Linearly Dependent Sets of Algebraic Curvature Tensors with a Nondegenerate Inner Product

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Abstract

This paper examines the effects of linear dependence on a set of three algebraic curvature tensors constructed from symmetric bilinear forms with respect to a nondegenerate inner product, and proceeds to show the possibility for simultaneous orthogonal diagonalization of these forms. Additionally the capability for simultaneous diagonalization lends itself to some useful results regarding the eigenvalues of two of the forms.

1 Introduction

Let V be a real vector space of finite dimension n, with elements $x, y, z, w \in V$. A multilinear function $R \in \otimes^4 V^*$ is known as an *algebraic curvature tensor* if it satisfies the following properties:

$$R(x, y, z, w) = -R(y, x, z, w),$$

$$R(x, y, z, w) = R(z, w, x, y),$$

$$R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0.$$

This last property is known as the **Bianchi Identity**. The set of all algebraic curvature tensors on a vector space V is notated $\mathcal{A}(V)$.

Definition 1. *A bilinear form φ is said to be **symmetric** if it satisfies $\varphi(x, y) = \varphi(y, x)$, and a bilinear form ψ is said to be **antisymmetric** if $\psi(x, y) = -\psi(y, x)$. The set of all symmetric bilinear forms on a vector space V is notated $S^2(V^*)$, and the set of all antisymmetric bilinear forms is notated $\Lambda^2(V^*)$.*

Definition 2. *A **model space** is an algebraic object $(V, < \cdot, \cdot >, R)$, where V is a vector space, $< \cdot, \cdot >$ is a bilinear form on V, and R is an algebraic curvature tensor as previously defined. A **weak model space** is defined to be (V, R).*

Definition 3. *A bilinear form φ on a vector space V is said to be **nondegenerate** if $\varphi(x, v) = 0$ for all $v \in V$ only holds when $x = 0$.*

Definition 4. *Given a basis $\{e_1, ..., e_n\}$ for a symmetric bilinear form φ, some basis vector e_i is said to be **spacelike** if $\varphi(e_i, e_i) > 0$, and **timelike** if $\varphi(e_i, e_i) < 0$.*

Definition 5. *If $\{e_1^-, ..., e_p^-, e_1^+, ...e_q^+\}$ is a basis for some nondegenerate symmetric bilinear form φ, then the basis is **orthonormal** if it satisfies $\varphi(e_i^\pm, e_j^\pm) = \pm\delta_{ij}$, and $\varphi(e_i^-, e_j^+) = \varphi(e_i^+, e_j^-) = 0$ for all i, j.*

Definition 6. *A symmetric bilinear form φ with p timelike basis vectors and q spacelike basis vectors is said to have **signature** (p, q).*

Theorem (Sylvester's Law of Inertia). *The signature of a symmetric bilinear form φ is uniquely determined, and is invariant under isometry.*

For a proof of this theorem, see [5].

Definition 7. *Given a linear map $A : V \to V$, $R_A \in \otimes^4(V^*)$ can be defined as*

$$R_A(x, y, z, w) = \varphi(Ax, w)\varphi(Ay, z) - \varphi(Ax, z)\varphi(Ay, w).$$

Theorem 1. *If A is an endomorphism satisfying $A^* = A$, then $R_A \in \mathcal{A}(V)$.*

This theorem has been proved in [3]. A similar result has been proved in [1] for the other direction provided a particular rank requirement is met.

Theorem 2. *Given $A : V \to V$, with $R_A \in \mathcal{A}(V)$. If Rank $A \geq 3$, then $A^* = A$.*

Theorem 3. *Suppose φ is positive definite, Rank $\tau = n$, and Rank $\psi \geq 3$. If $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then ψ and τ are simultaneously orthogonally diagonalizable with respect to φ.*

The two above theorems are proved in [1], and the primary goal of this paper will be the extension of Theorem 3 to the case where φ is nondegenerate. In the positive definite case, self-adjoint symmetric bilinear forms can be simultaneously diagonalized, however in the nondegenerate case, this may not necessarily be so. Consider the following example:

$$\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \tau = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Note that

$$\tau(e_1, e_2) = \varphi(\tau e_1, e_2) = 0 = \varphi(e_1, e_1) = \varphi(\tau e_2, e_1) = \tau(e_2, e_1).$$

