On the Computation and Dimension of Structure Groups of Algebraic Curvature Tensors

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Abstract

The groups of symmetries of algebraic objects of are of vital importance in multiple fields of mathematics; here, we examine a less well-known family of these objects which arise as the symmetry groups of algebraic curvature tensors. These tensors themselves are of interest because of their connection to the Riemann curvature tensor, which is central in the study of differential geometry. Here we focus on weak model spaces, which ignore the inner product, instead considering only the algebraic properties of the tensor itself.

Due to the complex nature of nature of these structure groups, which are subgroups of the general linear group acting on the space of algebraic curvature tensors, our goal is simply to illuminate some examples and basic properties. Many of their characteristics are known due to the fact that they are Lie groups, which attach to them a Lie algebra structure at the identity which can be used to calculate important information. We also seek to show that there exist structure groups of any arbitrary dimension, which we do through explicit construction.

1 Introduction

Often, we seek to calculate how an object curves in space at a given point. If we restrict the Riemann curvature tensor on a pesudo-Riemannian manifold to a single point, we obtain an algebraic object on the tangent space called an *algebraic curvature tensor*. Reducing to it's algebraic properties, we can define a general algebraic curvature tensor as follows:

Definition 1. *If V is an n-dimensional vector space, a tensor $R \in \otimes^4(V^*)$ is called an* algebraic curvature tensor *if it satifies:*

$$R(x, y, z, w) = -R(y, x, z, w) \tag{1}$$

$$R(x, y, z, w) = R(z, w, x, y) \tag{2}$$

$$R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0 \tag{3}$$

The third of these is called the *Bianchi Identity*.

The vector space of all algebraic curvature tensors on V is denoted $\mathcal{A}(V)$. An ordered triple (V, ϕ, R), where ϕ is an inner product on V is known as a *model space*, while a pair (V, R) without the inner product is known as a *weak model space*. In this paper, we will consider curvature tensors by their value at some basis; because of the multilinearity of tensors, this is all we need to uniquely define a curvature tensor.

We can also construct an algebraic curvature tensor R_ϕ on a model space from its inner product *phi* through the following formula:

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$$

Sometimes, we can decompose an algebraic curvature R into the direct sum of smaller algebraic curvature tensors $R_1 \oplus R_2$, $R_1 \in \otimes^4(V_1^*)$, $R_2 \in \otimes^4(V_2^*)$, where $V_1 \oplus V_2 = V$, and where the following property holds:

$$R(x, y, z, w) = 0$$

whenever one of x, y, z, w is in V_1 and another is in V_2. Alternatively, we can construct a new curvature tensor R from smaller ones R_1 and R_2 by declaring $R(x, y, z, w) = R_1(x, y, z, w)$ if $x, y, z, w \in V_1$, $R(x, y, z, w) = R_2(x, y, z, w)$ if $x, y, z, w \in V_2$, and $R(x, y, z, w) = 0$ otherwise.

Suppose $A : V \to V$ is nonsingular (i.e. $A \in GL(V)$) and $R \in \mathcal{A}(V)$. Then define a new algebraic curvature tensor A^*R by precomposition:

$$A^*R(x, y, z, w) = R(Ax, Ay, Az, Aw)$$

The set of all such linear maps forms a subgroup of $GL(n, R)$, the *structure group* of R, which is denoted G_R. These subgroups are Lie groups as well, being closed subgroups of a Lie group, so we can use the theory of Lie groups to get information about the dimension. In addition, for an inner product ϕ, let $O(\phi)$ be the *orthogonal group* of ϕ, the group of linear maps in $GL(V)$ which preserve ϕ in the same sense above.

