R(e_2, e_1, e_3, e_4) \text{ and}$$
$$R(e_1, e_2, e_3, e_4) + R(e_3, e_1, e_2, e_4) + R(e_2, e_3, e_1, e_4) = 0.$$

Using this tensor and its corresponding properties, we can find out useful information about the geometry and topology of the object in question.

We will utilize the 0-model throughout the paper, $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, R)$, where V is a vector space with dimension m, $\langle\cdot,\cdot\rangle$ is a nondegenerate symmetric bilinear form, called a metric, and R is the algebraic curvature tensor defined previously. We say a model space $\mathfrak{M}$ is *decomposable* if $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 = (V_1, \langle\cdot,\cdot\rangle_1, R_1) \oplus (V_2, \langle\cdot,\cdot\rangle_2, R_2)$, and *indecomposable* otherwise.

Let $\pi = \text{span}\{x, y\}$ be a nondegenerate 2-plane in V. We define the skew-symmetric curvature operator by

$$\mathcal{R}(\pi) :=\mid (x,x)(y,y) - (x,y)^2 \mid^{-1/2} R(x,y).$$

We utilize the Jordan normal form of $\mathcal{R}(\pi)$ to classify possibile forms of the operator.

We will define the real Jordan block as

$$J_{\mathbb{R}}(\lambda, k) := \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix} \text{ on } \mathbb{R}^k,$$

and the complex Jordan block as

$$J_{\mathbb{C}}(a + bi, k) := \begin{bmatrix} A & I_2 & 0 & \dots & 0 \\ 0 & A & I_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A & I_2 \\ 0 & 0 & \dots & 0 & A \end{bmatrix} \text{ on } \mathbb{R}^{2k}$$

$$\text{where } A := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ and } I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We say $\mathfrak{M}$ is *skew-Tsankov* if the skew symmetric curvature operator commutes, i.e. if $\mathcal{R}(\xi_1, \xi_2)\mathcal{R}(\xi_3, \xi_4) = \mathcal{R}(\xi_3, \xi_4)\mathcal{R}(\xi_1, \xi_2)$ for all ξ_i [1].

In the Riemannian setting, a metric φ and any skew-symmetric endomorphism ψ can be simultaneously diagonalized (with 1-dimensional complex Jordan blocks) with respect to one another. This is not always true in the pseudo-Riemannian setting, specifically in the Lorentzian setting. The simultaneous diagonalization of Riemannian metrics lends to many results regarding the structure of the corresponding algebraic curvature structure and its decomposition.

In [2], Brozos-Vázquez and Gilkey provide information about skew-Tsankov models that are also Jacobi-Tsankov, meaning the Jacobi operator commutes.

Theorem 1.1. *Let $\mathfrak{M}$ be a model. If $\mathfrak{M}$ is Jacobi-Tsankov, then (a) $J(x)^2 = 0$ for all x in V, (b) $R = 0$ if $\mathfrak{M}$ is Lorentzian or Riemannian, and (c) $\mathfrak{M}$ is Jacobi-Tsankov if $\mathfrak{M}$ is indecomposable with dim($\mathfrak{M}$) < 14, and is skew-Tsankov.*

In [3] Brozos-Vázquez and Gilkey present a full classification of skew-Tsankov Riemannian models.

Theorem 1.2. *If $\mathfrak{M}$ is a Riemannian skew-Tsankov model, then there exists an orthogonally direct sum decomposition $V = V_1 \oplus \ \dots \ \oplus V_k \oplus U$ where dim(V_i) = 2 and $R = R_1 \oplus \dots \oplus R_k \oplus 0$.*

While [2] classifies the decomposition of skew-Tsankov models that are also Jacobi-Tsankov in the higher signature setting (up to dimension 14) and [3] fully classifies the decomposition in the Riemannian case, little is known about the conditions of decomposition for skew-Tsankov models in the Lorentzian setting, i.e., with $p = 1$. One of the several obstacles we run in to while examing the skew-symmetric curvature operator in the pseudo-Riemannian setting is that the eigenvalues are no longer purely imaginary [3]. This creates the possibility of many more Jordan normal forms representing the skew-symmetric curvature operator than in the Riemannian case. By determining what the possible Jordan normal forms will be in the case of a 0-model whose metric is Lorentzian, we can conclude several results about their structure. In this paper we aim to describe conditions for a skew-adjoint operator when we have a Lorentzian skew-Tsankov model in dimension 4. We can then outrule possible forms the skew-symmetric curvature operator, and thus determine the possible ways a model space that has these restraints decomposes.