Thus, τ is self-adjoint with respect to the nondegenerate φ. However, for τ to be diagonalizable, it must be similar to a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvalues of τ. But,

$$\det(\tau - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2 = 0 \Leftrightarrow \lambda = 0$$

As a result, τ is similar to the zero matrix, but no P can satisfy the equation $P[0]P^{-1} = \tau$, and thus τ cannot be diagonalized.

Theorem 4. *A family $\mathcal{F} \subset M_n$ of diagonalizable matrices is a commuting family if and only if it is a simultaneously diagonalizable family.*

This theorem, as well as its accompanying proof, appear in [4].

Lemma 1. *Let φ be an inner product of signature (p, q) on a vector space V. There exists a self-adjoint linear map $C \colon V \to V$ satisfying $C^2 = I_n$ so that defining $\varphi^+(x, y) := \varphi(Cx, y)$ fulfills the following:*

1) φ^+ is a positive definite inner product on V.
2) $\varphi(x, y) = \varphi^+(Cx, y) = \varphi^+(x, Cy)\ \forall x, y \in V$.
3) $\varphi^+(x, y) = \varphi(Cx, y) = \varphi(x, Cy)\ \forall x, y \in V$.

This lemma, as well as its proof, can be found in [3]. The C demonstrated in this lemma will be referred to as the **change-of-signature endomorphism**, and such a C is used to move from a nondegenerate inner product to a positive definite one, while satisfying $C = C^{-1} = C^*$.
More explicitly, given some orthonormal basis $\{e_1^-, ..., e_p^-, e_1^+, ..., e_q^+\}$ for an inner product φ, C can be defined to satisfy

$$\varphi(Ce_i^+, e_j^+) = \varphi(e_i^+, e_j^+) = \delta_{ij},$$

$$\varphi(Ce_i^\pm, e_j^\mp) = 0,$$

$$\varphi(Ce_i^-, e_j^-) = -\varphi(e_i^-, e_j^-) = -(-\delta_{ij}) = \delta_{ij}.$$

Before presenting the upcoming lemma, note that R_{CAC}^+ refers to the algebraic curvature tensor constructed from CAC with respect to the inner product φ^+, and this notation will continue to be used throughout the remainder of this paper.

Lemma 2. *If A is an endomorphism, and C the change-of-signature endomorphism as defined above, then*

1. $R_\varphi(Cx, Cy, z, w) = R_C(x, y, z, w) = R_{\varphi^+}(x, y, z, w)$
2. $R_A(Cx, Cy, z, w) = R_{AC}(x, y, z, w) = R_{CAC}^+(x, y, z, w)$

The first relationship follows from the fact that $R_\varphi(Cx, Cy, z, w) = \varphi(Cx, w)\varphi(Cy, z) - \varphi(Cx, z)\varphi(Cy, w) = R_C(x, y, z, w)$, and using the knowledge that $\varphi(Cx, w) = \varphi^+(x, w)$, the second equality in 1) follows trivially. Additionally, $R_A(Cx, Cy, z, w) = \varphi(ACx, w)\varphi(ACy, z) - \varphi(ACx, z)\varphi(ACy, w) = R_{AC}(x, y, z, w)$. For the second equality, $\varphi(ACx, w)\varphi(ACy, z) - \varphi(ACx, z)\varphi(ACy, w) = \varphi^+(CACx, w)\varphi^+(CACy, z) - \varphi^+(CACx, z)\varphi^+(CACy, w) = R^+_{CAC}(x, y, z, w)$.

Theorem 5. *Let $C : V \to V$ be a linear transformation under a positive definite inner product. If C is both self-adjoint and orthogonal, then there is an orthonormal basis for V of the eigenvectors of C with corresponding eigenvalues of absolute value 1.*

This theorem appears in [4], and demonstrates that the eigenvalues of C as described in the previous lemma are all ± 1.

