Jordan Normal Forms for a Skew-Adjoint Operator in the Lorentzian Setting

Lillian Saunders
Research Experience for Undergraduates
California State University, San Bernardino
San Bernardino, California 92407

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Abstract

The purpose of this paper is to identify which Jordan normal forms are possible for a skew-adjoint operator in the Lorentzian setting. We will go through each form and the possible cases that arise within that form.

1 Introduction

For our purposes we will define V as a real vector space of finite dimension, n, and φ as a nondegenerate inner product.

Definition: Let $A : V \Rightarrow V$ be linear. The adjoint of A, denoted A^* is characterized by the equation $\varphi(Ax, y) = \overline{(x, A^*y)}$. If $A^* = -A$, we call A skew-adjoint. If $A^* = A$, we call A self-adjoint.

For the remainder of the paper, we assume A to be skew-adjoint with respect to a nondegenerate inner product φ. We also assume $n = \dim V = 4$.

Jordan normal forms are used to find possible forms of the operator A. These forms are found by using Jordan blocks [1] for A. These Jordan normal forms can be used because it is known that every matrix, A, has some unique collection of Jordan blocks associated to it. That is, for every A, there exists a T so that $A = TJT^{-1}$ where J is the Jordan normal form [3]. This means that we are able to consider every case by using these forms since we choose a basis that puts A into its Jordan normal form. The possible Jordan blocks are defined below.

We define the real Jordan block as:

$$J_{\mathbb{R}}(\lambda, k) := \begin{bmatrix} \lambda & 1 & 0 & ... & 0 \\ 0 & \lambda & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & ... & 0 & \lambda \end{bmatrix} \text{ on } \mathbb{R}^k,$$

and the complex Jordan block as:

$$J_{\mathbb{C}}(a + \sqrt{-1}b, k) := \begin{bmatrix} A & I_2 & 0 & ... & 0 \\ 0 & A & I_2 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & A & I_2 \\ 0 & 0 & ... & 0 & A \end{bmatrix} \text{ on } \mathbb{R}^{2k},$$

where $A := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

These blocks can be combined in various ways by using direct sums to get the Jordan normal forms we used, and in general, any operator relative to a suitably chosen basis.

Example:

$$J_{\mathbb{C}}(a + \sqrt{-1}b, 1) \oplus J_{\mathbb{R}}(\lambda, 2) := \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Definition: If φ is a nondegenerate inner product, there exists a basis $\{e_1, e_2, ..., e_p, f_1, f_2, ..., f_q\}$ such that $\varphi(e_i, e_i) = -1$ and $\varphi(f_i, f_i) = 1$ and 0 otherwise. We call such a basis orthonormal. The pair (p, q) is independant of the orthonormal basis chosen and (p, q) is called the signature of φ.

A signature of $(0, n)$ is known as positive definite while a signature of $(n, 0)$ is known as negative definite. If $p = q$ then the signature is called balanced. What we are looking for in this project is a signature of $(1, n-1)$ or $(n-1, 1)$ which is called Lorentzian.

Given φ, and any basis $\{e_i\}$ for φ, $\exists$ a change of basis $F_i = Pe_i$. The information φ offers can be expressed as a matrix, $[\varphi(e_i, e_j)]$, and $P^T[\varphi(e_i, e_j)]P = [\varphi(f_i, f_j)]$. So if $\{f_i\}$ is an orthonormal basis, we can assume that there is some orthonormal change of basis, $Pe_i = f_i$ that will take $[\varphi(e_i, e_j)]$ to $[\varphi(f_i f_j)]$ that will have the look of:

$$[\varphi(f_i, f_j)] = \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & 0 \\ 0 & 0 & 0 & \varepsilon_4 \end{bmatrix} \text{ where } \varepsilon_i = \pm 1,$$

where the determinant is $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = \pm 1$, and det $[\varphi(f_i, f_j)] =$ det $(P^T[\varphi(e_i, e_j)]P) =$ (det $P)^{\sim}$ det $[\varphi(e_i, e_j)]$, so that det $[\varphi(f_i, f_j)]$ has the same signature as det $[\varphi(e_i, e_j)]$.

If either all or none of the ε's are negative then we get a determinant of 1 and we can also see that the signature would be either positive or negative definite. If two are negative and two are positive then we again get a determinant of 1 with a balanced signature. The only way to get a negative determinant would be to have one or three negative ε's which also gives a Lorentzian signature. Therefore we can eliminate cases that result in a positive determinant.

These results are known already if φ is assumed to be positive definite or negative definite.

Theorem: There exists an orthonormal basis $\{f_1, f_2, ..., f_n\} = \beta$ such that $[A]_\beta = \oplus_{i=1}^{k} J(\sqrt{-1}b, 1) \oplus J(0, 1)$ that will have the following look:

$$\left[\begin{array}{cccc|c} B_1 & 0 & 0 & ... & 0 \\ 0 & B_2 & 0 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & ... & B_k & ... \\ \hline 0 & 0 & 0 & 0 & 0... \\ ... & ... & ... & ... & ... \end{array}\right] \text{ where } B_j = \left[\begin{array}{cc} 0 & b_j \\ -b_j & 0 \end{array}\right], b_j > 0.$$

We can eliminate cases that have an odd rank because these are not possible in the skew-symmetric case.

Theorem: If $A^* = -A$ then A has even rank [2].

Also, for the complex Jordan blocks $(a + \sqrt{-1}b, k)$ we can see that if $b = 0$ then this is equivalent to some combination of real Jordan blocks so we can disregard these cases in the complex Jordan block situation so that they are not repeated.

Example: $J_{\mathbb{C}}(a + 0\sqrt{-1}, 1) = J_{\mathbb{R}}(a, 1) \oplus J_{\mathbb{R}}(a, 1)$

So for each form in the results we have only listed the possible cases after eliminating the cases that fall into one of the above categories.We can also eliminate cases that result in a degenerate matrix, where φ has a determinant of 0, since φ is assumed to be nondegenerate. These will be demonstrated in the results below.

The results are classified by the Jordan normal forms. Within each form, the equations for it are given. These are used to calculate the system of equations that would preserve the skew-adjointness of A. Then we go through each case within the form to see whether it is possible. The equations are referenced when they are used to help calculate the metric. Then it is stated whether it is degenerate, has a positive determinant, or is a possibility. The forms are split into two sections: the first are the forms with no possibilities in the Lorentzian setting and the next section lists those situations that could work in the Lorentzian setting. At the end, our results are summarized.

