Is Every Invertible Linear Map in the Structure Group of some Algebraic Curvature Tensor?

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August 24, 2012

Abstract

We study the elements in the structure group of an algebraic curvature tensor R by analyzing Jordan normal forms. Because every matrix has a unique Jordan normal form representation, up to a permutation of the Jordan Blocks, we are able to determine which matrices taking on a specific form will be in the structure group of some algebraic curvature tensor. A method for analyzing these forms is developed and explained.

1 Introduction and Motivation

Let V be a finite dimensional vector space and $R : V \times V \times V \times V \to \mathbb{R}$ be linear in each input.

Definition 1. *R is an **algebraic curvature tensor** if the following properties are satisfied for all $x, y, z, w \in V$:*

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. *(Bianchi Identity)* $R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0$.

Let $\mathcal{A}(V)$ be the vector space of algebraic curvature tensors. It is known [3] that $dim(\mathcal{A}(V)) = \frac{n^2(n^2-1)}{12}$.

Given a manifold M paired with a smooth metric g there exists an algebraic curvature tensor at each point p in M. Conversely, if we are given some R on a vector space V, there will exist some manifold M with smooth metric g such that R is the Riemann curvature tensor at some point p on that manifold. Thus, the major motivation of studying these objects is to better understand the various types of curvature one could encounter on a manifold M with some metric g.

Definition 2. *Let V be a vector space, ϕ an inner product, and $R \in \mathcal{A}(V)$. Then, (V, ϕ, R) is a **full model space**, and (V, R) is a **weak model space**.*

Proposition 1. *Let (M, g) be a smooth manifold and $R \in \mathcal{A}(V)$. Then, $(T_pM, g|_p, R|_p)$ is a full model space where T_pM is the tangent space, $g|_p$ is the metric, and $R|_p$ is the algebraic curvature tensor at some point $p \in M$.*

1.1 Structure Groups

Let α be a multilinear function from k copies of some vector space V to $\mathbb{R}$. Now, if $A \in GL(n)$ where $GL(n)$ is the set of invertible matrices, then one can consider the difference between $\alpha(x_1, \ldots, x_n)$ and $\alpha(Ax_1, \ldots, Ax_n)$.

Definition 3. *Let $A \in GL(n)$.The **precomposition** of A, denoted A^*, on some tensor α has A act on the arguments before α operates on the arguments itself. That is, $A^*\alpha$ is the map,*

$$A^*\alpha : V \times \cdots \times V \stackrel{A^{(k)}}{\to} V \times \cdots \times V \stackrel{\alpha}{\to} V \times \cdots \times V \to \mathbb{R},$$

where $A^{(k)}(x_1, \ldots, x_k) = (Ax_1, \ldots, Ax_k)$.

Definition 4. *The **structure group** G_α is defined as $G_\alpha = \{A \in GL(n) | A^*\alpha = \alpha\}$.*

We now consider a few examples of known structure groups.

Example 1. *Let α be a positive definite inner product. Then,*

$$G_\alpha = \{A \in GL(n) | \alpha(Ax, Ay) = \alpha(x, y) \text{ for all } x, y, \in V\} = \mathcal{O}(n),$$

where $\mathcal{O}(n)$ is the familiar set of matrices whose transpose is their inverse.

Example 2. *Let α be the zero algebraic curvature tensor. Then, $G_\alpha = GL(n)$.*

The study of structure groups is important to the understanding of algebraic curvature tensors as they are the symmetries of these objects. Considering Proposition 1, one can see that there will be a corresponding structure group at each point p in the given manifold. Therefore, by studying these structure groups, one can draw various other conclusions about the manifold itself.

There are a variety of structure group characteristics that have been previously studied. For example [5] studied the decomposibility, and also characterized elements of these groups. Decomposibility was also studied by [2], and [1] studied structure groups to construct new invariants not of Weyl type on curvature homogeneous manifolds, which are manifolds with the same full model space at each point.

In the study of structure groups, it is natural to begin by taking an $R \in \mathcal{A}(V)$ and construct its corresponding structure group based on the properties of that tensor. We pose the opposite question.

Question 1. *Given a closed subgroup $H \leq GL(n)$ does there exist an $R \in \mathcal{A}(V)$ such that $G_R = H$?*

From Example 2 above, one can see that this is trivially true if $H = GL(n)$ when R is the zero tensor. At the other extreme, in [4] it is demonstrated that very many algebraic curvature tensors have structure groups containing only finitely many elements.