2 Preliminaries

In order to algorithmically compute structure groups, we need a standardized listing of the possible combinations of basis elements. According to [1], there are $\frac{n^2(n^2-1)}{12}$ independent curvature components; for convenience, we will not consider the reduction in independent components that come as a result of the Bianchi identity. We can do this by considering first pairs of basis elements (i, j), ordering them with $i < j$, and then taking pairs of these to create the quadruples necessary as inputs of R. Order the pairs uniquely as well by the dictionary order:

$$(i, j) < (k, l) \Leftrightarrow i < k \text{ or } (i = k \text{ and } j < l)$$

. That is to say, we consider all quadruples of the form

$$(i,j,k,l), i<j, k<l, (i,j) \leq (k,l)$$

. This gives us $\binom{n}{2}$ different combinations for each pair, and we choose (allowing repeats) two of these pairs, leaving $\frac{(\binom{n}{2})(\binom{n}{2}+1)}{2}$ combinations. Denote the set of these quadruples in n dimensions L_n. For example, $L_2 = \{(1,2,1,2)\}$ and $L_3 = \{(1,2,1,2),(1,2,1,3),(1,2,2,3),(1,3,1,3),(1,3,2,3),(2,3,2,3)\}$. We will continue to think about these quadruples as pairs of pairs, referring to the first and second pairs of these elements. We will also use the notation $L_n[i]$ to denote the i-th element in this ordered list.

These G_R are Lie groups, so we may consider the *Lie algebra* $\mathrm{Lie}(G_R)$ associated to them, which is just the tangent space of the manifold at the identity matrix I. In addition, the *exponential map* for matrices is given by

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

where $A^0 = I$. This map restricts to a diffeomorphism from a neighborhood of 0 in the Lie algebra to a neighborhood of I in the Lie group, which tells us the dimension of the group and gives us a family of one-parameter subgroups e^{tA}, where A is in $\mathrm{Lie}(G_R)$. This family in fact generates the component of the identity in the group.

2.1 Equations Defining Structure Groups

Recall that $A \in G_R$ exactly when $A^*R(x,y,z,w) = R(x,y,z,w)$. Suppose we are given R by its values on a basis $\{e_1, ...e_n\}$ of V. Denote $R(e_i,e_j,e_k,e_l)$ by R_{ijkl}, and let $A = [a_{ij}]$ be an element of G_R. Then, for any quadruple of indices (p,q,r,s) ordered as above, we have

$$R_{pqrs} = R(a_{1p}e_1+\ldots+a_{np}e_n, a_{1q}e_1+\ldots+a_{nq}e_n, a_{1r}e_1+\ldots+a_{nr}e_n, a_{1s}e_1+\ldots+a_{ns}e_n)$$

Expanding out in all possible combinations gives us

$$R_{pqrs} = \sum_{i,j,k,l}^{n} a_{ip}a_{jq}a_{kr}a_{ls}R_{ijkl}$$

But by standardizing the orders with equation (1) above we can group related R_{ijkl} in four ways, corresponding to $(i,j,k,l),(j,i,l,k),(j,i,k,l),(i,j,l,k)$, picking up negative signs accordingly:

$$R_{pqrs} = \sum_{i=1}^{j-1}\sum_{j=1}^{n}\sum_{k=1}^{l-1}\sum_{l=1}^{n}(a_{ip}a_{jq}a_{kr}a_{ls}+a_{jp}a_{iq}a_{lr}a_{ks}-a_{jp}a_{iq}a_{kr}a_{ls}-a_{ip}a_{jq}a_{lr}a_{ks})R_{ijkl}$$

If $i = k$ and $j = l$, the above coefficients are it. Otherwise, we could also get distinct coefficients of R_{ijkl} through equation (2) above, by considering combinations with the order of the pairs reversed:

$$\begin{aligned}
R_{pqrs} &= \sum_{(i,j)=(k,l)} (a_{ip}a_{jq}a_{kr}a_{ls} + a_{jp}a_{iq}a_{lr}a_{ks} - a_{jp}a_{iq}a_{kr}a_{ls} - a_{ip}a_{jq}a_{lr}a_{ks})R_{ijkl} \\
&+ \sum_{(i,j)\neq(k,l)} (a_{ip}a_{jq}a_{kr}a_{ls} + a_{jp}a_{iq}a_{lr}a_{ks} + a_{kp}a_{lq}a_{ir}a_{js} + a_{lp}a_{kq}a_{jr}a_{is})R_{ijkl} \\
&- \sum_{(i,j)\neq(k,l)} (a_{jp}a_{iq}a_{kr}a_{ls} + a_{ip}a_{jq}a_{lr}a_{ks} + a_{lp}a_{kq}a_{ir}a_{js} + a_{kp}a_{lq}a_{jr}a_{is})R_{ijkl}
\end{aligned}$$

