Linear Ropelength Upper Bounds of Twist Knots

Michael Swearingin

August 2010

California State University, San Bernardino
Research Experience for Undergraduates

Abstract

This paper provides an upper bound on the ropelength of certain families of knots which is linear in the crossing number. A modified version of the notation described by Dowker and Thistlethwaite is used to identify desired structures in knot diagrams that allow for algorithmic construction of certain knots on a cubic lattice.

1 Introduction

A knot is a closed, non-intersecting curve in 3-space. There is interest in knowing how much rope is needed to form particular knots given a specified radius and whether there are patterns for certain types of knots. To that end, upper bounds on the minimal ropelength of particular knots or families of knots are examined. This paper will not consider links of more than one knot component although the following results could reasonably be modified to accomodate such structures.

A conformation of a knot is a physical realization of the knot. The ropelength of a particular conformation is the ratio of the arclength of the rope to its radius. The ropelength of a knot K, or $Rop(K)$, is the smallest ratio over all possible conformations of K. The ropelength of a knot is scale invariant so we will fix the radius of the rope to always be one which also means that the curvature at any point on the knot can not exceed one.

We will use the term planar projection of a knot to mean a conformation that has been projected onto a plane. We will say that a planar projection of a knot is R-2 reduced if the projection presents no opportunities for a crossing-removing Reidemeister 2 move and if the conformation is reduced in the usual sense: a conformation is reduced if crossings can not be removed simply by fixing one part of the knot and rotating or untwisting the other part as in the figure below.

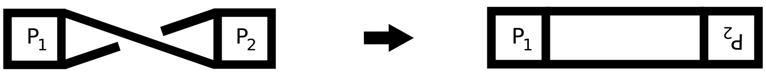
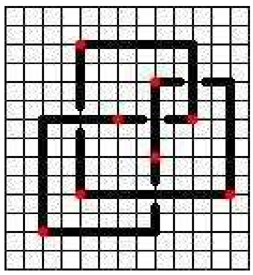


Figure 1: P_1 and P_2 represent knotted arcs of the knot.

1.1 The Cubic Lattice

As it pertains to upper bounds for ropelength, any known ropelength of a conformation is an upper bound on the knot itself and conformations can be constructed or embedded on a cubic lattice. Embedding a knot on a cubic lattice is done by drawing it in three-space with edges of unit length only in directions that are perpendicular to the x, y, or z plane. As with the original knot, the curve must be closed and non-intersecting. Since we are using a radius of one, we must also demand that the distance between any two parallel edges must be at least two. The length of this curve is simply the number of edges used to construct it. Though the corners exceed the curvature specifications, it is easy to see that a spacing of at least two units between parallel strands of the knot allows us to curve each corner into a quarter circle with one unit radius without making two strands closer than two units away from each other. The length of the knot once its corners have been smoothed into quarter circles will be shorter than the original curve on the cubic lattice with the right-angled corners, so the total number of edges becomes an upper bound on the ropelength of the knot.

For the purposes of this paper, we will only be using two parallel planes of the cubic lattice which allows us to simplify our diagram even more by projecting it onto one of those planes. The planes will be two units apart in order to satisfy the rope radius of one. In the event that two different points on the lattice are mapped to the same point via the projection, the resulting diagram will show a crossing where one strand is segmented to denote that it travels underneath. Also, our diagrams will use red dots to signify what would be a change of plane on the cubic lattice meaning each red dot also represents two additional edges. The figure to the right is an example of the 4-crossing knot, or figure-8 knot, having been embedded on a cubic lattice and projected down to one plane. The total number of edges for this diagram of the figure-8 knot is 80.



1.2 Modified Dowker Sets

Let P be a planar projection of a knot K. Beginning at any point on P and heading in a fixed direction, we assign integers consecutively beginning with 0 to each crossing every time a crossing is encountered in a manner similar to Dowker and Thistlethwaite [1]. Hence, each crossing will be assigned two numbers: one for the undercrossing and one for the overcrossing.

Definition 1.1: A crossing pair $C \in \mathbb{Z} \times \mathbb{Z}$ of P consists of any two numbers assigned to the same crossing of P in the scheme described above, where the first entry is filled by the number assigned to the undercrossing.