Because our question is somewhat extensive, we approach it by considering a single $A \in H$ and ask if there exists some nonzero R such that $A \in G_R$.

2 Preliminaries

We review the Jordan decomposition of linear endomorphisms as they are important to our method of analysis in dimension 3.

Definition 5. *A* ***Jordan Block*** *of size k corresponding to some eigenvalue $\lambda \in \mathbb{R}$ on $\mathbb{R}^k$ is defined as:*

$$J(k,\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

The Jordan Block corresponding to the pair of complex conjugate eigenvalues $a \pm b\sqrt{-1}$ is defined in the construction of the size $2k$ matrix where $A := \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on $\mathbb{R}^2$ and

$$J(k,a,b) = \begin{bmatrix} A & I & 0 & \cdots & 0 & 0 \\ 0 & A & I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A & I \\ 0 & 0 & 0 & \cdots & 0 & A \end{bmatrix} \text{ on } \mathbb{R}^{2k}.$$

The following definition provides a common operation used with Jordan block matrices.

Definition 6. *Let A_i for $i = 1, \ldots, n$ be a set of square matrices. The* ***direct sum*** *of A_i is:*

$$\bigoplus_{i=1}^{n} A_i = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{bmatrix}.$$

Making use of definitions 5 and 6 in its construction, the following lemma is very important to our method of analysis.

Lemma 1. *Let A be a linear transformation of a vector space V. Choosing an appropriate basis for V, A will decompose as the direct sum of Jordan blocks. The unordered collection of these blocks is determined by A.*

Definition 7. *The* ***Jordan normal form*** *of A is the unordered collection of Jordan blocks from Lemma 1.*

To illustrate Lemma 1, we consider an example in dimension 3.

Example 3. *Let A be a 3×3 matrix with real entries. Then, $det(A - tI)$ will be a polynomial of degree 3, which implies that A will have at least one real eigenvalue. Let λ be this real eigenvalue. Then, the following Jordan normal forms are possible:*

1. $J(3, \lambda)$,
2. $J(2, \lambda) \oplus J(1, \eta)$,
3. $J(1, \lambda) \oplus J(1, \eta) \oplus J(1, \gamma)$,
4. $J(2, a, b) \oplus J(1, \lambda)$.

The construction of 2, 3, and 4 make use of Lemma 1. For example, consider 2.

$$J(2,\lambda) \oplus J(1,\eta) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} \eta \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \eta \end{bmatrix}$$

Now that we have reviewed Jordan normal form, we present our method of analysis.

3 Method

In first considering dimension 2, we were able draw conclusions using standard calculation because of the trivial nature of the case. When moving to dimension 3, a different approach was developed to accomodate the increasing complexity.

Lemma 2. *Let V be a vector space and let $\{e_i\}$ be a basis for V. Enumerate $\{R(e_i, e_j, e_k, e_l)\} = \{x_1, \ldots, x_p\} = \mathbf{x}$ where $dim(\mathcal{A}(V)) = p$. Given an $A \in GL(n)$, the equations $A^*R(e_i, e_j, e_k, e_l) = R(e_i, e_j, e_k, e_l)$ can be expressed as $K\mathbf{x} = \mathbf{x}$ for some matrix K, or equivalently, $(K - I)\mathbf{x} = 0$.*

It follows from Lemma 2 that $(K - I)\mathbf{x} = 0$ will have solutions if either $\mathbf{x} = 0$ or $det(K - I) = 0$. If $\mathbf{x}$ is the zero vector, we are dealing with the trivial case where R is the zero tensor. This implies that the solution space of this equation is where $det(K - I) = 0$, and the dimension of this space is the nullity of $K - I$.

Theorem 1. *If there exists a nonzero R such that $A \in G_R$, then $K\mathbf{x} = \mathbf{x}$ has a nontrivial solution.*

Corollary 1. *The solution space of $K\mathbf{x} = \mathbf{x}$ is the set of all algebraic curvature tensors R such that $A \in G_R$.*

Therefore, using this method, we can learn valuable information about both the matrix A and algebraic curvature tensor R being considered.

4 Results

Beginning with dimension 2, we use standard calculations to determine what must be preserved by an element in the structure group of a curvature tensor R and provide a proof of a known result. We then look at dimension 3 where Lemma 2 is applied to Jordan decompositions and used to analyze which matrices will or will not be in the structure group of some R. We emphasize that in dimension 3, we were able to answer the question that is the title of this paper as no, although a complete study of this situation is far from complete.