Theorem 6. *Suppose dim(V) ≥ 4, φ is positive definite, Rank $\tau = n$, and Rank $\psi \geq 3$. The set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent if and only if one of the following holds:*

1) $|Spec(\psi)| = |Spec(\tau)| = 1$.
2) $Spec(\tau) = \{\eta_1, \eta_2, \eta_2, ...\}$, and $Spec(\psi) = \{\lambda_1, \lambda_2, \lambda_2, ...\}$, with $\eta_1 \neq \eta_2, \lambda_2^2 = \epsilon(\delta\eta_2^2 - 1)$, and $\lambda_1 = \frac{\epsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1)$ for $\epsilon, \delta = \pm 1$.

This theorem was another result of [1], and will be used to compare the spectrum of the diagonalized symmetric bilinear forms under a nondegenerate inner product to their spectrum in the positive definite case.

The proof of this paper's main theorem will rely on the use of this C to transform a nondegenerate inner product into a positive definite one. Next, it will be shown that a linear dependence relationship between a set of three algebraic curvature tensors in the nondegenerate case implies a different linear dependence under the new positive definite inner product. From this point, it will be demonstrated that each piece of this new linear dependence relationship is indeed an algebraic curvature tensor, as well as that certain rank requirements are met, and thus given this new positive definite inner product, the results of [1] may be applied to conclude that the symmetric bilinear forms are simultaneously diagonalizable with respect to φ^+ as described above. Finally, it will be shown that the ability to simultaneously diagonalize the symmetric bilinear forms under the positive definite inner product implies that they can also be simultaneously diagonalized under the original nondegenerate inner product.

Following this, a discussion of the spectrum of each of the symmetric bilinear forms is in order, and it will be shown that the relationships presented in Theorem 6 also hold in the event that φ is nondegenerate.

2 Results

Theorem 7. *Suppose φ is nondegenerate, Rank $\psi \geq 3$, and Rank $\tau = n$. Additionally, assume that τ is diagonalizable with respect to φ. If the set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent so that R_φ is not a real multiple of R_ψ or R_τ, then ψ and τ are simultaneously diagonalizable with respect to φ.*

Proof. Given some basis $\{e_1^-, ..., e_p^-, e_1^+, ..., e_q^+\}$, φ satisfies $\varphi(e_i^\pm, e_j^\pm) = \delta_{ij}$ and $\varphi(e_i^-, e_j^+) = \varphi(e_i^+, e_j^-) = 0$. In order for us to consider the simultaneous diagonalization of τ and ψ, we need the symmetric bilinear form that we use for our inner product to be positive definite, rather than merely nondegenerate. Consider φ^+, defined by $\varphi^+(x, y) = \varphi(Cx, y)$, with $C : V \to V$ is a linear transformation defined as in Lemma 1. Additionally, Lemma 1 demonstrates that C is self-adjoint with respect to both φ and φ^+. Next, consider the two different cases under which the set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent. First, it is possible that the linear dependence relationship is between only two of the curvature tensors. However, since R_φ is not a multiple of R_ψ or R_τ, we simply need to check the case where $c_2 R_\psi + c_3 R_\tau = 0$. This simplifies to $R_\psi = \lambda R_\tau$ for $\lambda \in \mathbb{R}$, in which case ψ is a real multiple of τ. Given that τ is diagonalizable, it follows that ψ is diagonalizable as well on the same choice of orthonormal basis, and the result holds. Therefore, we need only consider the case where the linear dependence involves all three curvature tensors. If this is true, we have $c_1 R_\varphi + c_2 R_\psi = c_3 R_\tau$, which can be simplified to

$$R_\varphi + \epsilon R_\psi = \delta R_\tau$$

for ϵ, $\delta = \pm 1$. Since C is an isomorphism on the vector space, we can consider $R_\varphi(Cx, Cy, z, w) + \epsilon R_\psi(Cx, Cy, z, w) = \delta R_\tau(Cx, Cy, z, w)$, which using Lemma 2, is equivalent to

$$R_{\varphi^+} + \epsilon R_{\psi C} = \delta R_{\tau C}$$

For Theorem 1.6 of [1] to be applicable, we need ψC and τC to be self-adjoint with respect to φ.