2 Results 1: Forms that aren't possible

2.1 Form 1

$$J_{\mathbb{R}}(\lambda, 4) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

This matrix gives the following equations:
$Ae_1 = \lambda e_1$
$Ae_2 = e_1 + \lambda e_2$
$Ae_3 = e_2 + \lambda e_3$
$Ae_4 = e_3 + \lambda e_4$

These can be used to calculate the following system of equations:
1.) $(Ae_1, e_1) = (\lambda e_1, e_1) = \lambda\varphi_{11} = 0$
2.) $(Ae_2, e_2) = (e_1 + \lambda e_2, e_2) = \varphi_{12} + \lambda\varphi_{22} = 0$
3.) $(Ae_3, e_3) = (e_2 + \lambda e_3, e_3) = \varphi_{23} + \lambda\varphi_{33} = 0$
4.) $(Ae_4, e_4) = (e_3 + \lambda e_4, e_4) = \varphi_{34} + \lambda\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = \lambda\varphi_{12} = -\varphi_{11} - \lambda\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = \lambda\varphi_{13} = -\varphi_{12} - \lambda\varphi_{13}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = \lambda\varphi_{14} = -\varphi_{13} - \lambda\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = \varphi_{13} + \lambda\varphi_{23} = -\varphi_{22} - \lambda\varphi_{23}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = \varphi_{14} + \lambda\varphi_{24} = -\varphi_{23} - \lambda\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \varphi_{24} + \lambda\varphi_{34} = -\varphi_{33} - \lambda\varphi_{34}$

2.2 Case 1, $\lambda \neq 0$.

By [1] we get that $\varphi_{11} = 0$. Using this, [5] becomes $\lambda\varphi_{12} = -\lambda\varphi_{12}$ so $\varphi_{12} = 0$. This makes [6] $\lambda\varphi_{13} = -\lambda\varphi_{13}$ so $\varphi_{13} = 0$. This simplifies [7] to $\lambda\varphi_{14} = -\lambda\varphi_{14}$ so $\varphi_{14} = 0$. Using these facts we can see that [2] makes $\varphi_{22} = 0$, [8] makes $\varphi_{23} = 0$, and [9] makes $\varphi_{24} = 0$. Also, [3] makes $\varphi_{33} = 0$, [10] makes $\varphi_{34} = 0$, and lastly [4] makes $\varphi_{44} = 0$. Therefore the resulting matrix φ is degenerate.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.3 Form 2

$$J_{\mathbb{R}}(\lambda, 2) \oplus J_{\mathbb{R}}(\eta, 2) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & \eta \end{pmatrix}$$

This matrix gives the equations:
$Ae_1 = \lambda e_1$
$Ae_2 = e_1 + \lambda e_2$
$Ae_3 = \eta e_3$
$Ae_4 = e_3 + \eta e_4$

These can be used to derive the following system of equations:
1.) $(Ae_1, e_1) = (\lambda e_1, e_1) = \lambda\varphi_{11} = 0$
2.) $(Ae_2, e_2) = (e_1 + \lambda e_2, e_2) = \varphi_{12} + \lambda\varphi_{22} = 0$
3.) $(Ae_3, e_3) = (\eta e_3, e_3) = \eta\varphi_{33} = 0$
4.) $(Ae_4, e_4) = (e_3 + \eta e_4, e_4) = \varphi_{34} + \eta\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = \lambda\varphi_{12} = -\varphi_{11} - \lambda\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = \lambda\varphi_{13} = -\eta\varphi_{13}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = \lambda\varphi_{14} = -\varphi_{13} - \eta\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = \varphi_{13} + \lambda\varphi_{23} = -\eta\varphi_{23}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = \varphi_{14} + \lambda\varphi_{24} = -\varphi_{23} - \eta\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \eta\varphi_{34} = -\varphi_{33} - \eta\varphi_{34}$

2.3.1 Case 1, $\lambda = \eta = 0$.

Using [2] we see that $\varphi_{12} = 0$, [4] shows that $\varphi_{34} = 0$, [5] shows that $\varphi_{11} = 0$, [7] shows that $\varphi_{13} = 0$, and [10] shows that $\varphi_{33} = 0$. Also, [9] shows that $\varphi_{14} = -\varphi_{23}$. This case has a positive determinant, ${\varphi_{14}}^4$.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & \varphi_{14} \\ 0 & x & -\varphi_{14} & y \\ 0 & -\varphi_{14} & 0 & 0 \\ \varphi_{14} & y & 0 & z \end{pmatrix}$$

2.3.2 Case 2, $\lambda \neq 0$ and $\eta \neq 0$.

Using [1] we show that $\varphi_{11} = 0$,and [3] shows $\varphi_{33} = 0$. Using these two we see that [5] makes $\lambda\varphi_{12} = -\lambda\varphi_{12}$ so $\varphi_{12} = 0$ and [10] makes $\lambda\varphi_{34} = -\lambda\varphi_{34}$ so $\varphi_{34} = 0$. Using these, [2], and [4] we see that $\varphi_{22} = 0$ and $\varphi_{44} = 0$ too. Using [6] we can see that either $\varphi_{13} = 0$ or $\lambda = -\eta$ We can look at these individually.

a.) $\varphi_{13} = 0$ and $\lambda \neq -\eta$.
Then using these facts and [7] we get $\varphi_{14} = 0$, these and [8] give us $\varphi_{23} = 0$. So these with [9] give us $\varphi_{24} = 0$. This is degenerate as shown below:

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

b.) $\varphi_{13} \neq 0$ and $\lambda = -\eta$.
But by simplifying [7] we would get $\varphi_{13} = 0$ so we have a contradiction.

c.) $\varphi_{13} = 0$ and $\lambda = -\eta$.
So we can use [9] to see that $\varphi_{14} = -\varphi_{23}$. This creates the matrix below which has a positive determinant, ${\varphi_{14}}^4$.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & \varphi_{14} \\ 0 & 0 & -\varphi_{14} & x \\ 0 & -\varphi_{14} & 0 & 0 \\ \varphi_{14} & x & 0 & 0 \end{pmatrix}$$

2.4 Form 3

$$J_{\mathbb{C}}(a + \sqrt{-1}b, 2) = \begin{pmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}$$

This results in the equations:
$Ae_1 = ae_1 - be_2$
$Ae_2 = be_1 + ae_2$
$Ae_3 = e_1 + ae_3 - be_4$
$Ae_4 = e_2 + be_3 + ae_4$