4.1 Dimension 2

Considering dimension 2, we note that $dim(\mathcal{A}(\mathbb{R}^2)) = 1$. Let the vector space $V = \mathbb{R}^2$ with basis $\{e_1, e_2\}$. Then, the only output that must be preserved is $R(e_1, e_2, e_2, e_1)$ by the properties of an algebraic curvature tensor. The following result is general knowledge [3], but we give a proof to keep this presentation self-contained.

Theorem 2. *For any nonzero $R \in \mathcal{A}(\mathbb{R}^2)$,*

$$G_R \cong SL(2)^{\pm} = \{A \in M_2(\mathbb{R}) | det(A) = \pm 1\}.$$

Proof. Let $A \in GL(2)$ where $Ae_1 = ae_1 + be_2$ and $Ae_2 = ce_1 + de_2$. Consider the following calculation:

$$\begin{aligned}
A^*R(e_1, e_2, e_2, e_1) &= R(Ae_1, Ae_2, Ae_2, Ae_1) \\
&= R(ae_1 + be_2, ce_1 + de_2, ce_1 + de_2, ae_1 + be_2) \\
&= R(ae_1, de_2, de_2, ae_1) + R(be_2, ce_1, de_1, be_2) - R(be_2, ce_1, de_2, ae_1) - R(ae_1, de_2, ce_1, be_2) \\
&= a^2d^2R(e_1, e_2, e_2, e_1) + b^2c^2R(e_2, e_1, e_1, e_2) - abcdR(e_2, e_1, e_2, e_1) - abcdR(e_1, e_2, e_1, e_2) \\
&= (ad - bc)^2R(e_1, e_2, e_2, e_1) \\
&= (detA)^2R(e_1, e_2, e_2, e_1)
\end{aligned}$$

Therefore, if $detA = \pm 1$, $A^*R = R$ and $A \in G_R$. □

Thus, if $detA \neq \pm 1$, there is no algebraic curvature tensor R for which $A \in G_R$.

Corollary 2. *The only subgroups H of $GL(2)$ that are structure groups for any algebraic curvature tensor over a vector space of dimension 2 are $GL(2)$ itself, when R is the zero tensor, and $SL(2)^{\pm}$.*

Because in this case the dimension of $\mathcal{A}(\mathbb{R}^2)$ is one, it may be considered trivial. However, it is important to understand the process involved here, so that it can be applied to higher dimensional cases.

4.2 Dimension 3

In dimension 3, we know that $dim(\mathcal{A}(\mathbb{R}^3)) = 6$. Now, let $R \in \mathcal{A}(\mathbb{R}^3)$ where for $i = 1, 2, 3, 4, 5, 6$. R_i is defined as follows:

$$\begin{aligned}
R_1 &= R(e_1, e_2, e_2, e_1), \\
R_2 &= R(e_1, e_3, e_3, e_1), \\
R_3 &= R(e_2, e_3, e_3, e_2), \\
R_4 &= R(e_1, e_2, e_3, e_1), \\
R_5 &= R(e_2, e_1, e_3, e_2), \\
R_6 &= R(e_3, e_1, e_2, e_3).
\end{aligned}$$

We will focus on Jordan decompositions 1 and 2 found in Example 3 and determine when matrices of this form will be members of the structure group for some curvature tensor R.

4.2.1 $J(3, \lambda)$.

Suppose

$$A = J(3, \lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

for a nonzero λ. Precomposing this matrix A with each R_i produces the 6 equations listed below. The solution of which will determine the R_i that are nonzero along with the λ value that preserves the nonzero

R_i.

$$\begin{aligned}
R_1 &= \lambda^4 R_1\\
R_2 &= \lambda^2 R_1 + \lambda^4 R_2 + 2\lambda^3 R_4\\
R_3 &= R_1 + \lambda^2 R_2 + \lambda^4 R_3 + 2\lambda R_4 + 2\lambda^3 R_6 - 2\lambda^2 R_5\\
R_4 &= \lambda^3 R_1 + \lambda^4 R_4\\
R_5 &= \lambda^4 R_5 - \lambda^2 R_1 - \lambda^3 R_4\\
R_6 &= \lambda^4 R_6 + \lambda R_1 - \lambda^3 R_2 + 2\lambda^2 R_4
\end{aligned}$$

To help the reader see how these equations were constructed, we illustrate an example of how the equation $R_4 = \lambda^3 R_1 + \lambda^4 R_4$ was formed.