Theorem 8. *ψC is self-adjoint with respect to φ if and only if $C\psi C$ is self-adjoint with respect to the positive definite φ^+, and similarly τC is self-adjoint with respect to φ if and only if $C\tau C$ is self-adjoint with respect to φ^+.*

Proof. Using Lemma 1, $\varphi(\psi Cx, y) = \varphi^+(C\psi Cx, y)$, and similarly $\varphi(\tau Cx, y) = \varphi^+(C\tau Cx, y)$, and the above equivalencies follow immediately from this fact. $\square$

Let us briefly consider τ, which we know can be diagonalized with respect to φ. Since we can choose a basis on which C is also diagonal, it becomes clear that both $C\tau$ and τC will only be nonzero on the diagonal, and their diagonal entries will be equal. Thus we can say that τ and C commute with one another Using this fact, we are able to show that $C\tau C$ is self-adjoint with respect to φ^+ in the following way:

$$\varphi^+(C\tau Cx, y) = \varphi(C^2\tau Cx, y) = \varphi(\tau Cx, y) = \varphi(C\tau x, y)$$

$$= \varphi(\tau x, Cy) = \varphi(x, \tau Cy) = \varphi^+(Cx, \tau Cy) = \varphi^+(x, C\tau Cy)$$

From the self-adjoint nature of $C\tau C$, we can conclude that $R^+_{C\tau C} \in \mathcal{A}(V)$. Since we have that $R_{\varphi^+} + \epsilon R^+_{C\psi C} = \delta R^+_{C\tau C}$, we can say that:

$$R^+_{C\psi C} = \epsilon(\delta R^+_{C\tau C} - R_{\varphi^+}).$$

Since the right-hand side of this equation is a linear combination of algebraic curvature tensors, it too is an algebraic curvature tensor, and as a result we can conclude that $R^+_{C\Psi C} \in \mathcal{A}(V)$. Once this is known, it follows that $C\psi C$ is self-adjoint with respect to φ^+. Given this,

$$\varphi^+(C\psi Cx, y) = \varphi^+(x, C\psi Cy)$$

$$= \varphi(\psi Cx, y) = \varphi(x, \psi Cy)$$

$$= \varphi(Cx, \psi y) = \varphi(x, \psi Cy)$$

$$= \varphi(x, C\psi y) = \varphi(x, \psi Cy).$$

Thus, it follows from the self-adjoint nature of $C\psi C$ with respect to φ^+ that C and ψ commute.

Using the fact that $R^+_{C\psi C} \in \mathcal{A}(V)$, the equation $R_{\varphi^+} + \epsilon R^+_{C\psi C} = \delta R^+_{C\tau C}$ becomes a linear dependence relation between three algebraic curvature tensors with respect to a positive definite inner product. However, given the knowledge that $\psi C = C\psi$ and $\tau C = C\tau$, it becomes clear that $C\psi C = \psi$ and $C\tau C = \tau$, demonstrating that the rank requirements for ψ and τ are met and thus Theorem 1.6 of [1] may be applied to conclude that ψ and τ are simultaneously orthogonally diagonalizable with respect to this new positive definite inner product.

On the other hand, it remains to be seen that ψ and τ will be simultaneously diagonalizable with respect to φ, and to check this we must consider whether or not the basis that diagonalizes both forms with respect to φ^+ will still be orthonormal with respect to φ. However, if we consider the family $\{\psi, \tau, C\}$ with respect to φ^+, we notice that it is a family of commuting self-adjoint operators. It has already been shown that $C\psi = \psi C$ and $C\tau = \tau C$. Also, the ability to simultaneously diagonalize comes with the knowledge that $\psi\tau = \tau\psi$ by Theorem 4. Thus it follows that the result of simultaneous diagonalization with respect to some chosen basis for φ^+ can be extended to include C.