This can be rearranged to

$$\begin{aligned}
R_{pqrs} &= \sum_{(i,j)=(k,l)} (a_{ip}a_{jq} - a_{jp}a_{iq})(a_{kr}a_{ls} - a_{lr}a_{ks})R_{ijkl} \\
&+ \sum_{(i,j)\neq(k,l)} ((a_{ip}a_{jq} - a_{jp}a_{iq})(a_{kr}a_{ls} - a_{lr}a_{ks}) + (a_{kp}a_{lq} - a_{lp}a_{kq})(a_{ir}a_{js} - a_{jr}a_{is}))R_{ijkl}
\end{aligned}$$

So, for each element $(p, q, r, s) \in L_n$ we obtain a fourth degree homogeneous polynomial in the components of A. This is all the equations the curvature tensor must satisfy; if we consider expanding one of the determined components, such as one with repeated entries in a single pair, we get a degenerate equation, as all the coefficients above become zero.

3 Results

3.1 Computation of Structure Groups

Given the above equations, we can (theoretically) calculate the Lie algebra of G_R by the taking an arbitrary path through the group $A(t)$ with $A(0) = I$ and differentiating at zero to find $A'(o)$, an arbitrary tangent vector in the tangent space $T_I G_R$. The set of all these tangent vectors is the Lie algebra, so by considering the above matrix components $a_{ij}(t)$ as functions of t, we can simply differentiate implicity and evaluate at zero. In dimension n, we have differentiated $|L_n|$ different equations in n^2 unknowns, which after evaluating leaves us $|L_n|$ different *linear* equations in the $a'_{ij}(0)$, whose solutions can be found with basic Gaussian reduction. Since the left hand side above is constant with respect to t, we can this linear system as $\vec{0} = M\vec{a}$, where $\vec{a} = (a'_{11}(0), a'_{12}(0), \ldots, a'_{1,n}(0), a'_{21}(0), \ldots, a'_{nn}(0))$. So, the Lie algebra is simply the null space of M, where we think about the matrices as vectors instead.

Computing every derivative would be extremely resource consuming, but the matrix M can be computed with nothing but simple comparisons, making it feasible to compute structure groups on a computer. It stems from the observation that if consider all the possible products $a_{ip}a_{jq}a_{kr}a_{ls}$ above, when differentiated and evaluated at zero, no more than one of the factors can be nondiagonal, by the product rule there would always be a nondiagonal factor which evaluated to zero at $t = 0$. This gives us the following result:

Theorem 1. *$Lie(G_R) = Null(M)$, where M is the coefficient matrix of the system of linear equations given by $S_R\vec{P} = \vec{0}$. Here, $\vec{P} = (R_{L_n[i]})_{i=1}^{|L_n|}$ and $S_R = [S_R]_{\alpha\beta}$ is the $|L_n|$ by $|L_n|$ matrix given in terms of its $\alpha\beta$ entry according to the following rules: Let*

$$d(\alpha) = \begin{cases} 2, & \text{if } L_\alpha = (p,q,p,q) \\ 1, & \text{otherwise} \end{cases}$$

1. *Suppose $L_\alpha = L_\beta$. Then,*

   $$[S_R]_{\alpha\beta} = a'_{L_\alpha(1)L_\alpha(1)}(0) + a'_{L_\alpha(2)L_\alpha(2)}(0) + a'_{L_\alpha(3)L_\alpha(3)}(0) + a'_{L_\alpha(4)L_\alpha(4)}(0)$$

   .

2. *Suppose L_α and L_β share* ***exactly one*** *pair (that is, either the first pair of L_α is equal to the first pair of L_β, the first pair of L_α is equal to the second pair of L_β, the second pair of L_α is equal to the first pair of L_β, or the second pair of L_α is equal to the second pair of L_β. Then, if the other, nonequal pairs share* ***exactly one*** *element,*

   $$[S_R]_{\alpha\beta} = \pm d(\alpha) a_{ip}$$

   where i and p are the nonequal indices in the nonequal pair, negative if and only if the show up in opposite places (i.e. i comes first in its pair, while p comes second, or vice versa).