2 Results

Theorem 2.1 *The possible Jordan normal forms for a skew-adjoint operator in dimension 4 with a Lorentzian metric are as follows: $J_{\mathbb{C}}(a+bi,1) \oplus J_{\mathbb{C}}(a+bi,1)$, $J_{\mathbb{C}}(a+bi,1) \oplus J_{\mathbb{R}}(\lambda,2)$, $J_{\mathbb{C}}(a+bi,1) \oplus J_{\mathbb{R}}(\lambda,1) \oplus J_{\mathbb{R}}(\eta,1)$, $J_{\mathbb{R}}(\lambda,3) \oplus J_{\mathbb{R}}(\eta,1)$, $J_{\mathbb{R}}(\lambda,2) \oplus J_{\mathbb{R}}(\eta,1) \oplus J_{\mathbb{R}}(\tau,1)$, and $J_{\mathbb{R}}(\lambda,1) \oplus J_{\mathbb{R}}(\eta,1) \oplus J_{\mathbb{R}}(\tau,1) \oplus J_{\mathbb{R}}(\delta,1)$.*

We know that any matrix can be written in its respective Jordan normal form, so we begin by considering all of the possible Jordan normal forms for a 4x4 matrix. We will determine whether it is possible to obtain each of these forms from a skew-adjoint operator when the metric has signature (1,3). The possible Jordan normal forms for a 4 x 4 matrix are $J_{\mathbb{R}}(\lambda,4)$, $J_{\mathbb{R}}(\lambda,3) \oplus J_{\mathbb{R}}(\eta,1)$, $J_{\mathbb{R}}(\lambda,2) \oplus J_{\mathbb{R}}(\eta,2)$, $J_{\mathbb{R}}(\lambda,2) \oplus J_{\mathbb{R}}(\eta,1) \oplus J_{\mathbb{R}}(\delta,1)$, $J_{\mathbb{R}}(\lambda,1) \oplus J_{\mathbb{R}}(\eta,1) \oplus J_{\mathbb{R}}(\delta,1) \oplus J_{\mathbb{R}}(\gamma,1))$, $J_{\mathbb{C}}(a+bi,1) \oplus J_{\mathbb{R}}(\lambda,1) \oplus J_{\mathbb{R}}(\eta,1)$, $J_{\mathbb{C}}(a+bi,1) \oplus J_{\mathbb{R}}(\lambda,2)$ $J_{\mathbb{C}}(a+bi,1) \oplus J_{\mathbb{C}}(c+di,1)$, and $J_{\mathbb{C}}(a+bi,2)$.

We begin by considering the case when a skew-adjoint operator has Jordan normal form

$$A = J_{\mathbb{C}}(a+bi,2) = \begin{pmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}$$

We calculate
$Ae_1 = ae_1 - be_2$
$Ae_2 = be_1 + ae_2$
$Ae_3 = e_1 + ae_3 - be_4$
$Ae_4 = e_2 + be_3 + ae_4$

We will use the notation φ_{ij} to represent the inner product we get from our metric φ for $\langle e_i, e_j \rangle$.

We then use these equations to calculate the folowing:

$0 = (Ae_1, e_1) = a\varphi_{11} - b\varphi_{12}$
$0 = (Ae_2, e_2) = a\varphi_{22} + b\varphi_{12}$
$0 = (Ae_3, e_3) = \varphi_{13} + a\varphi_{33} - b\varphi_{34}$
$0 = (Ae_4, e_4) = \varphi_{24} + b\varphi_{34} + a\varphi_{44}$
$(Ae_1, e_2) = -(e_1, Ae_2) = b\varphi_{22} = b\varphi_{11}$
$(Ae_1, e_3) = -(e_1, Ae_3) = a\varphi_{13} - b\varphi_{23} = -\varphi_{11} - a\varphi_{13} - b\varphi_{14}$
$(Ae_1, e_4) = -(e_1, Ae_4) = a\varphi_{14} - b\varphi_{24} = -\varphi_{12} - b\varphi_{13} - a\varphi_{14}$
$(Ae_2, e_3) = -(e_2, Ae_3) = a\varphi_{23} + b\varphi_{13} = -\varphi_{12} - a\varphi_{23} - b\varphi_{24}$
$(Ae_2, e_4) = -(e_2, Ae_4) = a\varphi_{24} + b\varphi_{14} = -\varphi_{22} - b\varphi_{23} - a\varphi_{24}$
$(Ae_3, e_4) = -(e_3, Ae_4) = \varphi_{14} - b\varphi_{44} + a\varphi_{34} = -\varphi_{23} - b\varphi_{33} - a\varphi_{34}$

from which we derive the following list of equations:

1.) $a\varphi_{11} = a\varphi_{22} = b\varphi_{12}$
2.) $a\varphi_{13} = a\varphi_{22} + b\varphi_{34}$
3.) $\varphi_{24} = -b\varphi_{34} - a\varphi_{11}$
4.) $b\varphi_{23} = \varphi_{11} + 2a\varphi_{13} - b\varphi_{14}$
5.) $b\varphi_{24} = \varphi_{12} + b\varphi_{13}$
6.) $b\varphi_{13} = -\varphi_{12} + b\varphi_{24}$
7.) $b\varphi_{14} = -\varphi_{22} - b\varphi_{23} + 2a\varphi_{24}$
8.) $\varphi_{14} + b\varphi_{11} = -\varphi_{23} + b\varphi_{22}$
9.) $b\varphi_{11} = b\varphi_{22}$

We will consider the cases where $a \neq 0$ and $b \neq 0$, $a = 0$ and $b = 0$, $a = 0$ and $b \neq 0$, and $a \neq 0$ and $b = 0$.

Case 1: Suppose $a = b = 0$. Then it immediately follows that $\varphi_{11} = \varphi_{12} = \varphi_{13} = \varphi_{22} = \varphi_{24} = 0$ and $\varphi_{14} = -\varphi_{23}$.

Case 2: Suppose $a = 0$ and $b \neq 0$. Then by 1), we have $\varphi_{11} = \varphi_{22} = 0$. We then conclude by 2), 3), and 5) that $\varphi_{13} = \varphi_{24} = \varphi_{12} = 0$. By 8), we have $\varphi_{14} = -\varphi_{23}$.

Case 3: Suppose $a \neq 0$ and $b = 0$. Then by 1), 2), 3), and 5), we have $\varphi_{12} = \varphi_{34} = \varphi_{24} = \varphi_{13} = 0$. It follows from 7) that $\varphi_{14} = -\varphi_{23}$.

Case 4: Suppose $a \neq 0$ and $b \neq 0$. By 9), we have $\varphi_{11} = \varphi_{22}$, but by 4) and 7), we have $\varphi_{11} = -\varphi_{22}$. Thus $\varphi_{11} = \varphi_{22} = 0$. Then by 7), we have $\varphi_{14} = -\varphi 23$. Thus, by the previous statement and 7), we have $2ab\varphi_{24} = b\varphi_{12} = a\varphi_{22}$. Therefore we have $a\varphi_{22} = a\varphi_{11} = b\varphi_{12}$. From 1), we have $b\varphi_{22} = b\varphi_{11} = a\varphi_{12}$. So either $a = b$ or $\varphi_{22} = \varphi_{11} = \varphi_{12} = 0$. If $a = b$, then by 3) 4), and 7), we find a must equal 1, therefore $a = b = 1$. Then by 4) and 7) we have $\varphi_{11} = \varphi_{24}$. Since we already have $\varphi_{11} = \varphi_{22} = 2\varphi_{24}$, we now have $\varphi_{11} = \varphi_{12} = \varphi_{13} = \varphi_{22} = \varphi_{24} = 0$. If $\varphi_{22} = \varphi_{11} = \varphi_{12} = 0$, then by 4) and 7), $\varphi_{13} = \varphi_{24} = 0$ also.

Thus, for any a and b, we have $\varphi_{11} = \varphi_{12} = \varphi_{13} = \varphi_{22} = \varphi_{24} = 0$ and $\varphi_{14} = -\varphi_{23}$. Therefore we can represent our metric using the array below.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & \varphi_{14} \\ 0 & 0 & -\varphi_{14} & 0 \\ 0 & -\varphi_{14} & x & y \\ \varphi_{14} & 0 & y & w \end{pmatrix}$$

We next perform a change of basis to find an orthonormal basis for φ.

We set $d = \varphi_{14}$. Then we perform a change of basis to form the matrix f, an orthnormal matrix for φ, by performing the following transformation.