We can use these to calculate the following:
1.) $(Ae_1, e_1) = (ae_1 - be_2, e_1) = a\varphi_{11} - b\varphi_{12} = 0$
2.) $(Ae_2, e_2) = (be_1 + ae_2, e_2) = b\varphi_{12} + a\varphi_{22} = 0$
3.) $(Ae_3, e_3) = (e_1 + ae_3 - be_4, e_3) = \varphi_{13} + a\varphi_{33} - b\varphi_{34} = 0$
4.) $(Ae_4, e_4) = (e_2 + +be_3 + ae_4, e_4) = \varphi_{24} + b\varphi_{34} + a\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = a\varphi_{12} - b\varphi_{22} = -b\varphi_{11} - a\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = a\varphi_{13} - b\varphi_{23} = -\varphi_{11} - a\varphi_{13} + b\varphi_{14}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = a\varphi_{14} - b\varphi_{24} = -\varphi_{12} - b\varphi_{13} - a\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = b\varphi_{13} + a\varphi_{23} = -\varphi_{12} - a\varphi_{23} + b\varphi_{24}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = b\varphi_{14} + a\varphi_{24} = -\varphi_{22} - b\varphi_{23} - a\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \varphi_{14} + a\varphi_{34} - b\varphi_{44} = -\varphi_{23} - b\varphi_{33} - a\varphi_{34}$

If you add [6] and [9] and subtract [7] and [8] you get two more equations:
11.) $2a\varphi_{13} + \varphi_{11} = -2a\varphi_{24} - \varphi_{22}$
12.) $2a\varphi_{14} = 2a\varphi_{23}$

2.4.1 Case 1, $a = 0$ and $b \neq 0$.

Using [1 or 2] $\varphi_{12} = 0$. Using [5] $\varphi_{11} = \varphi_{22}$ but using [11] $\varphi_{11} = -\varphi_{22}$ so $\varphi_{11} = \varphi_{22} = 0$. [6] gives $\varphi_{14} = -\varphi_{23}$. Simplifying [7 or 8] gives $\varphi_{13} = \varphi_{24}$ but using [3 and 4] gives $\varphi_{13} = -\varphi_{24}$ so $\varphi_{13} = \varphi_{24} = 0$. Using this and [3 or 4] $\varphi_{34} = 0$ and using [10] we see $\varphi_{33} = \varphi_{44}$. This gives a similar matrix that also has a positive determinant, ${\varphi_{14}}^4$.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & \varphi_{14} \\ 0 & 0 & -\varphi_{14} & 0 \\ 0 & -\varphi_{14} & x & 0 \\ \varphi_{14} & 0 & 0 & x \end{pmatrix}$$

2.4.2 Case 2, $a \neq 0$ and $b \neq 0$.

Using [1 and 2] we see that $a\varphi_{11} = b\varphi_{12} = -a\varphi_{22}$ so $\varphi_{11} = -\varphi_{22}$ and $\varphi_{12} = \frac{-a}{b}\varphi_{22}$. Substituing φ_{22} into [5] for the φ_{11} creates $2a\varphi_{12} = 2b\varphi_{22}$ which can simplify to $\varphi_{12} = \frac{b}{a}\varphi_{22}$. Therefore $\frac{-a}{b}\varphi_{22} = \frac{b}{a}\varphi_{22}$. Multiplying both sides by a and b gives $-a^2\varphi_{22} = b^2\varphi_{22}$. This means that $\varphi_{22} = \varphi_{11} = \varphi_{12} = 0$. By [11] we get $\varphi_{13} = -\varphi_{24}$ and by [12] we get $\varphi_{14} = \varphi_{23}$. We can manipulate [6] to get $2a\varphi_{13} = b(\varphi_{14} + \varphi_{23}) \Rightarrow 2a\varphi_{13} = 2b\varphi_{14} \Rightarrow \varphi_{13} = \frac{b}{a}\varphi_{14}$, and we can manipulate [8] to get $2a\varphi_{23} = b(\varphi_{24} - \varphi_{13}) \Rightarrow 2a\varphi_{14} = -2b\varphi_{13} \Rightarrow \varphi_{13} = \frac{-a}{b}\varphi_{14}$. So $\frac{b}{a}\varphi_{14} = \frac{-a}{b}\varphi_{14} \Rightarrow b^2\varphi_{14} = -a^2\varphi_{14}$ so $\varphi_{14} = \varphi_{13} = \varphi_{23} = \varphi_{24} = 0$. With a similar manipulation on [3, 4, and 10] we can see $\varphi_{33} = \varphi_{34} = \varphi_{44} = 0$. So we again get a matrix of all zeros that is degenerate.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.5 Form 4

$$J_{\mathbb{C}}(a + \sqrt{-1}b, 1) \oplus J_{\mathbb{R}}(\lambda, 2) = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

This form gives us the equations:
$Ae_1 = ae_1 - be_2$
$Ae_2 = be_1 + ae_2$
$Ae_3 = \lambda e_3$
$Ae_4 = e_3 + \lambda e_4$

Using these gives us the equations:
1.) $(Ae_1, e_1) = a\varphi_{11} - b\varphi_{12} = 0$
2.) $(Ae_2, e_2) = b\varphi_{12} + a\varphi_{22} = 0$
3.) $(Ae_3, e_3) = \lambda\varphi_{33} = 0$

4.) $(Ae_4, e_4) = \varphi_{34} + \lambda\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = a\varphi_{12} - b\varphi_{22} = -b\varphi_{11} - a\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = a\varphi_{13} - b\varphi_{23} = -\lambda\varphi_{13}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = a\varphi_{14} - b\varphi_{24} = -\varphi_{13} - \lambda\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = b\varphi_{13} + a\varphi_{23} = -\lambda\varphi_{23}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = b\varphi_{14} + a\varphi_{24} = -\varphi_{23} - \lambda\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \lambda\varphi_{34} = -\varphi_{33} - \lambda\varphi_{34}$

2.5.1 Case 1, $a = 0, b \neq 0, \lambda \neq 0$.

By [1 or 2] we see that $\varphi_{12} = 0$ and therefore [5] gives us that $\varphi_{22} = \varphi_{11}$, by [3] we get that $\varphi_{33} = 0$, and by [10] we can see that $\lambda\varphi_{34} = -\lambda\varphi_{34}$ so $\varphi_{34} = 0$ which lets us see by [4] that $\varphi_{44} = 0$. We can simplify [6] so that $b\varphi_{23} = \lambda\varphi_{13} \Rightarrow \varphi_{23} = \frac{\lambda}{b}\varphi_{13}$. [8] simplifies to $b\varphi_{13} = -\lambda_{23} \Rightarrow \varphi_{23} = \frac{-b}{\lambda}\varphi_{13}$. This means that $\frac{\lambda}{b}\varphi_{13} = \frac{-b}{\lambda}\varphi_{13} \Rightarrow \lambda^2\varphi_{13} = -b^2\varphi_{13}$ so $\varphi_{13} = 0 = \varphi_{23}$. By using a similar process with [7 and 9] we get that $\varphi_{14} = \varphi_{24} = 0$. This gives us another degenerate matrix.

$$\varphi = \begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ 0 & \varphi_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.5.2 Case 2, $a \neq 0, b \neq 0, \lambda \neq 0$.