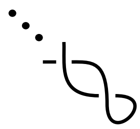
Definition 1.2: A Modified Dowker Set (MDS) D of P is an unordered set of all the crossing pairs of P given a starting point on P and a fixed direction in which to traverse and label the crossings of P. As with other sets, we allow $|D|$ to signify the number of elements in D, which is the number of crossings in P.

Definition 1.3: Let $U = \{(u_1, o_1), (u_2, o_2)\} \in D$, where D is an MDS. U is a twist unit if $|u_1 - o_2| = 1$ or $|D| - 1$ and $|u_2 - o_1| = 1$ or $|D| - 1$. In other words, U is a twist unit if each pair's undercrossing is consecutive modulo $|D|$ to the other pair's overcrossing.

Definition 1.4: Let T be a subset of an MDS, D, where $2 \leq |T| \leq |D|$. We will define T to be a twist if there exists an ordering of T, say $T = \{p_1, p_2, ...p_{|T|}\}$, such that $\forall i \in \mathbb{N}$ where $i < |T|$, $\{p_i, p_i + 1\}$ is a twist unit and $\forall q \in D \backslash T$, both $\{q, p_1\}$ and $\{q, p_{|T|}\}$ are not twist units. p_1 and $p_{|T|}$ are called the ends of the twist. By this definition, any two distinct twists from the same MDS are necessarily disjoint.

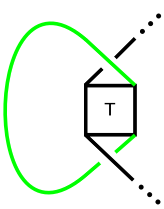
We make a note that our definition of twist requires at least 2 crossings though other authors have allowed twists to consist of only 1 crossing.

Lemma 1.4.1: Suppose a planar projection of a knot P admits a twist T. If an end of T connects back to itself without crossing anything else in between, then P is not a reduced diagram.



Proof 1.4.1: The figure to the right portrays what happens at the end of the twist if it loops back on itself without crossing anything else in between. This presents a Reidemeister 1 move opportunity so the diagram can not be a reduced diagram.

Lemma 1.4.2: If a twist has an even number of crossings and one end connects to the other end, then that twist can not be part of a link with one component.



Proof 1.4.2: The figure to the left illustrates this scenario where the box with T inside represents an even number of crossings in the twist. T having an even number of crossings means that the strand coming into it from the top-left will exit out the bottom-left, so it can not be connected to the other strand in the twist as it is a closed loop.

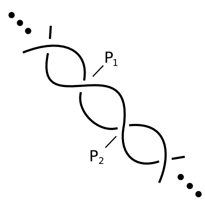
Definition 1.5: A knot K can be called an n - twist knot if it admits an R-2 reduced planar projection P with MDS D such that there are distinct $T_1, T_2, ...T_n \in D$ where $D = T_1 \cup T_2 \cup ... \cup T_n$.

Lemma 1.5.1: It can not be the case that one end of a twist connects to the end of a different twist twice.

Proof 1.5.1: Let T_1 and T_2 denote the twists that are connected twice end to end. Let p_1 and p_2 be the ends in question of T_1 and T_2 respectively. There are two possible cases in which this could happen:

Case 1: They connect in such a way that the projection of T_1 and T_2 is alternating. Since T_1 and T_2 are disjoint, $p_1 \notin T_2$ and $p_2 \notin T_1$. If the projection is still alternating, then the undercrossing strand of p_1 would become the overcrossing strand of p_2 and vice versa. However, this would make $\{p_1, p_2\}$ a twist unit so T_1 and T_2 could not be twists by definition which is a contradiction.

Case 2: They connect in such a way that the projection of T_1 and T_2 is not alternating. T_1 and T_2 themselves must remain alternating so if combined they are no longer alternating, the undercrossing strand and overcrossing strand of p_1 must become the undercrossing strand and overcrossing strand respectively of p_2. This results in a Reidemeister 2 move opportunity to decrease the number of crossings as the figure to the right shows. However, since we have required our projections to be R-2 reduced, this leads to a contradiction.