Example 4. *Writing each Ae_i as a linear combination, we have:*

$$\begin{aligned}
Ae_1 &= \lambda e_1,\\
Ae_2 &= e_1 + \lambda e_2,\\
Ae_3 &= e_2 + \lambda e_3.
\end{aligned}$$

Consider $R_4 = R(e_1, e_2, e_3, e_1)$. Now, precompose R_4 with A and make use of the defining characteristics of R in Definition 1.

$$\begin{aligned}
A^* R(e_1, e_2, e_3, e_1) &= R(Ae_1, Ae_2, Ae_3, Ae_1)\\
&= R(\lambda e_1, e_1 + \lambda e_2, e_2 + \lambda e_3, \lambda e_1)\\
&= R(\lambda e_1, \lambda e_2, e_2, \lambda e_1) + R(\lambda e_1, \lambda e_2, \lambda e_3, \lambda e_1)\\
&= \lambda^3 R(e_1, e_2, e_2, e_1) + \lambda^4 R(e_1, e_2, e_3, e_1)\\
&= \lambda^3 R_1 + \lambda^4 R_4
\end{aligned}$$

Thus, if A is in the structure group of this particular R, we have that $R_4 = \lambda^3 R_1 + \lambda^4 R_4$.

Once these equations were established, we applied Lemma 2, which produced the following matrix equation.

$$\begin{bmatrix} R_1\\ R_2\\ R_3\\ R_4\\ R_5\\ R_6 \end{bmatrix} = \begin{bmatrix} \lambda^4 & 0 & 0 & 0 & 0 & 0\\ \lambda^2 & \lambda^4 & 0 & 2\lambda^3 & 0 & 0\\ \lambda & \lambda^2 & \lambda^4 & 2\lambda & -2\lambda^2 & 2\lambda^3\\ \lambda^3 & 0 & 0 & \lambda^4 & 0 & 0\\ -\lambda^2 & 0 & 0 & -\lambda^3 & \lambda^4 & 0\\ \lambda & \lambda^3 & 0 & 2\lambda^2 & 0 & \lambda^4 \end{bmatrix} \begin{bmatrix} R_1\\ R_2\\ R_3\\ R_4\\ R_5\\ R_6 \end{bmatrix}.$$

The determinant of $K - I$ is calculated to be $(\lambda^4 - 1)^6$. Therefore, the determinant equals zero if and only if $\lambda = \pm 1$. Thus, there exists some nontrivial curvature tensor R with A in its structure group. However, if $\lambda \neq \pm 1$, only the trivial solution of $R = 0$ exists.

Calculating the rank of $K - I$ when $\lambda = \pm 1$, we have $Rank(K - I) = 4$. Thus, the nullity is 2, which implies that there exists a 2 dimensional subspace of algeraic curvature tensors that are preserved by this matrix A.

We can give a more thorough analysis of this case regardless of what λ is by determining which R_i are preserved by a matrix taking on this form.

Claim 1. *R_1 is zero.*

Proof. Suppose R_1 is nonzero. Then, $\lambda^4 = 1$. Consider $R_4 = \lambda^3 R_1 + \lambda^4 R_4$. Then, substituting we have $R_4 = \lambda^3 R_1 + R_4$ which implies that $\lambda^3 R_1 = 0$. However, since λ is nonzero, R_1 must be zero. Thus, we have a contradiction, and R_1 is zero. □

Claim 2. *R_4 is zero.*

Proof. Suppose R_4 is nonzero. Then, $\lambda^4 = 1$. Consider $R_5 = \lambda^4 R_5 - \lambda^3 R_4$. Then, $R_5 = R_5 - \lambda^3 R_4$ and $\lambda^3 R_4 = 0$. However, since λ is nonzero, R_4 is zero. Thus, we have a contradiction, and R_4 is zero. □

Claim 3. *R_2 is zero.*

Proof. Suppose R_2 is nonzero. Then, $\lambda^4 = 1$. Consider, $R_6 = \lambda^4 R_6 - \lambda^3 R_2$. Now, $R_6 = R_6 - \lambda^3 R_2$ and $\lambda^3 R_2 = 0$. However, since λ is nonzero, R_2 must be zero. Thus, we have a contradiction, and R_2 is zero. □

Now, it appears that the only outputs preserved by an R with A in its structure group are R_3, R_5, and R_6. However, because we have also determined that the subspace of algebraic curvature tensors that could be preserved by A is 2 dimensional, at least one of R_3, R_5, or R_6 must be dependent on another. Thus, we will have some degree of freedom in determining the particular R with A in its structure group.