Using [3, 10 and 4] we can get that $\varphi_{33} = \varphi_{34} = \varphi_{44} = 0$. Using [1 and 2] we see that $a\varphi_{11} = b\varphi_{12} = -a\varphi_{22}$ which also means that $\varphi_{11} = -\varphi_{22}$ and $\varphi_{12} = \frac{-a}{b}\varphi_{22}$. Using [5] we see that $2a\varphi_{12} = b\varphi_{22} - b\varphi_{11} \Rightarrow 2a\varphi_{12} = 2b\varphi_{22} \Rightarrow \varphi_{12} = \frac{b}{a}\varphi_{22}$ so $\frac{-a}{b}\varphi_{22} = \frac{b}{a}\varphi_{22} \Rightarrow -a^2\varphi_{22} = b^2\varphi_{22}$. This means that $\varphi_{22} = \varphi_{11} = \varphi_{12} = 0$. Using a similar process with [6 and 8] we can get $\varphi_{13} = \varphi_{23} = 0$ and with [7 and 9] we can get $\varphi_{14} = \varphi_{24} = 0$. This gives us a degenerate matrix with all zeros.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.6 Form 5

$$J_{\mathbb{R}}(\lambda, 2) \oplus J_{\mathbb{R}}(\eta, 1) \oplus J_{\mathbb{R}}(\tau, 1) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \tau \end{pmatrix}$$

This form gives the equations:
$Ae_1 = \lambda e_1$

$Ae_2 = e_1 + \lambda e_2$
$Ae_3 = \eta e_3$
$Ae_4 = \tau e_4$

This results in the following equations:
1.) $(Ae_1, e_1) = \lambda\varphi_{11} = 0$
2.) $(Ae_2, e_2) = \varphi_{12} + \lambda\varphi_{22} = 0$
3.) $(Ae_3, e_3) = \eta\varphi_{33} = 0$
4.) $(Ae_4, e_4) = \tau\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = \lambda\varphi_{12} = -\varphi_{11} - \lambda\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = \lambda\varphi_{13} = -\eta\varphi_{13}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = \lambda\varphi_{14} = -\tau\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = \varphi_{13} + \lambda\varphi_{23} = -\eta\varphi_{23}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = \varphi_{14} + \lambda\varphi_{24} = -\tau\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \eta\varphi_{34} = -\tau\varphi_{34}$

2.6.1 Case 1, $\lambda = \eta = 0, \tau \neq 0$.

By [2 and 4] we can see that $\varphi_{12} = \varphi_{44} = 0$. Using [5, 7, 8, 9, and 10] we can also see that $\varphi_{11} = \varphi_{14} = \varphi_{13} = \varphi_{24} = \varphi_{34} = 0$. This gives the degenrate matrix below.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varphi_{22} & \varphi_{23} & 0 \\ 0 & \varphi_{23} & \varphi_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.6.2 Case 2, $\lambda = \tau = 0, \eta \neq 0$.

Using [2 and 3] we can see that $\varphi_{12} = \varphi_{33} = 0$. Using [5, 6, 8, 9, and 10] we can also see that $\varphi_{11} = \varphi_{13} = \varphi_{23} = \varphi_{14} = \varphi_{34} = 0$. This gives the following degenerate matrix.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varphi_{22} & 0 & \varphi_{24} \\ 0 & 0 & 0 & 0 \\ 0 & \varphi_{24} & 0 & \varphi_{44} \end{pmatrix}$$

2.6.3 Case 3, $\eta = \tau = 0, \lambda \neq 0$.

Using [1 or 5] we see that $\varphi_{11} = 0$. Using [6-9] we can see that $\varphi_{13} = \varphi_{14} = \varphi_{23} = \varphi_{24} = 0$. Using [5] we can see that $\lambda\varphi_{12} = -\lambda\varphi_{12}$ so $\varphi_{12} = 0$. This means that with [2] $\varphi_{22} = 0$. This yet again give a degenerate matrix.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{33} & \varphi_{34} \\ 0 & 0 & \varphi_{34} & \varphi_{44} \end{pmatrix}$$

2.6.4 Case 4, $\lambda, \eta, \tau \neq 0$.

Using [1, 3, and 4] we can see that $\varphi_{11} = \varphi_{33} = \varphi_{44} = 0$. Using [5 and 2] we can also see that $\varphi_{12} = \varphi_{22} = 0$. Using [6] we see that either $\lambda = -\eta$ or $\varphi_{13} = 0$, using [7] we see that either $\lambda = -\tau$ or $\varphi_{14} = 0$, and using [10] we see that either $\eta = -\tau$ or $\varphi_{34} = 0$. But [8] says that if $\lambda = -\eta$ then $\varphi_{13} = 0$ and [9] says that if $\lambda = -\tau$ then $\varphi_{14} = 0$. This means that φ_{13} and φ_{14} will always $= 0$ so we have a degenerate matrix with either $\eta = -\tau$ [φ_1] or $\varphi_{34} = 0$ [φ_2].

$$\varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{23} & \varphi_{24} \\ 0 & \varphi_{23} & 0 & \varphi_{34} \\ 0 & \varphi_{24} & \varphi_{34} & 0 \end{pmatrix} \text{ or } \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{23} & \varphi_{24} \\ 0 & \varphi_{23} & 0 & 0 \\ 0 & \varphi_{24} & 0 & 0 \end{pmatrix}$$

2.7 Form 6

$$J_{\mathbb{C}}(a + \sqrt{-1}b, 1) \oplus J_{\mathbb{C}}(c + \sqrt{-1}d, 1) = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{pmatrix}$$

This results in the equations:
$Ae_1 = ae_1 - be_2$
$Ae_2 = be_1 + ae_2$
$Ae_3 = ce_3 - de_4$
$Ae_4 = de_3 + ce_4$

We can use these to calculate the following:
1.) $(Ae_1, e_1) = a\varphi_{11} - b\varphi_{12} = 0$
2.) $(Ae_2, e_2) = b\varphi_{12} + a\varphi_{22} = 0$
3.) $(Ae_3, e_3) = c\varphi_{33} - d\varphi_{34} = 0$
4.) $(Ae_4, e_4) = d\varphi_{34} + c\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = a\varphi_{12} - b\varphi_{22} = -b\varphi_{11} - a\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = a\varphi_{13} - b\varphi_{23} = -c\varphi_{13} + d\varphi_{14}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = a\varphi_{14} - b\varphi_{24} = -d\varphi_{13} - c\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = b\varphi_{13} + a\varphi_{23} = -c\varphi_{23} + d\varphi_{24}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = b\varphi_{14} + a\varphi_{24} = -d\varphi_{23} - c\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = c\varphi_{34} - d\varphi_{44} = -d\varphi_{33} - c\varphi_{34}$