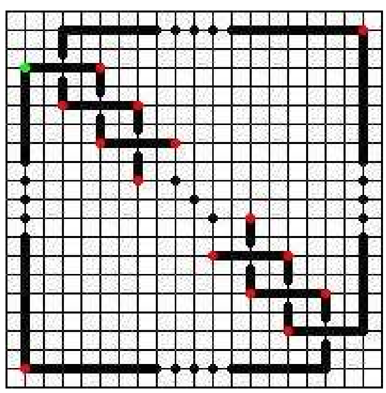
2 One-Twist Knots

Lemma 2.1: Suppose K is a 1 - twist knot, that is K admits a planar projection P with MDS D such that D is a twist. Then $Rop(K) \leq 20 * |D|$.

Proof 2.1: It is sufficient to show that K can be constructed on a cubic lattice with $20 * |D|$ edges.

We first observe that $|D|$ must be odd because if it were even, then K would be a link of two components by lemma 1.4.2.

The algorithm proceeds as follows:



Step 1: We begin at the origin on the upper plane, which is denoted with the green dot on the figure to the right.
Step 2: Do steps 3 and 4 a combined total of $|D| - 1$ times.
Step 3: Go East 4 units, change planes.
Step 4: Go South 4 units, change planes.
Step 5: Go East 4 units.
Step 6: Go North $2*|D|$ units, change planes
Step 7: Go West $2 * |D|$ units.
Step 8: Do steps 9 and 10 a combined total of $|D| - 1$ times.
Step 9: Go South 4 units, change planes.
Step 10: Go East 4 units, change planes.
Step 11: Go South 4 units.
Step 12: Go West $2 * |D|$ units, change planes.

Step 13: Go North $2 * |D|$ units. End.

From step 1 to step 7 we have not yet made any crossings and we end up exactly 2 edges East and North from where we started in the opposite plane. Thus, every time we execute either step 8 or step 9 we add a crossing which gives $|D| - 1|$ crossings. In addition, step 11 also produces a crossing before looping back around to the origin for a total of $|D|$ crossings.

From step 2 to step 5 we will have gone East $2*|D|+2$ times and South $2*|D|-2$ times. Similarly, from step 8 to step 11 we will go East an additional $2*|D|-2$ times and South an additional $2*|D|+2$ times. Thus, we go both East and South a total of $4*|D|$ times each. Steps 6, 7, 12, and 13 tell us to go both West and North a total of $4*|D|$ times each making the total of units travelled in the cardinal directions $16*|D|$. In addition, we change planes $2*|D|$ times, and since each change requires 2 units to be travelled, the total length of the arc is $20*|D|$.

Theorem 2.2: If K is a 1 - twist knot then $Rop(K) \leq 20 * c$ where c is the minimal crossing number of K.

This results directly from the fact that the 1 - twist projection of a knot is alternating and reduced. Kauffman showed that in such cases the number of crossings is the minimal crossing number [2].

3 Two-Twist Knots

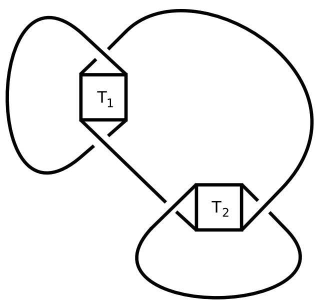
Analyzing 2-twist knots is more tedious than 1-twist knots as we will have to consider the different ways in which 2-twist knots can be connected diagrammatically. Also, we will need to consider whether there are restrictions on the number of crossings in the twists themselves. Once we have narrowed down the possibilities, we will then produce the algorithms for constructing 2-twist knots on a case by case basis. Since the diagrams may not be alternating, we will also need to show that the upper bound is linear in the minimal crossing number and not the number of crossings in the diagram.

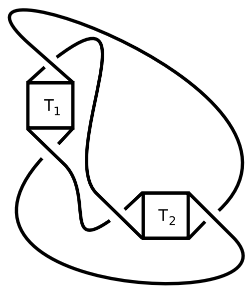
Lemma 3.1: Suppose a prime knot K is a 2 - twist knot, that is K admits a planar projection P with MDS D such that $D = T_1 \cup T_2$ where T_1 and T_2 are distinct twists. Then $Rop(K) < 20 * |D|$.

Proof 3.1: It is sufficient to show that K can be constructed on a cubic lattice with less than $20 * |D|$ edges.

