ROPELENGTH OF TWISTS AND ALTERNATING KNOTS

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Abstract. We show that the ropelength of any alternating conformation is at least 4 times the crossing number of its knot type. This is a generalization of the known result for reduced alternating conformations. In addition we give lower bounds on the ropelength contribution of twists.

1. Introduction

A *knot* is a simple closed curve in $\mathbb{R}^3$. A *knot diagram* is a two dimensional representation of a knot, formed by projecting the knot onto a plane. A *crossing* is a point in a knot diagram where two arcs of the knot projection intersect. An *overpass* is an arc for which at each of its crossings it passes over the other arc. An alternating knot is is one which admits a diagram such that the crossings alternate between overpass and underpass when traveling along the orientation of the knot. A crossing is said to be reducible if it can be removed as in figure 1. We will refer to the distance between an overpass and an underpass at their crossing as the their *crossing distance*. A knot diagram is reduced if it has no reducible crossings.

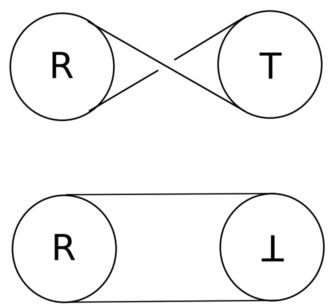


Figure 1. Reducible Crossing

To a given conformation, k, we associate a quantity called the injectivity radius, which is the maximum thickness that can be assigned to the curve before it self-intersects. We define its *ropelength* to be the ratio of k's arclength to its injectivity radius, r, and we write

$$\mathrm{Rop}(k) := \frac{\text{arclength}}{r}.$$

Supposing K is a knot type, we define the ropelength of K to be

$$\mathrm{Rop}(K) := \inf_{k \in K} \mathrm{Rop}(k).$$

Since ropelength is scale invariant we will choose $r = \frac{1}{2}$. The only knot type for which the ropelength is known precisely is the unknot, with a ropelength of 2π.

2. Ropelength of Alternating Conformations

In order to obtain a ropelength lower bound for all alternating conformations we will employ Sadjadi's methods. Lemmas 2.1 through 2.4 are a reproduction of her work. We will then employ a result by Denne, Diao, and Sullivan.

For an alternating conformation k we label each overpass, p_i, and its underpass, q_i, where $1 \leq i \leq cr(k)$. We assume that its diagram is projected onto the xy-plane. Let o_i denote the height of p_i along the z-axis at its crossing and u_i the height of q_i.

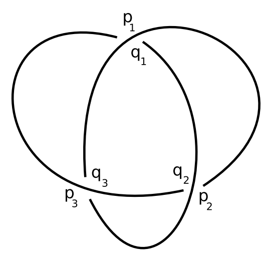


Figure 2. Labeled Diagram of a Trefoil

Definition Define the height function $h : \mathbb{R}^3 \to \mathbb{R}$ by $h(x, y, z) := z$. We will denote the image of k under h with $h(k)$. Observe that $l(h(k)) \leq l(k)$.

To find a lower bound for $l(h(k))$, we construct what Sadjadi called the taut image of k. The point of crossing of each overpass of k and the point of crossing of each underpass are selected, and each point is connected by a line segment to the next point along the orientation of k. Thus, the image under h of the resulting polygonal curve, which we will write as $t(k)$, has arc length less than or equal to that of $h(k)$.

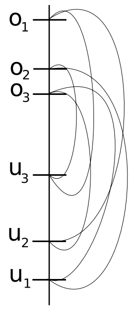


FIGURE 3. Graph of $h(k)$ for the trefoil above, with paths bent for viewing

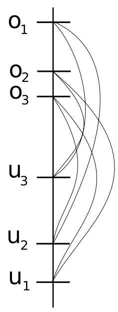


FIGURE 4. Graph of $t(k)$

Definition A pair (p_i, q_i) is said to be split by a point $z_0 \in \mathbb{R}$ if $o_i \geq z_0 \geq u_i$.