2.7.1 Case 1, $a = c = 0, b, d \neq 0$.

Using [1 and 3] we can see that $\varphi_{12} = \varphi_{34} = 0$. Using [5 and 10] we can see that $\varphi_{22} = \varphi_{11}$ and $\varphi_{44} = \varphi_{33}$. Using [6-9] we can see that either $b = d$ [φ_1] or $b = -d$ [φ_2] or $\varphi_{13} = \varphi_{14} = \varphi_{23} = \varphi_{24} = 0$ [φ_3]. φ_3 gives the positive determinant ${\varphi_{11}}^2{\varphi_{33}}^2$. φ_1 has determinant $\varphi_{11}(-\varphi_{33}{\varphi_{13}}^2+\varphi_{11}{\varphi_{33}}^2-\varphi_{33}{\varphi_{14}}^2)+\varphi_{13}(\varphi_{13}({\varphi_{13}}^2 +{\varphi_{14}}^2)-\varphi_{11}\varphi_{13}\varphi_{33})-\varphi_{14}(\varphi_{14}\varphi_{11}\varphi_{33}-\varphi_{14}({\varphi_{13}}^2+{\varphi_{14}}^2))$ which reduces to $({\varphi_{13}}^2$

$+{\varphi_{14}}^2-\varphi_{11}\varphi_{33})^2$ which is positive. φ_2 has determinant $\varphi_{11}(-\varphi_{33}{\varphi_{13}}^2+\varphi_{11}{\varphi_{33}}^2 -\varphi_{33}{\varphi_{14}}^2)+\varphi_{13}(-\varphi_{13}(-{\varphi_{13}}^2-{\varphi_{14}}^2)-\varphi_{11}\varphi_{33}\varphi_{13}) - \varphi_{14}(\varphi_{14}(-{\varphi_{13}}^2-{\varphi_{14}}^2)+\varphi_{14} \varphi_{11}\varphi_{33})$ which also reduces to $({\varphi_{13}}^2+{\varphi_{14}}^2-\varphi_{11}\varphi_{33})^2$ which is positive.

$$\varphi_1 = \begin{pmatrix} \varphi_{11} & 0 & \varphi_{13} & \varphi_{14} \\ 0 & \varphi_{11} & -\varphi_{14} & \varphi_{13} \\ \varphi_{13} & -\varphi_{14} & \varphi_{33} & 0 \\ \varphi_{14} & \varphi_{13} & 0 & \varphi_{33} \end{pmatrix} \text{ or } \varphi_2 = \begin{pmatrix} \varphi_{11} & 0 & \varphi_{13} & \varphi_{14} \\ 0 & \varphi_{11} & \varphi_{14} & -\varphi_{13} \\ \varphi_{13} & \varphi_{14} & \varphi_{33} & 0 \\ \varphi_{14} & -\varphi_{13} & 0 & \varphi_{33} \end{pmatrix} \text{ or}$$
$$\varphi_3 = \begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ 0 & \varphi_{11} & 0 & 0 \\ 0 & 0 & \varphi_{33} & 0 \\ 0 & 0 & 0 & \varphi_{33} \end{pmatrix}$$

2.7.2 Case 2, $a = 0, b, c, d \neq 0$.

Using [1 and 5] we can see that $\varphi_{12} = 0$ and $\varphi_{11} = \varphi_{22}$. Using [3, 4, and 10] we can determine that $\varphi_{33} = \varphi_{34} = \varphi_{44} = 0$. The determinant of this matrix is again $(\varphi_{13}\varphi_{24} - \varphi_{23}\varphi_{14})^2$.

$$\varphi = \begin{pmatrix} x & 0 & \varphi_{13} & \varphi_{14} \\ 0 & x & \varphi_{23} & \varphi_{24} \\ \varphi_{13} & \varphi_{23} & 0 & 0 \\ \varphi_{14} & \varphi_{24} & 0 & 0 \end{pmatrix}$$

2.7.3 Case 3, $c = 0, a, b, d \neq 0$.

Using [3 and 10] we can determine that $\varphi_{34} = 0$ and $\varphi_{33} = \varphi_{44}$. Using [1, 2, and 5] we can determine that $\varphi_{11} = \varphi_{12} = \varphi_{22} = 0$. Yet again we get the positive determinant $(\varphi_{13}\varphi_{24} - \varphi_{23}\varphi_{14})^2$.

$$\varphi = \begin{pmatrix} 0 & 0 & \varphi_{13} & \varphi_{14} \\ 0 & 0 & \varphi_{23} & \varphi_{24} \\ \varphi_{13} & \varphi_{23} & x & 0 \\ \varphi_{14} & \varphi_{24} & 0 & x \end{pmatrix}$$

2.7.4 Case 4, $a, b, c, d \neq 0$.

Using [1, 2, and 5] we can again determine that $\varphi_{11} = \varphi_{12} = \varphi_{22} = 0$. Using [3, 4, and 10] we can again determine that $\varphi_{33} = \varphi_{34} = \varphi_{44} = 0$. This matrix also has the positive determinant $(\varphi_{13}\varphi_{24} - \varphi_{23}\varphi_{14})^2$.

$$\varphi = \begin{pmatrix} 0 & 0 & \varphi_{13} & \varphi_{14} \\ 0 & 0 & \varphi_{23} & \varphi_{24} \\ \varphi_{13} & \varphi_{23} & 0 & 0 \\ \varphi_{14} & \varphi_{24} & 0 & 0 \end{pmatrix}$$

3 Results 2: Forms that are possible

3.1 Form 7

$$J_{\mathbb{C}}(a+\sqrt{-1}b,1)\oplus J_{\mathbb{R}}(\lambda,1)\oplus J_{\mathbb{R}}(\eta,1)=\begin{pmatrix} a & b & 0 & 0\\ -b & a & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & \eta \end{pmatrix}$$

This form gives the equations:
$Ae_1 = ae_1 - be_2$
$Ae_2 = be_1 + ae_2$
$Ae_3 = \lambda e_3$
$Ae_4 = \eta e_4$