$f_1 = e_1$
$f_2 = e_2$
$f_3 = e_3 + x_1 e_1 + x_2 e_2$
$f_4 = e_4 + y_1 e_1 + y_2 e_2$

$$f = \begin{pmatrix} 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \\ 0 & -d & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$$

We then perform another change of basis to create a matrix g that represents φ using only ± 1.

$g_1 = \frac{1}{\sqrt{|2d|}}(f_1 + f_4)$
$g_2 = \frac{1}{\sqrt{|2d|}}(f_1 - f_4)$
$g_3 = \frac{1}{\sqrt{|2d|}}(f_2 + f_3)$
$g_4 = \frac{1}{\sqrt{|2d|}}(f_2 - f_3)$

$$\text{So } g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The matrix g is an orthonormal basis for φ. Since there are 2 positive and 2 negative eigenvalues, we know that φ must have balanced signature, i.e. signature = (2,2). Thus, the matrix representation $J_{\mathbb{C}}(a+bi,2)$ is not possible in the Lorentzian case for a skew-adjoint operator.

Next, we move on to the Jordan normal form $J_{\mathbb{R}}(\lambda,4)$. We calculate

$Ae_1 = \lambda e_1$
$Ae_2 = e_1 + \lambda e_2$
$Ae_3 = e_2 + \lambda e_3$
$Ae_4 = e_3 + \lambda e_4$

We use these equations to calculate the following system of equations:

1.) $0 = \lambda\varphi_{11}$
2.) $0 = \varphi_{12} + \lambda\varphi_{22}$
3.) $0 = \varphi_{23} + \lambda\varphi_{33}$
4.) $0 = \varphi_{34} + \lambda\varphi_{44}$
5.) $\lambda\varphi_{12} = -\varphi_{11} - \lambda\varphi_{12}$
6.) $\lambda\varphi_{13} = -\varphi_{12} - \lambda\varphi_{13}$
7.) $\lambda\varphi_{14} = -\varphi_{13} - \lambda\varphi_{14}$
8.) $\varphi_{13} + \lambda\varphi_{23} = -\varphi_{22} - \lambda\varphi_{23}$
9.) $\varphi_{14} + \lambda\varphi_{24} = -\varphi_{23} - \lambda\varphi_{24}$
10.) $\varphi_{24} + \lambda\varphi_{34} = -\varphi_{33} - \lambda\varphi_{34}$

We know that $\lambda \neq 0$ or else we would have odd rank, which is not possible in the skew-symmetric case [4], so by equation 1 $\varphi_{11} = 0$. By equation 5, 6, and 7, it follows that $\varphi_{12} = \varphi_{12} = \varphi_{12} = 0$. We continue solving this system of equations until we see that $\varphi_{ij} = 0$ for all i and j, thus φ is degenerate, so this case may be eliminated.

The next case we consider is $J_{\mathbb{R}}(\lambda,2) \oplus J_{\mathbb{R}}(\eta,2)$. We then calculate
$Ae_1 = \lambda e_1$
$Ae_2 = e_1 + \lambda e_2$
$Ae_3 = \eta e_3$
$Ae_4 = e_3 + \eta e_4$

and derive the following system of equations:

1.) $0 = \lambda\varphi_{11}$
2.) $0 = \varphi_{12} + \lambda\varphi_{22}$
3.) $0 = \eta\varphi_{33}$
4.) $0 = \varphi_{34} + \eta\varphi_{44}$
5.) $\lambda\varphi_{12} = -\varphi_{11} - \lambda\varphi_{12}$
6.) $\lambda\varphi_{13} = -\eta\varphi_{13}$
7.) $\lambda\varphi_{14} = -\varphi_{13} - \eta\varphi_{14}$
8.) $\varphi_{13} + \lambda\varphi_{23} = -\eta\varphi_{23}$
9.) $\varphi_{14} + \lambda\varphi_{24} = -\varphi_{23} - \eta\varphi_{24}$
10.) $\eta\varphi_{34} = -\varphi_{33} - \eta\varphi_{34}$

In order to preserve even rank, $\lambda \neq 0$ implies $\eta \neq 0$ and $\lambda = 0$ implies $\eta = 0$, so we will examine each case.

Case 1: Suppose $\lambda, \eta \neq 0$. Then by equations 1 and 3, we have $\varphi_{11} = \varphi_{33} = 0$. By equations 5 and 10, $\varphi_{34} = \varphi 12 = 0$. It then follows from equations 2 and 4 that $\varphi_{44} = \varphi_{22} = 0$. By equation 6, we have $\lambda\varphi_{13} = -\eta\varphi_{13}$, so $\varphi_{13} = 0$ or $\lambda = -\eta$. We will examine each case.

