CLASSIFYING CLOSED THREE- AND FOUR-BRAIDS AS HYPERBOLIC

Marjorie Kasten
Wellesley College
California State University, San Bernardino REU
Summer 2011

July 22, 2011

Abstract

We identify which closed three- and four-braids are hyperbolic. We also make general claims about n-stranded closed braids and their hyperbolic nature.

1 Introduction

We partition all knots into three types: Torus, Satellite, and Hyperbolic knots. A torus knot is a knot that sits on the surface of a torus in $\mathbb{R}^3$. A satellite knot is a knot which contains an incompressible, non-boundary parallel torus in its complement. A hyperbolic knot is a knot that has a complement that can be given a metric of constant curvature -1 [**1**].

Prime knots, knots that cannot be written as the connect sum of two non-trivial knots, can be any of the three types of knots. However, all connected sums are satellite knots, by definition. These satellite knots are called "swallow follow". A torus consumes one of the components of the connect sum and traverses the other component. This ensures the the torus is incompressible and non-boundary parallel. We refer to this torus as the companion of the knot K.

It is well known that, no matter what the type, any knot can be represented by a closed braid. A braid is a set of n strings, all of which are attached to a horizontal bar at the top and at the bottom. If we bring the bottom bar and "attach" it to the top bar, the braid is considered to be closed [**1**]. The question is, how can we tell which closed braids are of which type? More specifically, hyperbolic? We will look at the case of closed three- and four-braids (possibly five-braids).

We describe a braid algebraically by its word. Given an n-braid, its word will be composed of a number of $\{\sigma_1, \sigma_2, ..., \sigma_{n-1}\}$. The subscript indicates which two strands are crossing. For example, σ_1 means that the first strand is being crossed over the second strand. Additionally, σ_1^{-1} means that the second strand is crossing over the first strand. The figure-8 knot is represented by the braid word $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$, which looks like

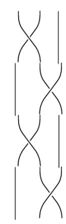

Figure 1: Figure-8 knot braid representation (not closed)

In order to close the braid shown above, we would simply attach the top of the braid to the bottom of the braid.

There are proven facts about closed braids that will be useful in the proofs that follow. We can relate the bridge number of a knot, $br(K)$, the number of local maxima in the reduced alternating projection, to the number of strands in the minimal braid representation of a knot, b(K), the braid index

$$br(K) \leq b(K)$$

Therefore, we can say that, no matter what the projection, the bridge number will be less than or equal to the number of strands in the braid representation.

It it important to note, when given a satellite knots, $br(V)$, where V is the companion to the knot K, has $br(V) > 1$ because V can not be the unknot.

2 Closed Three-Braids

Lemma: *If a closed three-braid is not of the form $\sigma_1^k\sigma_2, \sigma_1^k\sigma_2^{-1}, (\sigma_1\sigma_2)^k$, or $\sigma_1^p\sigma_2^q$, where $k, p, q \in \mathbb{Z}$, then it is hyperbolic*

Proof. Only $T_{2,k}$ and $T_{3,k}$ torus knots can be represented by a closed three-braid ($\sigma_1^k\sigma_2$ or $\sigma_1^k\sigma_2^{-1}$ and $(\sigma_1\sigma_2)^k$, respectively).

We now consider satellite knots, which we define to be a knot which contains an incompressible torus in its complement.

Birman and Menasco have completely classified prime satellite knots as

$$(\sigma_2)^k(\sigma_1\sigma_2^2\sigma_1)^q\text{, where } |p| \geq 2,\ |q| \geq 1$$

It is known that, given knots K_1 and K_2, the connect sum $K_1\#K_2$ is a satellite knot. Using Birman and Menasco's (??) result, we know that the braid of a connect sum is represented by $n+(m-1)$ strands, where the first component requires n strands and the second component requires m strands. Using the first n strands, the first component is braided completely. Then, using the n^{th} strand and the remaining $(m-1)$ strands, the second component is completely braided.