This results in the following equations:
1.) $(Ae_1, e_1) = a\varphi_{11} - b\varphi_{12} = 0$
2.) $(Ae_2, e_2) = b\varphi_{12} + a\varphi_{22} = 0$
3.) $(Ae_3, e_3) = \lambda\varphi_{33} = 0$
4.) $(Ae_4, e_4) = \eta\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = a\varphi_{12} - b\varphi_{22} = -b\varphi_{11} - a\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = a\varphi_{13} - b\varphi_{23} = -\lambda\varphi_{13}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = a\varphi_{14} - b\varphi_{24} = -\eta\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = b\varphi_{13} + a\varphi_{23} = -\lambda\varphi_{23}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = b\varphi_{14} + a\varphi_{24} = -\eta\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \lambda\varphi_{34} = -\eta\varphi_{34}$

3.1.1 Case 1, $\lambda = \eta = 0, a, b \neq 0$.

Using [1 and 2] we see that $a\varphi_{11} = b\varphi_{12} = -a\varphi_{22}$ which also means that $\varphi_{11} = -\varphi_{22}$ and $\varphi_{12} = \frac{-a}{b}\varphi_{22}$. Using [5] we see that $2a\varphi_{12} = b\varphi_{22} - b\varphi_{11} \Rightarrow 2a\varphi_{12} = 2b\varphi_{22} \Rightarrow \varphi_{12} = \frac{b}{a}\varphi_{22}$ so $\frac{-a}{b}\varphi_{22} = \frac{b}{a}\varphi_{22} \Rightarrow -a^2\varphi_{22} = b^2\varphi_{22}$. This means that $\varphi_{22} = \varphi_{11} = \varphi_{12} = 0$. Using [6] we can see that $a\varphi_{13} = b\varphi_{23} \Rightarrow \varphi_{13} = \frac{b}{a}\varphi_{23}$ but using [8] we can see that $b\varphi_{13} = -a\varphi_{23} \Rightarrow \varphi_{13} = \frac{-a}{b}\varphi_{23}$. Therefore $\frac{b}{a}\varphi_{23} = \frac{-a}{b}\varphi_{23} \Rightarrow b^2\varphi_{23} = -a^2\varphi_{23}$ so $\varphi_{23} = \varphi_{13} = 0$. Using [7 and 9] and the same process we see that $\varphi_{14} = \varphi_{24} = 0$. This leaves us with another degenerate matrix.

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \varphi_{33} & \varphi_{34}\\ 0 & 0 & \varphi_{34} & \varphi_{44} \end{pmatrix}$$

3.1.2 Case 2, $a = \lambda = \eta = 0, b \neq 0$.

By [1 or 2] we see that $\varphi_{12} = 0$ and by [5] we see that $\varphi_{11} = \varphi_{22}$. By [6-9] we can also see that $\varphi_{23} = \varphi_{24} = \varphi_{13} = \varphi_{14} = 0$. The determinant of this matrix would be ${\varphi_{11}}^2(\varphi_{33}\varphi_{44} - {\varphi_{34}}^2)$ and is therefore a possibility.

$$\varphi = \begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ 0 & \varphi_{11} & 0 & 0 \\ 0 & 0 & \varphi_{33} & \varphi_{34} \\ 0 & 0 & \varphi_{34} & \varphi_{44} \end{pmatrix}$$

3.1.3 Case 3, $a = 0, b, \lambda, \eta \neq 0$.

By [1 or 2] we see that $\varphi_{12} = 0$ and by [5] we see that $\varphi_{11} = \varphi_{22}$. By [3 and 4] we see that $\varphi_{33} = \varphi_{44} = 0$. Using [6] we can see that $b\varphi_{23} = \lambda\varphi_{13} \Rightarrow \varphi_{23} = \frac{\lambda}{b}\varphi_{13}$. Using [8] we can see that $b\varphi_{13} = -\lambda\varphi_{23} \Rightarrow \varphi_{23} = \frac{-b}{\lambda}\varphi_{13}$. So $\frac{\lambda}{b}\varphi_{13} = \frac{-b}{\lambda}\varphi_{13} \Rightarrow \lambda^2\varphi_{13} = -b^2\varphi_{13}$, therefore $\varphi_{13} = \varphi_{23} = 0$. By a similar process with [7 and 9] we can see that $\varphi_{14} = \varphi_{24} = 0$. Using [10] we see that either $\lambda = -\eta$ [φ_1] or $\varphi_{34} = 0$ [φ_2]. φ_2 would be degenerate and φ_1 could work because the determinant is $-{\varphi_{11}}^2{\varphi_{34}}^2$.

$$\varphi_1 = \begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ 0 & \varphi_{11} & 0 & 0 \\ 0 & 0 & 0 & \varphi_{34} \\ 0 & 0 & \varphi_{34} & 0 \end{pmatrix} \text{ or } \varphi_2 = \begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ 0 & \varphi_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.1.4 Case 4, $a, b, \lambda, \eta \neq 0$.

Using [3 and 4] we can see that $\varphi_{33} = \varphi_{44} = 0$. Using [1 and 2] we see that $a\varphi_{11} = b\varphi_{12} = -a\varphi_{22}$ which also means that $\varphi_{11} = -\varphi_{22}$ and $\varphi_{12} = \frac{-a}{b}\varphi_{22}$. Using [5] we see that $2a\varphi_{12} = b\varphi_{22} - b\varphi_{11} \Rightarrow 2a\varphi_{12} = 2b\varphi_{22} \Rightarrow \varphi_{12} = \frac{b}{a}\varphi_{22}$ so $\frac{-a}{b}\varphi_{22} = \frac{b}{a}\varphi_{22} \Rightarrow -a^2\varphi_{22} = b^2\varphi_{22}$. This means that $\varphi_{22} = \varphi_{11} = \varphi_{12} = 0$. Using [6] we can get $(a+\lambda)\varphi_{13} = b\varphi_{23} \Rightarrow \frac{(a+\lambda)}{b}\varphi_{13} = \varphi_{23}$. Using [8] we get $b\varphi_{13} = -(a+\lambda)\varphi_{23}$ so we can substitute $\frac{(a+\lambda)}{b}\varphi_{13}$ for the φ_{23}. So $b\varphi_{13} = \frac{-(a+\lambda)^2}{b}\varphi_{13} \Rightarrow b^2\varphi_{13} = -(a+\lambda)^2\varphi_{13}$, therefore $\varphi_{13} = \varphi_{23} = 0$. This same method can be used with [7 and 9] to get $\varphi_{14} = \varphi_{24} = 0$. [10] says that either $\lambda = -\eta$ [φ_1] or $\varphi_{34} = 0$ [φ_2]. Either way we get a degenerate matrix.

$$\varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi_{34} \\ 0 & 0 & \varphi_{34} & 0 \end{pmatrix} \text{ or } \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.2 Form 8

$$J_{\mathbb{R}}(\lambda, 1) \oplus J_{\mathbb{R}}(\eta, 1) \oplus J_{\mathbb{R}}(\tau, 1) \oplus J_{\mathbb{R}}(\delta, 1) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