4.2.2 $J(2, \lambda) \oplus J(1, \eta)$.

We now consider the Jordan decomposition $J(2, \lambda) \oplus J(1, \eta)$. Suppose,

$$A = J(2, \lambda) \oplus J(1, \eta) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \eta \end{bmatrix}$$

for some nonzero λ and η. Using our matrix A and the properties of our curvature tensors, each of the following equations for R_i were determined.

$$\begin{aligned} R_1 &= \lambda^4 R_1 \\ R_2 &= \lambda^2 \eta^2 R_2 \\ R_3 &= \eta^2 R_2 + \lambda^2 \eta^2 R_3 + 2\eta^2 \lambda R_6 \\ R_4 &= \lambda^3 \eta R_4 \\ R_5 &= \lambda^3 \eta R_5 - \lambda^2 \eta R_4 \\ R_6 &= \lambda^2 \eta^2 R_6 + \eta^2 \lambda R_2 \end{aligned}$$

Using the construction found in Lemma 2, we create the following matrix equation.

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} = \begin{bmatrix} \lambda^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2\eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & \eta^2\lambda^2 & 0 & 0 & 2\eta^2\lambda \\ 0 & 0 & 0 & \lambda^3\eta & 0 & 0 \\ 0 & 0 & 0 & -\lambda^2\eta & \lambda^3\eta & 0 \\ 0 & \eta^2\lambda & 0 & 0 & 0 & \eta^2\lambda^2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix}.$$

The determinant of $K-I$ was calculated to be $(\lambda^4-1)(\lambda^2\eta^2-1)^3(\lambda^3\eta-1)^2$. Thus, for our determinant to be zero either $\lambda^4, \lambda^2\eta^2$, or $\lambda^3\eta = 1$.

Now, we consider the rank of $K-I$ according to which of the three above equations satisfy the condition of our determinant being zero. If $\lambda^4-1, \lambda^2\eta^2-1$, or $\lambda^3\eta-1$ goes to zero, the rank of $K-I$ is 5. Thus, the nullity is 1 and there exists a one dimensional subspace of algebraic curvature tensors that A will preserve.

The only other case we must consider is when both λ^4-1 and $\lambda^2\eta^2-1$ are zero because any other combination implies that the remaining factored component would go to zero as well. Then, we would be dealing with a 6 dimensional subspace, which is the space with the zero tensor. So, when both λ^4-1 and $\lambda^2\eta^2-1$ are zero, $\lambda = \pm 1$ and $\eta = \pm 1$. We consider the four possible cases here and conclude that when $\lambda = \eta$ the rank of $K-I$ is 3. Thus, the nullity is also 3, and there is a 3 dimensional subspace of algebraic curvature tensors being preserved. Finally, if $\lambda = -\eta$, the rank of $K-I$ is 4, so the nullity is 2. Thus, there exists a 2 dimensional subspace of algebraic curvature tensors being preserved by A.

Using algebraic manipulation of the above equations and resulting contradictions, we determine that R_1, R_3, and R_5 will be the only nonzero elements.

Claim 4. *R_2 is zero.*

Proof. Suppose R_2 is nonzero. Then, $R_2 = \lambda^2\eta^2 R_2$ implies that $\lambda^2\eta^2 = 1$. Now, consider $R_6 = \lambda^2\eta^2 R_6 + \eta^2\lambda R_2$. Because $\lambda^2\eta^2 = 1$, we can rewrite $R_6 = R_6 + \eta^2\lambda R_2$, which implies that $\eta^2\lambda R_2 = 0$. However, both η and λ are nonzero. Thus, R_2 must be zero, and we have a contradiction. □

Claim 5. *R_4 is zero.*

Proof. Suppose R_4 is nonzero. Then, $R_4 = \lambda^3\eta R_4$ implies that $\lambda^3\eta = 1$. Now, consider $R_5 = \lambda^3\eta R_5 - \lambda^2\eta R_4$ which can be rewritten as $R_5 = R_5 - \lambda^2\eta R_4$. Then, $\lambda^2\eta R_4 = 0$. However, since both λ and η are nonzero, R_4 is zero. Thus, we have a contradiction and R_4 is zero. □

Claim 6. *R_6 is zero.*

Proof. Suppose R_6 is nonzero. Then, $R_6 = \lambda^2\eta^2 R_6$ implies that $\lambda^2\eta^2 = 1$. Now, consider$R_3 = \lambda^2\eta^2 R_3 + 2\eta^2\lambda R_6$, which can be rewritten as $R_3 = R_3 + 2\eta^2\lambda R_6$. Then, $2\eta^2\lambda R_6 = 0$. However, since both λ and η are nonzero, $R_6 = 0$. Thus, we have a contradiction and R_6 is zero. □

Therefore, we know that the outputs R_1, R_3, and R_5 are nonzero and must be preserved by some tensor R.