> Case a: Suppose $\varphi_{13} = 0$ and $\lambda \neq -\eta$. Then it directly follows from equations 7, 8, and 9 that $\varphi_{14} = \varphi_{23} = \varphi_{24} = 0$, so the metric is degenerate.
>
> Case b: Suppose $\lambda = -\eta$ and $\varphi_{13} \neq 0$. By equation 7, $\varphi_{13} = 0$, so we reach a contradiction.
>
> Case c: Suppose $\lambda = -\eta$ and $\varphi_{13} = 0$. We conclude that $\varphi_{14} = -\varphi_{23}$ and all other $\varphi ij = 0$. We can perform the same change of basis as in the Jordan normal form $J_{\mathbb{C}}(a+bi, 2)$ and conclude that this must have balanced signature, and therefore does not exist in the Lorentzian case.

Case 2: Suppose $\lambda = \eta = 0$. We find that $\varphi_{11} = \varphi_{12} = \varphi_{13} = \varphi_{14} = \varphi_{23} = \varphi_{33} = \varphi_{34} = 0$. Then φ is degenerate because there does not exist a j for every i such that $\varphi_{ij} \neq 0$. Thus, the Jordan normal forms we have not outruled are $J_{\mathbb{C}}(a+bi, 1) \oplus J_{\mathbb{C}}(a+bi, 1)$, $J_{\mathbb{C}}(a+bi, 1) \oplus J_{\mathbb{R}}(\lambda, 2)$, $J_{\mathbb{C}}(a+bi, 1) \oplus J_{\mathbb{R}}(\lambda, 1) \oplus J_{\mathbb{R}}(\eta, 1)$, $J_{\mathbb{R}}(\lambda, 3) \oplus J_{\mathbb{R}}(\eta, 1)$, $J_{\mathbb{R}}(\lambda, 2) \oplus J_{\mathbb{R}}(\eta, 1) \oplus J_{\mathbb{R}}(\tau, 1)$, and $J_{\mathbb{R}}(\lambda, 1) \oplus J_{\mathbb{R}}(\eta, 1) \oplus J_{\mathbb{R}}(\tau, 1) \oplus J_{\mathbb{R}}(\delta, 1)$. ■

Conjecture 2.2 *The skew-symmetric curvature operator is always block diagonalizable (with only blocks of size 2) in dimension 4 when the signature of the metric is Lorentzian.*

In Theorem 2.1, the only Jordan normal form that is possible for a skew-adjoint operator that is not block diagonalizable is $J_{\mathbb{R}}(\lambda, 3) \oplus J_{\mathbb{R}}(\eta, 1)$. While we have not yet been able to conclude that this Jordan normal form is not possible, we believe that it is not geometrically realizable. If we could eliminate this case, then the conjecture holds.

Conjecture 2.3 *Let $\mathfrak{M}$ be a skew-Tsankov Lorentzian 0-model of dimension 4. Then $\{\mathcal{R}(\pi) \mid \pi$ is a nondegenerate 2-plane$\}$ are simultaneously diagonalizable.*

If we knew that the Jordan normal forms of the skew-symmetric curvature operator were block diagonalizable (by the Conjecture 2.2) it would help determine if simultaneous diagonalizability would follow since we also know they commute (because they are skew-Tsankov).

Conjecture 2.4 *Let $\mathfrak{M}$ be a 4 dimensional Lorentzian 0-model. Then $\mathfrak{M}$ is skew-Tsankov if and only if there exists an orthogonal direct sum decomposition $V = V_1 \oplus \ldots \oplus V_k \oplus U$ where dimension $V_i = 2$ and $R = R_1 \oplus \ldots \oplus R_k \oplus 0$.*

In [4], Brozos-Vázquez and Gilkey show that Riemannian 0-models decompose this way. It seems likely that decomposition would occur in the same way if Conjecture 2.3 is true.

3 Open Questions

1. What are the possible Jordan normal forms for a skew-Tsankov 0-model with a Lorentzian metric when $dim(\mathfrak{M}) = 3$?
2. How does a skew-Tsankov Lorentzian 0-model decompose when $dim(\mathfrak{M}) > 4$?
3. Theorem 1.1 addresses the pseudo-Riemannian case up to dimension 14, in which a counterexample is found. It would be interesting to seek additional counterexamples and/or consider higher dimensions.

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