This form gives the equations:
$Ae_1 = \lambda e_1$
$Ae_2 = \eta e_2$
$Ae_3 = \tau e_3$
$Ae_4 = \delta e_4$

This results in the following equations:
1.) $(Ae_1, e_1) = \lambda\varphi_{11} = 0$
2.) $(Ae_2, e_2) = \eta\varphi_{22} = 0$
3.) $(Ae_3, e_3) = \tau\varphi_{33} = 0$
4.) $(Ae_4, e_4) = \delta\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = \lambda\varphi_{12} = -\eta\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = \lambda\varphi_{13} = -\tau\varphi_{13}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = \lambda\varphi_{14} = -\delta\varphi_{14}$
8.) $(Ae_2, e_3) = -(e_2, Ae_3) = \eta\varphi_{23} = -\tau\varphi_{23}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = \eta\varphi_{24} = -\delta\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \tau\varphi_{34} = -\delta\varphi_{34}$

3.2.1 Case 1, $\lambda = \eta = \tau = \delta = 0$.

In this case we have a starting matrix of all zeros which is skew-adjoint with respect to any metric so we don't need to consider it.

3.2.2 Case 2, $\lambda = \eta = 0, \tau, \delta \neq 0$.

Using [3 and 4] we can see that $\varphi_{33} = \varphi_{44} = 0$. Also using [6-9] we can see that $\varphi_{13} = \varphi_{14} = \varphi_{23} = \varphi_{24} = 0$. Lastly, using [10] we see that either $\tau = -\delta$ [φ_1] or $\varphi_{34} = 0$ [φ_2]. φ_2 is degenerate but φ_1 gives the determinant ${\varphi_{34}}^2({\varphi_{12}}^2 - \varphi_{11}\varphi_{22})$ so it is a possibility.

Because all of the cases where two variables are 0 and two are not are very similar we can disregard the others and just use this one example to express them all.

$$\varphi_1 = \begin{pmatrix} \varphi_{11} & \varphi_{12} & 0 & 0 \\ \varphi_{12} & \varphi_{22} & 0 & 0 \\ 0 & 0 & 0 & \varphi_{34} \\ 0 & 0 & \varphi_{34} & 0 \end{pmatrix} \text{ or } \varphi_2 = \begin{pmatrix} \varphi_{11} & \varphi_{12} & 0 & 0 \\ \varphi_{12} & \varphi_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.2.3 Case 3, $\lambda, \eta, \tau, \delta \neq 0$.

Using [1-4] we see that $\varphi_{11} = \varphi_{22} = \varphi_{33} = \varphi_{44} = 0$. The remaining equations result in with the following choices:
5.) $\lambda = -\eta$ or $\varphi_{12} = 0$
6.) $\lambda = -\tau$ or $\varphi_{13} = 0$
7.) $\lambda = -\delta$ or $\varphi_{14} = 0$
8.) $\eta = -\tau$ or $\varphi_{23} = 0$

9.) $\eta = -\delta$ or $\varphi_{24} = 0$
10.) $\tau = -\delta$ or $\varphi_{34} = 0$

This means there are 64 different possibilities to consider. Fortunately we can eliminate 42 of these due to contradictions or because they are degenerate. Out of the resulting 22 we can determine that 19 of them either have a determinant that is greater than or equal to 0 so they can be eliminated as well. This leaves us with just three possible cases. When $\lambda = -\tau = -\delta = \eta$ and $\varphi_{12} = \varphi_{34} = 0$ $[\varphi_1]$ then we get the determinant $\varphi_{13}\varphi_{24}(\varphi_{13}\varphi_{24} - \varphi_{23}\varphi_{14}) - \varphi_{23}\varphi_{14}(\varphi_{13}\varphi_{24} - \varphi_{23}\varphi_{14})$. But as we have seen before this can be reduced to see that it is positive. When $\lambda = -\eta = -\delta = \tau$ and $\varphi_{13} = \varphi_{24} = 0$ $[\varphi_2]$ then the determinant is $-\varphi_{14}(\varphi_{34}\varphi_{12}\varphi_{23} - {\varphi_{23}}^2\varphi_{14}) - \varphi_{12}(\varphi_{34}\varphi_{14}\varphi_{23} - {\varphi_{34}}^2\varphi_{12})$ which reduces to $(\varphi_{14}\varphi_{23} - \varphi_{12}\varphi_{34})^2$ which is positive. Lastly, when $\lambda = -\eta = -\tau = \delta$ and $\varphi_{14} = \varphi_{23} = 0$ $[\varphi_3]$ then the determinant is $\varphi_{13}(\varphi_{13}{\varphi_{24}}^2 - \varphi_{34}\varphi_{12}\varphi_{24}) - \varphi_{12}(\varphi_{34}\varphi_{13}\varphi_{24} - \varphi_{12}{\varphi_{34}}^2)$ which reduces to $(\varphi_{13}\varphi_{24} - \varphi_{12}\varphi_{34})^2$ which is also positive.

$$\varphi_1 = \begin{pmatrix} 0 & 0 & \varphi_{13} & \varphi_{14} \\ 0 & 0 & \varphi_{23} & \varphi_{24} \\ \varphi_{13} & \varphi_{23} & 0 & 0 \\ \varphi_{14} & \varphi_{24} & 0 & 0 \end{pmatrix} \text{ or } \varphi_2 = \begin{pmatrix} 0 & \varphi_{12} & 0 & \varphi_{14} \\ \varphi_{12} & 0 & \varphi_{23} & 0 \\ 0 & \varphi_{23} & 0 & \varphi_{34} \\ \varphi_{14} & 0 & \varphi_{34} & 0 \end{pmatrix} \text{ or}$$
$$\varphi_3 = \begin{pmatrix} 0 & \varphi_{12} & \varphi_{13} & 0 \\ \varphi_{12} & 0 & 0 & \varphi_{24} \\ \varphi_{13} & 0 & 0 & \varphi_{34} \\ 0 & \varphi_{24} & \varphi_{34} & 0 \end{pmatrix}$$

3.3 Form 9

$$J_{\mathbb{R}}(\lambda, 3) \oplus J_{\mathbb{R}}(\eta, 1) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix}$$

This form gives the equations:
$Ae_1 = \lambda e_1$
$Ae_2 = e_1 + \lambda e_2$
$Ae_3 = e_2 + \lambda e_3$
$Ae_4 = \eta e_4$