Now, since we have a basis on which C is diagonal, we can express it as follows:

$$C = \begin{pmatrix} \kappa_1 & 0 & \cdots & \\ 0 & \kappa_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & & \kappa_n \end{pmatrix}$$

As a result, we can consider $\varphi(e_i, e_j) = \varphi^+(Ce_i, e_j)$. Since C is now diagonal with respect to this basis,

$$\varphi^+(Ce_i, e_j) = \varphi^+(\kappa_i e_i, e_j) = c_i \delta_{ij}$$

However, we note that C is both self-adjoint and orthogonal with respect to the positive definite φ^+, and since it has been diagonalized, we can conclude by Theorem 4 that all $\kappa_i = \pm 1$.

Now, given our simultaneous diagonalization result with respect to φ^+, we can express ψ and τ as follows:

$$\psi = \begin{pmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & & \lambda_n \end{pmatrix}; \tau = \begin{pmatrix} \eta_1 & 0 & \cdots & \\ 0 & \eta_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & & \eta_n \end{pmatrix}.$$

Here, the λ_i represent the eigenvalues of ψ, while the η_i represent the eigenvalues of τ. It follows that:

$$\psi(e_i, e_j) = \varphi(\psi e_i, e_j) = \varphi^+(C\psi e_i, e_j) = \varphi^+(\psi C e_i, e_j) = \kappa_i \varphi^+(\psi e_i, e_j) = \kappa_i \lambda_i \delta_{ij}$$

Similarly, for τ,

$$\tau(e_i, e_j) = \varphi(\tau e_i, e_j) = \varphi^+(C\tau e_i, e_j) = \varphi^+(\tau C e_i, e_j) = \kappa_i \varphi^+(\tau e_i, e_j) = \kappa_i \eta_i \delta_{ij}$$

Thus, ψ and τ remain diagonal with respect to φ, and our result holds. $\square$

Theorem 9. *If φ is nondegenerate, Rank $\tau = n$, τ is diagonalizable with respect to φ, Rank $\psi \geq 3$, and dim(V) ≥ 4, then the set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent with φ not a real multiple of ψ or τ if and only if one of the following holds:*

1) $|Spec(\psi)| = |Spec(\tau)| = 1$.
2) $Spec(\tau) = \{\eta_1, \eta_2, \eta_2, ...\}$, and $Spec(\psi) = \{\lambda_1, \lambda_2, \lambda_2, ...\}$, with $\eta_1 \neq \eta_2, \lambda_2^2 = \epsilon(\delta\eta_2^2 - 1)$, and $\lambda_1 = \frac{\epsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1)$ for $\epsilon, \delta = \pm 1$.

The proof of this theorem follows the same method as the proof in the positive definite case, originally presented in [1]. First, it is assumed that |Spec(τ) ≥ 3|, and a contradiction is reached. Then, we consider the case where τ has two eigenvalues, both of which are repeated. The main difference lies in the inclusion of the eigenvalues of C, which we then divide out by to reach the same set of equations as in [1], and the remainder of the proof can be found there.

Proof. We begin by recalling the ability to express

$$\psi(e_i, e_j) = \kappa_i \lambda_i \delta_{ij}; \tau(e_i, e_j) = \kappa_i \eta_i \delta_{ij}.$$

Given this information, we can consider the linear dependence of $\{R_\varphi, R_\psi, R_\tau\}$ on (e_i, e_j, e_j, e_i) for $i \neq j$. Essentially, we are looking at the equation

$$R_\varphi(e_i, e_j, e_j, e_i) + \epsilon R_\psi(e_i, e_j, e_j, e_i) = \delta R_\tau(e_i, e_j, e_j, e_i)$$

$$\begin{aligned}\varphi(e_i, e_i)\varphi(e_j, e_j) - \varphi(e_i, e_j)^2 + \epsilon[\varphi(\psi e_i, e_i)\varphi(\psi e_j, e_j) - \varphi(\psi e_i, e_j)^2] \\ = \delta[\varphi(\tau e_i, e_i)\varphi(\tau e_j, e_j) - \varphi(\tau e_i, e_j)^2] \\ \kappa_i\kappa_j + \epsilon\kappa_i\kappa_j\lambda_i\lambda_j = \delta\kappa_i\kappa_j\eta_i\eta_j\end{aligned}$$