In the three-braid case, the entire braid of the first component (using strands 1 and 2), meaning it is a power of σ_1, is connected by the second strand to the entire braid of the second component (using strands 2 and 3), a power of σ_2. Therefore, we can write any braid word of a connect sum as

$$\sigma_1^p \sigma_2^q, \text{ where } p, q \in \mathbb{Z}.$$

The wrapping number, $w(K)$: the least number of times K crosses a meridional disk of the companion, V, of any connect sum is always 1. Without loss of generality, let the torus completely consume K_1 and traverse K_2. There is always a disk that can be drawn on the portion of the torus around K_2 that will only intersect K_2 once. This completely classifies connected sums

Claim: Any satellite knot whose wrapping number is greater than 1 cannot be presented by a closed three-braid.

Proof of Claim: From Schubert's result, we can use the inequality

$$br(K) \geq w \cdot br(V), \text{ where } K \text{ is a knot and } V \text{ is its companion}$$

We know that $br(V) > 2$. Because we are considering only those satellite knots whose wrapping number is greater than 1, $w \geq 2$. Therefore,

$$br(K) \geq 2 \cdot 2 = 4$$

However, $br(K) \leq 3$, where K is a closed three-braid. So there are no closed three-braids that are satellite knots with $w(K) > 1$.

Birman and Menasco have completely classified prime satellite knots as

$$(\sigma_2)^k(\sigma_1\sigma_2^2\sigma_1)^q, \text{ where } |p| \geq 2,\ |q| \geq 1$$

□

3 Closed Four-Braids

We now turn our attention to closed four-braids.

Lemma: If a closed four-braid representation of a knot is not of (or conjugate to) the form $(\sigma_1)^k\sigma_2\sigma_3$, $(\sigma_1)^k\sigma_2^{-1}\sigma_3$, $(\sigma_1)^k\sigma_2\sigma_3^{-1}$, $(\sigma_1)^k\sigma_2^{-1}\sigma_3^{-1}$,$(\sigma_1\sigma_2)^k\sigma_3$, $(\sigma_1\sigma_2)^k\sigma_3^{-1}$, $(\sigma_1\sigma_2\sigma_3)^k$, $W(\sigma_1,\sigma_2)(\sigma_3)^k$, $W(\sigma_1,\sigma_2)(\sigma_3^{-1})^k$ or $(\sigma_2)^p(\sigma_3)^q(\sigma_2\sigma_1\sigma_3\sigma_2)^r$, where $k,p,q,r \in \mathbb{Z}$, $|p| > 0$, $|q| > 0$, $|r| \geq 2$, $2|$ only one of p,q and $2 \nmid r$, then it is hyperbolic

Proof. We can classify torus knots in a similar way to the three-braid case. With four strands, we can produce $T_{2,k}$, $T_{3,k}$ and $T_{4,k}$ torus knots,

- $(\sigma_1)^k\sigma_2\sigma_3$, $(\sigma_1)^k\sigma_2^{-1}\sigma_3$, $(\sigma_1)^k\sigma_2\sigma_3^{-1}$, $(\sigma_1)^k\sigma_2^{-1}\sigma_3^{-1}$
- $(\sigma_1\sigma_2)^k\sigma_3$, $(\sigma_1\sigma_2)^k\sigma_3^{-1}$
- $(\sigma_1\sigma_2\sigma_3)^k$

respectively.

We now consider satellite knots. As stated before, any connect sum is a satellite knot. We construct connect sums in the same way.

Claim: In a four-braid representation of a connect sum $K_1\#K_2$, one of K_1 or K_2 must be a two braid.

Proof of Claim: By SOMEONE'S result, we know the bridge number of a connect sum

$$br(K_1\#K_2) = br(K_1) + br(K_2) - 1, \text{ where } K_1 \text{ and } K_2 \text{ are nontrivial}$$

Because we have a four braid, we know

$$br(K_1\#K_2) \leq 4$$

Let us consider the case where neither component is a two-braid. This means the bridge number of either component must be greater than two to guarantee that the component is not a two-braid. So we chose $br(K_1)$, $br(K_2) = 3$. We have

$$br(K_1\#K_2) = br(K_1) + br(K_2) - 1 = 3 + 3 - 1 = 5 > 4$$

Therefore, we cannot have both $br(K_1)$ and $br(K_2) > 2$. Now, without loss of generality, let $br(K_2) = 2$. We have

$$br(K_1\#K_2) = br(K_1) + br(K_2) - 1 = 3 + 2 - 1 = 4$$

Now we must show that K_1 must be a two braid. Using Birman and Menasco's result concerning connect sums of four-braids, we know that the braid index, $b(K)$, the smallest integer n such that a knot K can be represented by an n-braid, of a connect sum is additive (respectively additive minus 1) under disjoint union (respectively connected sum). Four-braids are a special case, and have another set of conditions. Birman and Menasco's result states

Let K be a 4-braid representative of a composite knot K. If K cannot be represented by a closed braid of braid index < 4, then K is conjugate to a composite 4-braid...