Lemma 2.1. *A point z_0 which splits b overpass-underpass pairs of a conformation k has at least $2b$ edges which cross it in $t(k)$.*

Proof. Let a be the number of unsplit pairs above z_0 and b the number split. This means that there are $a + b$ overpasses above z_0. The case that results in the fewest edges crossing z_0 is when every edge incident with an overpass above z_0 is from an underpass above z_0. In this case, since there are 2 edges incident with every overpass and underpass, we have $2a$ edges joining underpasses above z_0 to overpasses above z_0, leaving the remaining $2(a + b) - 2a = 2b$ edges to cross z_0. □

Definition We now give another labeling of the overpass and underpass heights, h_i for $1 \leq i \leq 2cr(k)$ where $h_i \geq h_{i+1}$ for all i. Define b_n to be the number of overpass-underpass pairs split by a point $z_0 \in [h_n, h_{n+1}]$ where $z_0 \in (h_n, h_{n+1})$ if $h_n \neq h_{n+1}$.

Lemma 2.2. *An alternating conformation k for which every crossing distance is at least* 1 *has arclength at least* $\sum_{n=1}^{cr(k)-1} 2b_n(h_n - h_{n+1})$.

Proof. There are b_n pairs split by a point $z_0 \in [h_n, h_{n+1}]$, so there are at least $2b_n$ edges crossing z_0 by Lemma 3.1. We know that there are $2cr(k) - 1$ intervals of the form $[h_n, h_{n+1}]$, and each edge crossing an interval $[h_n, h_{n+1}]$ has length at least $h_n - h_{n+1}$, so $l(t(k)) \geq \sum_{n=1}^{2cr(k)-1} 2b_n(h_n - h_{n+1})$. □

Lemma 2.3. *For an alternating conformation k where each crossing distance is at least 1,* $\sum_{i=1}^{n} 2(o_i - u_i) = \sum_{n=1}^{2cr(k)-1} 2b_n(h_n - h_{n+2})$.

Proof. For a particular o_i and u_i there is some $x, y \in \{1, ..., 2cr(k)\}$ such that $o_i = h_x$ and $u_i = h_y$. The interval $[h_x, h_y]$ is partitioned by $h_x \geq h_{x+1} \geq, ..., \geq h_{y-1} \geq h_y$ and the length of $[o_i, u_i]$ is $\sum_{j=x}^{y-1} h_j - h_{j+1}$. So

$$\sum_{i=1}^{cr(k)} (o_i - u_i) = \sum_{i=1}^{cr(k)} \left(\sum_{j=x}^{y-1} (h_j - h_{j+1}) \right).$$

Every term in this sum is of the form $h_n - h_{n+1}$, and since each term occurs once for every pair it splits, we know that each $h_n - h_{n+1}$ occurs b_n times in the sum. Hence,

$$\sum_{i=1}^{cr(k)} (o_i - u_i) = \sum_{n=1}^{2cr(k)-1} b_n(h_n - h_{n+1}).$$

This means

$$\sum_{i=1}^{cr(k)} 2(o_i - u_i) = \sum_{n=1}^{2cr(k)-1} 2b_n(h_n - h_{n+1}).$$

□

Lemma 2.4. *Let k be an alternating conformation for which each crossing distance is at least 1. Then arclength$(k) \geq 2cr(k)$.*

Proof. We know $o_i - u_i \geq 1$ for all i so

$$l(k) \geq l(t(k)) \geq \sum_{n=1}^{cr(k)-1} b_n(h_n - h_{n+1}) =$$

$$\sum_{i=1}^{cr(k)} 2(o_i - u_i) \geq \sum_{i=1}^{cr(k)} 2(1) = 2cr(k).$$

□

Lemma 2.5. *Let k be an alternating conformation with diagram D. If for some crossing with overpass o and underpass u, $|o-u|<1$, then there exists an arc of the conformation γ_{ou} from o to u which is unknotted and has no crossings with any other arc of k.*

Proof. A result from Denne, Diao, and Sullivan states that if ab is a secant of k with $|a-b|<1$, then the ball with diameter $\overline{ab}$ intersects k in a single unknotted arc whose length is at most $\arcsin|a-b|$. Therefore, when we construct the ball S with diameter $\overline{ou}$, its intersection with k is an unknotted arc since $|o-u|<1$. So there exists an arc from o to u, γ_{ou}, which is unknotted.