3. *Otherwise,*

   $$[S_R]_{\alpha\beta} = 0$$

Proof. The matrix S_R represents the derivative of coefficient of $R_{L[\beta]}$ in the expansion of the equation that fixes $R_{L[\alpha]}$, evaluated at zero. The product with $\vec{P}$ then gives all the linear equations which determine the Lie algebra.

Suppose L_α and L_β share no pairs at all. Then, since the coefficients in the equations in section 2.1 were all obtained by taking the different possible combinations of pairs and orderings of those pairs, there are no coefficients which have more than two diagonal factors. Then, when differentiating, by the product rule we will get something of the form

$$a'_{ip}a_{jq}a_{kr}a_{ls} + a_{ip}a'_{jq}a_{kr}a_{ls} + a_{ip}a_{jq}a'_{kr}a_{ls} + a_{ip}a_{jq}a_{kr}a'_{ls}$$

If $L_\alpha = (p,q,r,s)$ and $L_\beta = (i,j,k,l)$ share neither pair, then there are at least two factors of $a_{ip}a_{jq}a_{kr}a_{ls}$ which are not diagonal, meaning there is at least one nondiagonal term in each of the summands above. So, evaluated at zero, this gives zero.

Suppose L_α and L_β share a pair, and let (p,q) and (i,j) be the other pair. Then, there are several cases:

1. (p, q) shares no element with (i, j). In this case, we obtain zero again because we can only match as many as two in L_β with the indices in L_α, leaving two nondiagonal factors in every summand in the coefficient.

2. $(p, q) = (i, j)$. That is to say, $\alpha = \beta$. In this case, using the expansions in section 2.1, we can see directly by plugging $(i, j, k, l) = (p, q, r, s)$ that none of the summands besides the one which permutes nothing will have less than 2 nondiagonal factors in both cases in that section. In that coefficient, we will obtain all diagonal terms, so since the diagonal evaluates to 1, we will obtain the value $a'_{ii}(0) + a'_{jj}(0) + a'_{kk}(0) + a'_{ll}(0)$

3. $p = i, q \neq j$. Here, we seek to match at least 3 of the indices in L_β with those in L_α. There are eight possible permutations of a quadruple (i, j, k, l) which can be obtained with only the rules (1) and (2) above. Listed, they are

   $$(i, j, k, l), -(i, j, l, k), -(j, i, k, l), (j, i, l, k), (k, l, i, j), -(l, k, i, j), -(k, l, j, i), (l, k, j, i)$$

   with the negative signs denoting an attached change of sign because of rule (1). Of these, if $L_\alpha = (p, q, p, q)$, then there are two, $(i, j, k, l) = (p, j, p, q)$ and $(k, l, i, j) = (p, q, p, j)$. Each of these will have 3 diagonal factors, so when the derivative is taken, only one summand, the one in which the nondiagonal term is differentiated, will be nonzero; otherwise, when evaluated at $t = 0$, that term will cause the product to be zero. This nondiagonal term will be gotten by choosing the component of the j-th basis element of the vector to which e_q was sent, so we will get $2a'_{jq}(0)$ as the coefficient. If $L_\alpha \neq (p, q, p, q)$, then it is easy to check that only one of the terms will match 3 indices, giving instead $a'_{jq}(0)$. It will be the exact same when $p \neq i, q = j$.

4. $q = i$. In this case, the same as above, but the negative permutations instead will be the ones that match 3 indices. So, instead, we obtain $-a'_{jq}(0)$ or $-a'_{jq}(0)$. Again, it is almost exactly the same for $p = j$.