As stated earlier, we also know that $br(K_1\#K_2) \leq 4$. Because the braid index is the least number of strands needed to produce a given knot, we know that the bridge number will be less than or equal to the braid index. Therefore we must show that the braid index must equal four in order to satisfy both Birman and Menasco's AND SOMEONE'S constraints. Because no nontrivial knot can have a braid index of 1, we consider the connect sum of, without loss of generality, $b(K_1) = 3$ and $b(K_2) = 2$ so that we have

$$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1 = 3 + 2 - 1 = 4$$

The only knots that can be represented by only two strands with a bridge index of two are $T_{2,k}$ torus knots. Therefore, K_2 must be a two-braid.

Knowing that one component must be a two-braid and the other component must have bridge number equal to three, we can construct any connect sum by

$$W(\sigma_1, \sigma_2)(\sigma_3)^k \text{ or } W(\sigma_1, \sigma_2)(\sigma_3^{-1})^k$$

We resume our study of satellite knots by looking at the complete set of conjugacy classes for closed four-braids of prime knots (Birman and Menasco). Because we are only concerned with knots, we look at which conjugacy classes can produce knots. There is only one, namely $(\sigma_2)^p(\sigma_3)^q(\sigma_2\sigma_1\sigma_3\sigma_2)^r$, where $k, p, q, r \in \mathbb{Z}$, $|p| > 0$, $|q| > 0$, $|r| \geq 2$ (constraints provided by Birman and Menasco), and $2 \not| r$ (guarantees only one component). The other five classes strictly produce multi-component knots. So all closed four-braid prime satellite knots are of that form. Any braid word that is not conjugate to those forms stated in the proof are hyperbolic. □

4 Closed Five-Braids

We know the torus knots that can be represented as closed five-braids

- $(\sigma_1)^k\sigma_2\sigma_3\sigma_4$, $(\sigma_1)^k\sigma_2^{-1}\sigma_3\sigma_4$, $(\sigma_1)^k\sigma_2\sigma_3^{-1}\sigma_4$, $(\sigma_1)^k\sigma_2\sigma_3\sigma_4^{-1}$, $(\sigma_1)^k\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}$
- $(\sigma_1\sigma_2)^k\sigma_3\sigma_4$, $(\sigma_1\sigma_2)^k\sigma_3^{-1}\sigma_4^{-1}$
- $(\sigma_1\sigma_2\sigma_3)^k\sigma_4$, $(\sigma_1\sigma_2\sigma_3)^k\sigma_4^{-1}$
- $(\sigma_1\sigma_2\sigma_3\sigma_4)^k$

Now we must consider representations of prime knots as closed-five braids. I plan on taking the four-braid conjugacy classes and seeing which ones, when adding a strand, come out as knots. Then playing around with those and seeing whether those are hyperbolic. Hopefully, they are not.

5 In Progress

In the coming weeks, here are some possible questions to work on

- Consider a hyperbolic braid of n-strands. If you add another component, trivially or nontrivially, is the resulting link still hyperbolic?
- What happens when you connect two disjoint links by dehn surgery?
- Do links behave the same way that knots do, in the sense that any given multi-component link is one of a torus, satellite, or hyperbolic knot?
- What is the completely classification of closed 5-braids that allow for an incompressible, non-boundary parallel torus in its complement?

6 References

1. C. Adams, *The Knot Book*, FINISH CITATION
2. J. Birman and W. Menasco, *A Note on Closed 3-Braids*, World Scientific 10 (2008), 1033-1047
3. J. Birman and W. Menasco, *Special Positions for Essential Tori in knot Complements*, Topology 33 (1994), 525-556
4. J. Birman and W. Menasco, *Studying knots via Closed Braids IV: Composite knots and Split knots*, Inv. Math. 120 (1990), 115-139
5. W. Thurston, *Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry*, AMS 8.3 (1982)