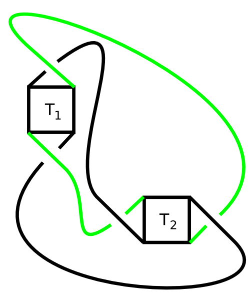
We will first examine the circumstances in which the two twists could be adjoined and possible values of $|T_1|$ and $|T_2|$. Lemma 1.4.1 tells us that no end of any twist can connect to itself and lemma 1.5.1 tells us that the twists can not be connected end to end. Alternating versus non-alternating aside, we are left with two possibilities of connecting the two twists as seen below. The boxes with T_1 and T_2 inside represent any additional crossings T_1 and T_2 might have except for the ones already depicted.

First, let us consider the diagram on the left. If either $|T_1|$ or $|T_2|$ is even, then the whole diagram represents a link of more than one component by lemma 1.4.2. If both $|T_1|$ and $|T_2|$ are odd, then the diagram represents a composite knot which can be seen if we were to cut the two strands connecting T_1 and T_2. The knot would then disintegrate into two 1 - twist knots: one with $|T_1|$ crossings and the other with $|T_2|$ crossings. Furthermore, both of these scenarios remain unchanged if the projection is non-alternating. Since we are only considering prime links of only one component, we can conclude that the diagram on the left can not occur.





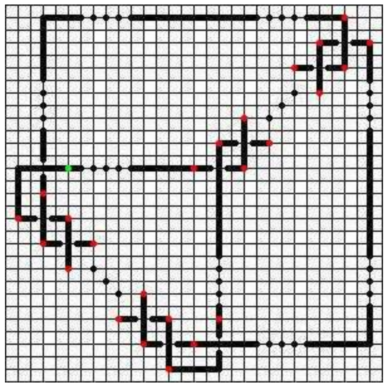
Thus, we are left with the diagram above on the right where neither twist is connected to itself in any way. However, it can not be the case that both $|T_1|$ and $|T_2|$ are odd in in this diagram either as it would also be a link of two components as depicted in the figure to the right. An odd number of crossings means that an incoming strand will come into the twist from one side and exit on the other side.



This scenario also remains unchanged if the projection is non-alternating so we are left with the following four cases: $|D|$ is even and P is alternating; $|D|$ is even and P is non-alternating; $|D|$ is odd and P is alternating; or $|D|$ is odd and P is non-alternating. Nevertheless, we have shown that in every case no twist can be connected to itself in any way, so the 2 - twist projections will have the same basic shape.

Case 1: $|D|$ is even and P is alternating

The case where $|T_1| = |T_2| = 2$ is the figure-8 knot which we were already able to embed on a cubic lattice with 80 or $20 * |D|$ edges in Section 1.1 of this paper. If $|T_1| \neq 2$ or $|T_2| \neq 2$, let $m = max\{|T_1|, |T_2|\}$ and $n = min\{|T_1|, |T_2|\}$. We can therefore assume that $m > 2$.



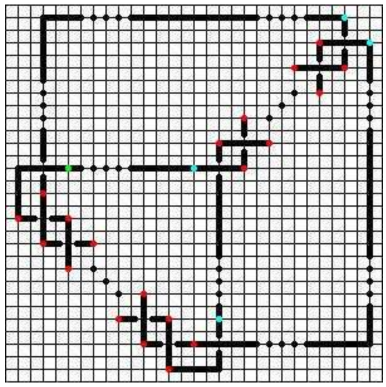
Step 1: We begin again at the origin on the upper plane, which is denoted with the green dot on the figure to the right.
Step 2: Go West 4 units, South 4 units, then change planes.
Step 3: Do steps 4 and 5 a combined total of $m - 2$ times.
Step 4: Go East 4 units, change planes.
Step 5: Go South 4 units, change planes.
Step 6: Go East 4 units, North 4 units, then change planes.
Step 7: Go North $2 * m - 6$ units.
Step 8: Do steps 9 and 10 a combined total of n times.
Step 9: Go North 4 units, change planes.
Step 10: Go East 4 units, change planes.
Step 11: Go South $2 * m + 2 * n - 4$ units.
Step 12: Go West $2 * n + 2$ units, change planes.
Step 13: Do steps 14 and 15 a combined total of $m - 2$ times.
Step 14: Go West 4 units, change planes.
Step 15: Go North 4 units, change planes.
Step 16: Go North $2 * n + 2$ units.
Step 17: Go East $2 * m + 2 * n - 4$ units.
Step 18: Do steps 19 and 20 a comined total of n times.
Step 19: Change planes, go South 4 units.
Step 20: Change planes, go West 4 units.
Step 21: Change planes, go West $2 * m - 6$. End.