The sphere S projects to a circle of diameter $|o-u|$ in D. When viewing γ_{ou} in D, since o and u have the same x and y coordinates, the projection of γ_{ou} in D is a simple closed curve, so any other arc which crosses it must cross an even number of times. Hence, such an arc would have either two consecutive overpasses, two consecutive underpasses, or one of each. The first two cases contradict that D is an alternating diagram. As for the last case, the arc would have to pass within the sphere S, contradicting that there can be only one arc in the sphere. □

Lemma 2.6. *Given an alternating conformation k with diagram D, let c be some crossing. If an arc along the orientation of k from the overpass of c to the underpass or vice versa is unknotted and has no crossings with any other arc of k, then when the arc is replaced by a line segment from the overpass at c to the underpass, the resulting conformation k' is alternating and of the same knot type.*

Proof. Supposing γ is the unknotted arc from the overpass o at c to the underpass u, we know that prior to o is an underpass and after u is an overpass since D is alternating. Hence, by replacing γ with $\overline{ou}$ the resulting conformation has an alternating diagram when projected onto the same plane as that of D. □

Theorem 2.1. *Let K be an alternating knot and $k \in K$ an alternating conformation. Then $Rop(k) \geq 4cr(K)$.*

Proof. If every crossing distance in D is at least 1, then we apply Lemma 3.1. Suppose that there are some crossings with crossing distance less than 1. Consider such a crossing with overpass o and underpass u. By Lemma 3.2 γ_{ou} is unknotted and has no crossings with any other arc of k. Thus, by Lemma 3.3 k' is alternating and $k' \in K$. This means that $\text{arclength}(k) \geq \text{arclength}(k') \geq 2cr(K)$. Dividing by the injectivity radius $\frac{1}{2}$, $\text{Rop}(k) \geq 4cr(K)$. □

3. Ropelength of Twists

Lemma 3.1. *It is either the case that every point on both arcs of a twist are at least a distance 1 away from every point on the other arc or the twist is unknotted.*

Proof. We view the twist from the side and label the overpasses and underpasses of the twist as in figure 5.

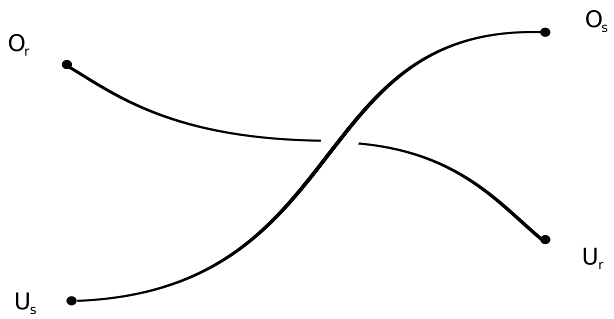


Figure 5

If a point on one arc were less than a distance 1 from some point on the other, then we construct the ball whose diameter is the line segment joining those two points. The intersection of the knot with this ball is a single unknotted arc. If this arc contains the portion of the conformation beyond the twist from O_s to U_r, or from O_r to U_s, then it is unknotted as in figure 6.

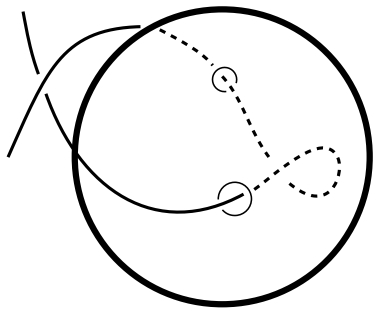


Figure 6

If the arc from O_r to O_s outside the twist is contained in the ball, then $|O_r - O_s| < 1$, so we construct the ball with diameter $\overline{O_r O_s}$. Viewed in the diagram the twist and ball appear as in the figure below.