Relating this back to the determinant, we see that if $R_2 = R_4 = R_6 = 0$, we are left with the equations $R_1 = \lambda^4 R_1, R_3 = \lambda^2\eta^2 R_3$, and $R_5 = \lambda^3\eta R_5$. Thus, we consider the following subcases.

Case 1. *Using the above equations, one can see that if either R_1, R_3, and R_5 are nonzero, only R_3 is zero, only R_1 is zero, or both R_1 and R_3 are zero, then $\lambda = \eta = \pm 1$.*

Case 2. *If $R_5 = 0$, then $\lambda = \pm 1$ and $\eta = \pm 1$.*

Case 3. *If $R_3 = R_5 = 0$, then $\lambda = \pm 1$ and η is a free variable.*

Case 4. *If $R_1 = R_5 = 0$, then $\eta = \pm \frac{1}{\lambda}$.*

Remark 1. *Because in cases 3 and 4, det$A$ is not forced to equal ± 1, $SL(3)^{\pm}$ is not the solution space in dimension 3.*

Therefore, given a matrix whose Jordan normal form takes this particular decomposition, we will have some degree of freedom in determining the R this matrix will be in the structure group of.

5 Conclusions

We studied the extent to which an arbitrary closed subgroup H of $GL(n)$ could be the structure group of some curvature tensor R. When $H = GL(n)$, we are dealing with the trivial case where R is the zero tensor. Beginning in dimension 2, we used standard calculations in our proof of the known result that for any nonzero $R \in \mathcal{A}(\mathbb{R}^2)$, the structure group of R is $SL(2)^{\pm}$. Thus, the only subgroups H of $GL(2)$ that can be realized as the structure group of some curvature tensor R are $GL(2)$ and $SL(2)^{\pm}$. In dimension 3, we studied two Jordan normal forms as candidates for elements in a structure group. This analysis was done using the method described Lemma 2. For $A = J(3, \lambda)$, we were able to conclude that if $\lambda = \pm 1$, there exist nontrivial curvature tensors R such that A is in the structure group. Also, this matrix A preserves a 2 dimensional subspace of algebraic curvature tensors. In considering $A = J(2, \lambda) \oplus J(1, \eta)$, we found much more flexibility in our choice of R and that depending on the choice of R it may preserve either a 1, 2, or 3 dimensional subspace of algebraic curvature tensors.

6 Questions

Through researching structure groups and their elements using Jordan normal form, we found many questions related to this topic that remain unresolved.

1. Although we have answered the following question for two Jordan normal forms in dimension 3, we would like to continue investigating whether if given an $A \in GL(n)$, can you find an R such that $A \in G_R$ by considering the other possible Jordan canonical forms.

2. As a follow up to 1, we would ask if you can find an R such that $A \in G_R$, then find all of G_R. In particular, we would like to do this with $J(3, \lambda)$ and $J(2, \lambda) \oplus J(1, \eta)$

3. Various algebraic curvature tensors are classified into certain types. For instance: canonical, Ricci flat, Weyl conformal, Osserman, decomposable, and others. If A is in the structure group of some curvature tensor R, what of these types could be found in the kernel of $K - I$?

4. How is the Jordan normal form of the matrix A related to the Jordan normal form of the corresponding matrix K?

5. C. Dunn poses the following question: Let $\mathcal{R} = \{R|_p | p \in M\}$ and $\mathcal{G} = (V, \langle , \rangle, \mathcal{R})$. Given $\mathcal{G}$, does there exist a (M, g) such that $\mathcal{R} = \{R|_p | p \in M\}$?

7 Acknowledgments

I would like to sincerely thank Dr. Dunn for his excellent guidance throughout this research process as well as Dr. Trapp for his valuable conversations. Also, I would like to thank California State University, San Bernardino and the NSF grant number DMS-1156608 for their generous support, which makes this program possible.

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