The total number of units travelled in each of the cardinal directions is $4 * m + 4 * n - 4$. Together with the $2 * m + 2 * n$ change of plane moves for $4 * m + 4 * n$ more units travelled, the total number of edges required to construct K on the cubic lattice is $20 * m + 20 * n - 16$.

Since $m + n = |T_1| + |T_2| = |D|$, $Rop(K) < 20 * |D|$.

Case 2: $|D|$ is even and P is non-alternating

The procedure for the non-alternating case only differs from the alternating case in that the non-alternating case does not need to change planes as many times. The twists themselves must remain alternating by definition so the only places where two overpasses or underpasses can occur consecutively is from an end of one twist to the end of another. Thus, we modify the above algorithm in the follow way:



Steps 6 and 21 should no longer change planes. The last instance of step 10 and the first instance of step 19 should no longer change planes. The diagram to the right illustrates where change of plane moves are omitted using blue dots.

With 4 fewer change of planes moves the resulting number of edges used becomes $20 * |D| - 24$.

It must also be noted that as the 4 crossing knot was the exception in the alternating case, if the crossings are adjusted to be non-alternating then the number of edges required is $20 * |D| - 8$.

Case 3: $|D|$ is odd and P is alternating

The diagram of the case when $|D|$ is odd is similar to the case when $|D|$ is even but the algorithm does differ. Either $|T_1|$ or $|T_2|$ is odd and the other is even, so let o be whichever one is odd and e be whichever one is even.

Step 1: Start at the origin which will be our upper plane.
Step 2: Go West 4 units, South 4 units, and change planes.
Step 3: Do steps 4 and 5 a combined total of $o - 2$ times, so step 4 will have been done one more time than step 5.
Step 4: Go East 4 units and change planes.
Step 5: Go South 4 units and change planes.
Step 6: Go East $2 * e + 2$ units.
Step 7: Go North $2 * o + 2 * e - 4$ units.
Step 8: Do steps 9 and 10 a combined total of e times.
Step 9: Change planes and then go West 4 units.
Step 10: Change planes and then go South 4 units.
Step 11: Go South $2 * o - 6$ units and change planes.
Step 12: Go South 4 units, West 4 units, and change planes.
Step 13: Do steps 14 and 15 a combined total of $o - 2$ times.
Step 14: Go North 4 units and change planes.

Step 15: Go West 4 units and change planes.
Step 16: Go North $e * 2 + 2$ units.
Step 17: Go East $2 * o + 2 * e - 4$ units.
Step 18: Do steps 19 and 20 a combined total of e times.
Step 19: Change planes and go South 4 units.
Step 20: Change planes and go West 4 units.
Step 21: Change planes and go West $2 * o - 6$ units. End.

The total number of units travelled in each of the cardinal directions is $4 * o + 4 * e - 4$. Together with the $2 * o + 2 * e$ change of plane moves for $4 * o + 4 * e$ more units travelled, the total number of edges required to construct K on the cubic lattice is $20 * o + 20 * e - 16$.

Since $o + e = |T_1| + |T_2| = |D|$, $Rop(K) < 20 * |D|$.

Case 4: $|D|$ is odd and P is non-alternating

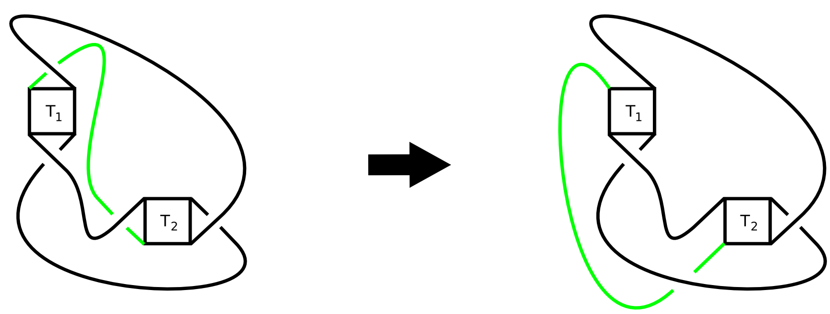
As with case 2, the only places that can be non-alternating are from an end of one twist to the end of the other. We use the same algorithm as in case 3 but with the following modifications:

Steps 11 and 21 should no longer change planes. The first instance of steps 9 and 19 should no longer change planes.