This results in the following equations:
1.) $(Ae_1, e_1) = \lambda\varphi_{11} = 0$
2.) $(Ae_2, e_2) = \varphi_{12} + \lambda\varphi_{22} = 0$
3.) $(Ae_3, e_3) = \varphi_{23} + \lambda\varphi_{33} = 0$
4.) $(Ae_4, e_4) = \eta\varphi_{44} = 0$
5.) $(Ae_1, e_2) = -(e_1, Ae_2) = \lambda\varphi_{12} = -\varphi_{11} - \lambda\varphi_{12}$
6.) $(Ae_1, e_3) = -(e_1, Ae_3) = \lambda\varphi_{13} = -\varphi_{12} - \lambda\varphi_{13}$
7.) $(Ae_1, e_4) = -(e_1, Ae_4) = \lambda\varphi_{14} = -\eta\varphi_{14}$

8.) $(Ae_2, e_3) = -(e_2, Ae_3) = \varphi_{13} + \lambda\varphi_{23} = -\varphi_{22} - \lambda\varphi_{23}$
9.) $(Ae_2, e_4) = -(e_2, Ae_4) = \varphi_{14} + \lambda\varphi_{24} = -\eta\varphi_{24}$
10.) $(Ae_3, e_4) = -(e_3, Ae_4) = \varphi_{24} + \lambda\varphi_{34} = -\eta\varphi_{34}$

3.3.1 Case 1, $\lambda, \eta \neq 0$.

Using [1 and 4] we can see that $\varphi_{11} = \varphi_{44} = 0$. Using this and [5] we can see that $\lambda\varphi_{12} = -\lambda\varphi_{12}$ therefore $\varphi_{12} = 0$. This same process shows with [6] that $\varphi_{13} = 0$. Using [2] we can see that $\varphi_{22} = 0$. Therefore with [8] we see that $\varphi_{23} = 0$. Using [3] we can see that $\varphi_{33} = 0$. [7] means that either $\varphi_{14} = 0$ or $\lambda = -\eta$. If $\lambda = -\eta$ then [9] says $\varphi_{14} = 0$ so either way it will. Since this is true [9] also says that either $\varphi_{24} = 0$ or $\lambda = -\eta$ but if $\lambda = -\eta$ then [10] says $\varphi_{24} = 0$ so it will be always 0 too. [10] will then say that either $\lambda = -\eta$ [φ_1] or $\varphi_{34} = 0$ [φ_2]. Either way we get a degenerate matrix.

$$\varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi_{34} \\ 0 & 0 & \varphi_{34} & 0 \end{pmatrix} \text{ or } \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.3.2 Case 2, $\lambda = \eta = 0$.

Using [2, 3, and 5] we can see that $\varphi_{12} = \varphi_{23} = \varphi_{11} = 0$. Using [9 and 10] we can see that $\varphi_{14} = \varphi_{24} = 0$. Lastly, using [8] we can see that $\varphi_{13} = -\varphi_{22}$. This results in the matrix below with determinant ${\varphi_{13}}^3\varphi_{44}$ which could possibly work.

$$\varphi = \begin{pmatrix} 0 & 0 & \varphi_{13} & 0 \\ 0 & -\varphi_{13} & 0 & 0 \\ \varphi_{13} & 0 & x & y \\ 0 & 0 & y & \varphi_{44} \end{pmatrix}$$

4 Results Summarized

Below are the specific cases that will work in the Lorentzian setting gathered from the above results. First, the specific form and its matrix are given and then the metric and determinant are given as well.

4.1 Form 7, Case 2, $a = \lambda = \eta = 0, b \neq 0$.

$$A = J_{\mathbb{C}}(\sqrt{-1}b, 1) \oplus J_{\mathbb{R}}(0, 1) \oplus J_{\mathbb{R}}(0, 1) = \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{The resulting metric is: } \varphi = \begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ 0 & \varphi_{11} & 0 & 0 \\ 0 & 0 & \varphi_{33} & \varphi_{34} \\ 0 & 0 & \varphi_{34} & \varphi_{44} \end{pmatrix}$$

The determinant of this matrix would be ${\varphi_{11}}^2(\varphi_{33}\varphi_{44} - {\varphi_{34}}^2)$.

4.2 Form 7, Case 3, $a = 0, b, \lambda, \eta \neq 0$.

For this case to work $\lambda = -\eta$ must be true.

$$A = J_{\mathbb{C}}(\sqrt{-1}b, 1) \oplus J_{\mathbb{R}}(\lambda, 1) \oplus J_{\mathbb{R}}(\eta, 1) = \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$$

$$\text{The resulting metric is: } \varphi = \begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ 0 & \varphi_{11} & 0 & 0 \\ 0 & 0 & 0 & \varphi_{34} \\ 0 & 0 & \varphi_{34} & 0 \end{pmatrix}$$

The determinant is $-{\varphi_{11}}^2{\varphi_{34}}^2$.

4.3 Form 8, Case 2, $\lambda = \eta = 0, \tau, \delta \neq 0$.

For this case to work $\tau = -\delta$ must be true.

$$A = J_{\mathbb{R}}(0, 1) \oplus J_{\mathbb{R}}(0, 1) \oplus J_{\mathbb{R}}(\tau, 1) \oplus J_{\mathbb{R}}(-\tau, 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & -\tau \end{pmatrix}$$

$$\text{The resulting metric is: } \varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & 0 & 0 \\ \varphi_{12} & \varphi_{22} & 0 & 0 \\ 0 & 0 & 0 & \varphi_{34} \\ 0 & 0 & \varphi_{34} & 0 \end{pmatrix}$$

The determinant is ${\varphi_{34}}^2({\varphi_{12}}^2 - \varphi_{11}\varphi_{22})$.

4.4 Form 9, Case 2, $\lambda = \eta = 0$.

$$A = J_{\mathbb{R}}(0, 3) \oplus J_{\mathbb{R}}(0, 1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The resulting metric is: $\varphi = \begin{pmatrix} 0 & 0 & \varphi_{13} & 0 \\ 0 & -\varphi_{13} & 0 & 0 \\ \varphi_{13} & 0 & x & y \\ 0 & 0 & y & \varphi_{44} \end{pmatrix}$

The determinant of the matrix is ${\varphi_{13}}^3\varphi_{44}$.

5 Open Questions

Possible questions for future research in this area:

1.) What Jordan normal forms are possibilities in signature (2,2) in dimension 4?

2.) What Jordan normal forms are possible in dimensions 5, 6, or higher for either Lorentzian or higher signatures?

3.) Can you use a change of basis to eliminate the variables that are unneccesary to find the determinant and thus aren't required to be known in the forms, for example the x, y variables in Form 9, Case 2?

4.) Can you construct an algebraic curvature tensor R on the Skew-Tsankov model in dimension 4 or greater in the Lorentzian setting? Since the known examples of these models in the positive definite setting are decomposible [5], does there exist one in the Lorentzian setting that is indecomposible?

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6 References

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