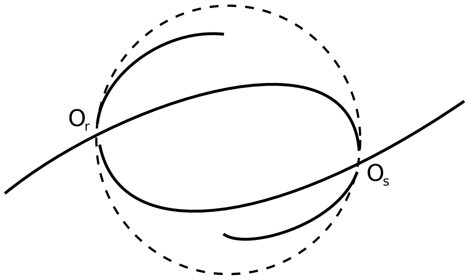


FIGURE 7

In order for the arc to travel from O_r to O_s it would have to cross the twist, contradicting that it is a twist in the diagram. The case where U_r to U_s remains in the ball leads to the same contradiction. If we have none of the previous cases then the intersection of the knot with the ball contains two arcs leading to a contradiction. So in all possible cases, if a point on one arc is less than a distance 1 from a point on the other arc, then the twist is unknotted. $\square$

Lemma 3.2. *For all $A, B, h, k \in \mathbb{R}$,*
$$\sqrt{1+(A+h)^2+(B+k)^2}+\sqrt{1+(A-h)^2+(B-k)^2} \geq 2\sqrt{1+A^2+B^2}.$$

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) := \sqrt{1+x^2+y^2}$. Then f is twice differentiable on $\mathbb{R}^2$ and its Hessian matrix H is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} \frac{y^2+1}{(1+x^2+y^2)^{\frac{3}{2}}} & \frac{-xy}{(1+y^2+x^2)^{\frac{3}{2}}} \\ \frac{-xy}{(1+x^2+y^2)^{\frac{3}{2}}} & \frac{x^2+1}{(1+x^2+y^2)^{\frac{3}{2}}} \end{pmatrix}$$

It is known that f is convex if and only if H is positive semi-definite for all $r \in \mathbb{R}^2$, so let $r = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Then $r^T H r =$

$$(a\ b) \begin{pmatrix} \frac{y^2+1}{(1+x^2+y^2)^{\frac{3}{2}}} & \frac{-xy}{(1+y^2+x^2)^{\frac{3}{2}}} \\ \frac{-xy}{(1+x^2+y^2)^{\frac{3}{2}}} & \frac{x^2+1}{(1+x^2+y^2)^{\frac{3}{2}}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \frac{a^2(y^2+1) - 2abxy + b^2(x^2+1)}{(1+x^2+y^2)^{\frac{3}{2}}}$$
$$\geq \frac{a^2y^2 - 2abxy + b^2x^2}{(1+x^2+y^2)^{\frac{3}{2}}} = \frac{(ay-bx)^2}{(1+x^2+y^2)^{\frac{3}{2}}} \geq 0.$$

So H is positive semi-definite for all $r \in \mathbb{R}^2$ and therefore f is convex.

By the definition of a convex function, for all $x, y \in \mathbb{R}^2$ and $\theta \in [0,1]$, we have $\theta f(x) + (1-\theta)f(y) \geq f(\theta x + (1-\theta)y)$. By choosing $\theta = \frac{1}{2}$, $x = (A+h, B+k)$, and $y = (A-h, B-k)$, we have

$$\frac{1}{2}\sqrt{1+(A+h)^2+(B+k)^2} + \frac{1}{2}\sqrt{1+(A-h)^2+(B-k)^2} =$$

$$\frac{1}{2}f((A+h, B+k)) + (1-\frac{1}{2})f((A-h, B-k)) \geq$$

$$f(\frac{1}{2}(A+h, B+k) + (1-\frac{1}{2})(A-h, B-k)) = \sqrt{1+A^2+B^2}.$$

Hence,

$$\sqrt{1+(A+h)^2+(B+k)^2}+\sqrt{1+(A-h)^2+(B-k)^2} \geq 2\sqrt{1+A^2+B^2}.$$

□

Definition For two curves $r(t) := \langle t, y_r(t), z_r(t)\rangle$ and $s(t) := \langle t, y_s(t), z_s(t)\rangle$ defined for $0 \leq t \leq d$ for some $d \geq 0$ define

$$r_0(t) := \left\langle t, \frac{y_r(t) - y_s(t)}{2}, \frac{z_r(t) - z_s(t)}{2} \right\rangle$$

and

$$s_0(t) := \left\langle t, \frac{y_s(t) - y_r(t)}{2}, \frac{z_s(t) - z_r(t)}{2} \right\rangle.$$

For convenience we will sometimes refer to the y-coordinate of $r_0(t)$ as $y_{r_0}(t)$ and the z-coordinate as $z_{r_0}(t)$, with the same convention for $s_0(t)$.