With 4 fewer change of planes moves, the number of edges used is $20 * |D| - 24$.

Lemma 3.2: If a prime knot K is a non-alternating 2 - twist knot, that is K admits a planar projection P with MDS D such that $D = T_1 \cup T_2$ where T_1 and T_2 are distinct twists, then the minimal crossing number of K is $|D| - 1$.

Proof 3.2: In this case, we can move one strand so that the diagram becomes alternating while eliminating one crossing and remaining reduced. The move is illustrated by the figures below:



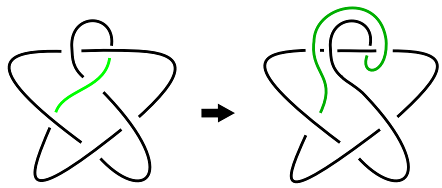
Even if there are no additional crossings in T_1 and T_2 other than the two for each already depicted, the diagram is still reduced. Now that it is also alternating, we can use Kauffman's results to conclude that $|D| - 1$ is the minimal crossing number [2].

Theorem 3.3: If a prime knot K is a 2 - twist knot, then $Rop(K) < 20 * c$ where c is the minimal crossing number of K.

This follows directly from lemmas 3.1 and 3.2:

In the alternating case, $c = |D|$ [2]. Thus, $Rop(K) = 20*|D|-16 \implies Rop(K) < 20 * c$. In the non-alternating case, $c = |D| - 1$ so $Rop(K) = 20 * |D| - 24 < 20 * |D| - 20 \implies Rop(K) < 20 * c$ as well. The only possible exception is the 4 crossing, non-alternating case, where the number of edges would come out to be $20 * c + 12$ if constructed as a 2-twist knot. However, the 4 crossing, non-alternating conformation can be reduced to the trefoil which is a 1-twist knot and we have already shown that the ropelength of 1-twist knots does not exceed $20 * c$.

Consequently, this means that the ropelength of the family of knots that can be conformed into 2-twist knots is also bounded above linearly in the minimal crossing number regardless of whether the 2-twist conformation is non-alternating. As an example, we can reverse the step used in lemma 3.2 to turn the alternating, 6-crossing, non-twist knot below into a non-alternating, 7-crossing, 2-twist knot and then the theorem would apply.



4 Further Study

4.1 N - Twist Knots

An appropriate question to ask at this point is whether it is possible to construct any n - twist knot on a cubic lattice so that the length is bounded above linearly in the minimal crossing number. One of the difficulties in defining a rigorous alogrithm for drawing knots with more than two twists is that more than two twists can be connected in more than one way. Nevertheless, a linear upper bound for these knots seems plausible though it might depend on how many twists there are and how they are connected.

4.2 Non-Twist Knots

The hope would be that the algorithms for twist knots could somehow be modified to accomodate some or all non-twist knots. One way this hope could be realized is if it could be shown every knot admits some projection in which it is an n - twist knot for some n. If n - twist knots were shown to have ropelengths which were bounded above linearly in the minimal crossing number then the linear upper bound question would be solved.

4.3 Sharper Upper Bounds

The upper bounds established by this paper are not sharp as there are ways of modifying the diagrams so that fewer edges are used while still only using two planes of the cubic lattice. For instance, 1 - twist knots constructed in a more circular shape require fewer edges than if constructed along a diagonal. It may prove necessary to have better upper bounds as the number of twists increases so that if a knot needed to be conformed to an n - twist knot by adding crossings, the upper bound would still be proportional to the minimal crossing number.

Acknowledgements

The author would like to thank the CSUSB REU advisors C. Dunn and R. Trapp for their support as well as www.analyzemath.com for their free downloadable graph paper. This research was jointly funded by the NSF grant DMS-0850959, and California State University, San Bernardino.

References

[1] C.H. Dowker and M.B. Thistlethwaite, Classification of Knot Projections. Topology and its Applications 16 (1983), 19-31.

[2] Kauffman, Louis H. State models and the Jones polynomial. Topology 26 (1987), no. 3, 395–407.