Since κ_i is nonzero for all i, we can divide out by $\kappa_i\kappa_j$ to obtain

$$1 + \epsilon\lambda_i\lambda_j = \delta\eta_i\eta_j.$$

Given this equation, the authors of [1] used the fact that $\dim(V) \geq 4$, plugging in different values for $i, j \in \{1, 2, 3, 4\}$. They first considered the case where $|\mathrm{Spec}(\tau)| \geq 3$ and were able to reach a contradiction regarding the number of distinct eigenvalues of ψ from the various equations that the linear dependence relationship leads to on different choices of basis vectors. Next, they considered the case where $\mathrm{Spec}(\tau) = \{\eta_1, \eta_1, \eta_2, \eta_2, ...\}$ for $\eta_1 \neq \eta_2$. This led to a similar contradiction, again based entirely off of evaluating the equation $1 + \epsilon\lambda_i\lambda_j = \delta\eta_i\eta_j$ on different basis vectors. From this, they conclude that $|\mathrm{Spec}(\tau) = 2|$, and in this case it follows that $|\mathrm{Spec}(\psi) = 2|$, since if τ has spectrum $\{\eta_1, \eta_2, \eta_2, ...\}$, then $1 + \epsilon\lambda_i\lambda_j = \delta\eta_2^2$ for all $i, j \geq 2$. Thus ψ has a spectrum of the form $\{\lambda_1, \lambda_2, \lambda_2, ...\}$, and the following equations must hold:

$$\begin{aligned}1 + \epsilon\lambda_1\lambda_2 = \delta\eta_1\eta_2 \\ 1 + \epsilon\lambda_2^2 = \delta\eta_2^2\end{aligned}$$

This leads to the equations presented in the second case:

$$\lambda_1 = \frac{\epsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1)$$

$$\lambda_2^2 = \epsilon(1 - \delta\eta_1\eta_2)$$

The converse of this has already been presented in [1], and the same methods may be applied in the case of nondegeneracy without any negative repercussions. □

3 Conclusion

We have made use of the change of signature matrix C to show that previous results regarding the simultaneous diagonalization of symmetric bilinear forms given a linear dependence relationship among the algebraic curvature tensors they form hold in the case that the inner product is merely nondegenerate, with the added assumption that τ is diagonalizable. In doing so, we have uncovered more valuable information regarding alternative ways of expressing a linear dependence relationship in inner products of different signature.

Additionally, we have established results regarding the eigenvalues of two of the forms involved that show that previously established relationships between the spectra of the symmetric bilinear forms also remain unchanged when the inner product is nondegenerate, again given the added assumption that τ is diagonalizable.

4 Open Questions

1. Do similar properties hold for the forms used to construct sets of four or more linearly dependent algebraic curvature tensors, and what requirements must be satisfied with respect to the signature of each form involved?

2. If τ is not diagonalizable, but instead has a particular Jordan normal form, what can be said about the Jordan normal form of ψ?

3. What conclusions can be drawn regarding a set of linearly dependent algebraic curvature tensors if the bilinear forms involved are antisymmetric?

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References

[1] A. Diaz, C. Dunn, *The linear independence of sets of two and three canonical algebraic curvature tensors*, Electronic Journal of Linear Algebra, Volume 20 (2010), 436-448.

[2] S. Friedberg, A. Insel, and L. Spence, *Linear Algebra (4th edition)*, Prentice Hall (2002), 381.

[3] P. Gilkey, *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific (2001), 13-14.

[4] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press (1990), 51-53.

[5] J.J. Sylvester, *A Demonstration of the Theorem that Every Homogeneous Quadratic Polynomial is Reducible by Real Orthogonal Substitutions to the Form of a Sum of Positive and Negative Squares*, Philosophical Magazine IV (1852), 138-142.