Lemma 3.3. *Two curves $r(t) := \langle t, y_r(t), z_r(t)\rangle$ and $s(t) := \langle t, y_s(t), z_s(t)\rangle$ defined for $0 \leq t \leq d$ have a total arclength greater than or equal to that of $r_0(t)$ and $s_0(t)$.*

Proof. By Lemma 3.2, we have that for all t

$$
\begin{aligned}
|r'(t)| + |s'(t)| &= \sqrt{1 + y_r'^2 + z_r'^2} + \sqrt{1 + y_s'^2 + z_s'^2} \\
&= \sqrt{1 + \left(\frac{y_r' - y_s'}{2} + \frac{y_r' + y_s'}{2}\right)^2 + \left(\frac{z_r' - z_s'}{2} + \frac{z_r' + z_s'}{2}\right)^2} \\
&+ \sqrt{1 + \left(\frac{y_r' - y_s'}{2} - \frac{y_r' + y_s'}{2}\right)^2 + \left(\frac{z_r' - z_s'}{2} - \frac{z_r' + z_s'}{2}\right)^2} \\
&\geq \sqrt{1 + \left(\frac{y_r'(t) - y_s'(t)}{2}\right)^2 + \left(\frac{z_r'(t) - z_s'(t)}{2}\right)^2} \\
&+ \sqrt{1 + \left(\frac{y_s'(t) - y_r'(t)}{2}\right)^2 + \left(\frac{z_s'(t) - z_r'(t)}{2}\right)^2} \\
&= |r_0'(t)| + |s_0'(t)|\,.
\end{aligned}
$$

□

Definition For $r(t) := \langle t, y_r(t), z_r(t)\rangle$ we may parametrize $r_0(t)$ in cylindrical coordinates as $\langle t, R(t)\cos\theta(t), R(t)\sin\theta(t)\rangle$ where $R(t) := \sqrt{y_{r_0}(t)^2 + z_{r_0}(t)^2}$ and $\theta(t) := \tan^{-1}\left(\frac{z_{r_0}(t)}{y_{r_0}(t)}\right)$. Define

$$r_c(t) := \left\langle t, \frac{1}{2}\cos\theta(t), \frac{1}{2}\sin\theta(t)\right\rangle.$$

Lemma 3.4. *If $R(t) \geq \frac{1}{2}$ for all t, then the arclength of $r_0(t)$ is greater than or equal to the arclength of $r_c(t)$.*

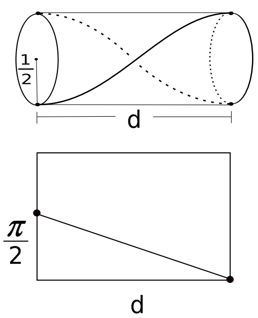
Proof. Observe that for all t,

$$
\begin{aligned}
|r_0'| &= \sqrt{1 + (R'\cos\theta - R\theta'\sin\theta)^2 + (R'\sin\theta + R\theta'\cos\theta)^2} \\
&= \sqrt{1 + R'^2 + (R\theta')^2} \geq \sqrt{1 + \left(\frac{1}{2}\theta'\right)^2} = |r_c'|\,.
\end{aligned}
$$

□

Lemma 3.5. *The minimum arclength of all arcs from $r_c(0)$ to $r_c(d)$ on the cylinder of radius $\frac{1}{2}$ and of height d is $\sqrt{\left(\frac{\pi}{2}\right)^2 + d^2}$.*

Proof. Every such path along the cylinder with the desired endpoints corresponds to a path along a rectangle with side lengths d and π. A straight line between the endpoints on the rectangle is the hypotenuse of a right triangle with legs of length $\frac{\pi}{2}$ and d.



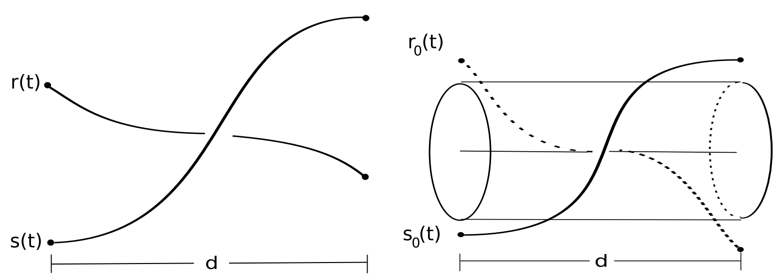
Therefore, the least possible arclength is $\sqrt{\left(\frac{\pi}{2}\right)^2 + d^2}$.

$\square$

In the following theorem, we will choose that the origin lie at the midpoint of the first crossing of the twist. The x-axis extends toward the other crossing of the twist. The y-axis extends into the page and the z-axis extends vertically upward from the xy-plane.

Theorem 3.1. *If every plane orthogonal to the xy-plane and parallel to the y-axis intersects each arc of the twist at exactly one point, then the ropelength contribution of the twist is at least* $2\sqrt{\pi^2 + 4d^2}$.

Proof. The arcs can be parametrized as $r(t) := \langle t, y_r(t), z_r(t)\rangle$ and $s(t) := \langle t, y_s(t), z_s(t)\rangle$ for $0 \le t \le d$, and by lemma 4.3 the arcs $r_0(t)$ and $s_0(t)$ have total arclength less than or equal to that of $r(t)$ and $s(t)$.



Since $r(t) - s(t) = r_0(t) - s_0(t)$ and $|r(t) - s(t)| \ge 1$ for all t, we have $|r_0(t) - s_0(t)| \ge 1$ for all t. Then since $\frac{r_0(t)+s_0(t)}{2}$ lies on the axis of the cylinder, Lemma 3.5 and Lemma 3.6 guarantee that the total arclength of $r_0(t)$ and $s_0(t)$ is at least that of $r_c(t)$ and $s_c(t)$, meaning at least $\sqrt{\left(\frac{\pi}{2}\right)^2 + d^2} + \sqrt{\left(\frac{\pi}{2}\right)^2 + d^2} = \sqrt{\pi^2 + 4d^2}$. Therefore the total arclength of $r(t)$ and $s(t)$ is at least $\sqrt{\pi^2 + 4d^2}$ and they contribute at least $2\sqrt{\pi^2 + 4d^2}$ to the ropelength. $\square$

There is an immediate application of Theorem 3.1 to $(n, 2)$ torus knots. Provided the twists satisfy the criteria of Theorem 3.1 we can guarentee a ropelength of at least 2π times the number of twists in a diagram, which comes from the minimum possible ropelength contribution when $d = 0$. The lower bound can be sharper provided we know the values of d for all the twists. Since any minimal crossing diagram of such a knot consists entirely of twists this may be useful in finding lower bounds for ropelength among all minimal crossing conformations of$(n, 2)$ torus knots.

A potential way to obtain a ropelength contribution of 2π from an arbitrary twist would be to parametrize its two arcs as $r(t) := \langle x_r(t), y_r(t), z_r(t)\rangle$ and $s(t) := \langle x_s(t), y_s(t), z_s(t)\rangle$ and prove that the arc length of $r_0(t)$ and $s_0(t)$ is less than that of $r(t)$ and $s(t)$ by the convexity of the function $f(x, y, z) := \sqrt{x^2 + y^2 + z^2}$. The next step would be to project $r_0(t)$ and $s_0(t)$ onto a sphere of radius $\frac{1}{2}$ to get the desired arclength lower bound of π.

4. Acknowledgments

I would like to thank Dr. Trapp for his guidance, advice, and direction, and I would like to thank Dr. Dunn for his helpful insight. This research was jointly funded by NSF grant DMS-0850959 and California State University at San Bernardino.

References

[1] Adams, Colin C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* W.H. Freeman and Company, 2001. Print.

[2] Elizabeth Denne, Yuanan Diao, and John M. Sullivan. "Quadrisecants give new lower bounds for the ropelength of a knot,"' Geometry and Topology, 10 (2006): 1-26.

[3] Sadjadi, Shala. "On Ropelength of Alternating Links." California State University, San Bernardino, 2007