□

3.2 Dimension 3

The structure groups of $R \in \mathcal{A}(V)$, where the dimension of V is 3, can be classified into 3 nontrivial types. By [2], we may choose a basis which reduces to zero three of the six curvature components: R_{1213}, R_{1223}, and R_{1323}. Further, we can normalize that basis so that the rest of the components, R_{1212}, R_{1313}, and R_{2323}, are ± 1 or 0. We do this by scaling each individual component by:

$$\begin{aligned} e_1 &\mapsto \left(\frac{|R_{2323}|}{|R_{1313}||R_{1212}|}\right)^{\frac{1}{4}} e_1 \\ e_2 &\mapsto \left(\frac{|R_{1313}|}{|R_{2323}||R_{1212}|}\right)^{\frac{1}{4}} e_2 \\ e_3 &\mapsto \left(\frac{|R_{1212}|}{|R_{1313}||R_{2323}|}\right)^{\frac{1}{4}} e_3 \end{aligned}$$

assuming that none are nonzero. If some are, then simply divide.

1. If all three are nonzero, $R = R_\phi$ for some nondegenerate inner product ϕ. To help show this, we use the following lemma:

   Lemma 1. *If $R' = cR$ where $c \in R$, then $G_{R'} = G_R$.*

   This is apparent since if $R(Ax, Ay, Az, Aw) = R(x, y, z, w)$, then $cR(Ax, Ay, Az, Aw) = cR(x, y, z, w) = R'(x, y, z, w)$ and vice versa. Because of this, we can just consider the cases where exactly one of the curvature components is -1; otherwise, it is simply the negative of one of these. If $R_{1212} = R_{1313} = R_{2323} = 1$, $-R$ is the curvature tensor generate by $\phi(i, j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, so $G_R = O(\phi)$ which has dimension $(3)(3-1)/2 = 3$. If exactly one is negative, then

   $$\phi = \begin{pmatrix} R_{1212} & 0 & 0 \\ 0 & R_{1313} & 0 \\ 0 & 0 & R_{2323} \end{pmatrix}$$

   generates R. In any event, $G_R = G_{R_\phi}$ for some ϕ, so by [3], $G_R = O(\phi)$, hence always has dimension 3.

2. If one of the curvature components is zero, then there are two possiblities: the remaining two have the same sign or opposite signs. We could, by symmetry, relabel our basis elements to make any of the three nonzero and we would have essentially the same curvature tensor, so we can consider the case where $R_{2323} = 0$. Using theorem 1, we can easily compute the nullspace of the system of equations: if $R_{1212} = R_{1313}$,

   $$\begin{pmatrix} x_1 & 0 & 0 \\ x_3 & -x_1 & x_2 \\ x_4 & -x_2 & -x_1 \end{pmatrix}$$

   If $R_{1212} = -R1313$, we obtain

   $$\begin{pmatrix} x_1 & 0 & 0 \\ x_3 & -x_1 & x_2 \\ x_4 & -x_2 & -x_1 \end{pmatrix}$$

   In either case, the dimension is 4. The other possibilities would be obtained just by switching around the basis elements, and hence switching columns of these matrices.

3. If all but one curvature component is zero, then again we need only consider the case where R_{1212}, given symmetry. In this case, with the algorithm in theorem 1 above, we find the nullspace is 6 dimensional:

   $$\begin{pmatrix} x_1 & x_2 & 0 \\ x_3 & -x_1 & 0 \\ x_4 & x_5 & x_6 \end{pmatrix}$$

So, all of the structure groups in dimension 3 falls into one of these 3 cases; in particular, it is just a relabeling of basis elements away from one of the given matrices.

3.3 Dimension 4

In dimension 4, the number of possible curvature components is 20, which gives us 20 equations in 16 unknowns, meaning in a general curvature tensor the structure group is likely to be zero dimensional. In addition, it permits the possiblity of the existence of a 1-dimensional structure group. This structure group is a 1-dimensional Lie group, so the connected component of the identity must be diffeomorphic to either R or the circle S^1. In fact, both are possible: if the nonzero curvature components are $R_{1213} = R_{2324} = R_{1334} = R_{1424} = 1$, we get the following matrix in Lie algebra, which exponentiates to a compact curve:

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \overset{e^A}{\mapsto} \begin{pmatrix} \cos(t) & 0 & -\sin(t) & 0 \\ 0 & \cos(t) & 0 & -\sin(t) \\ \sin(t) & 0 & \cos(t) & 0 \\ 0 & \sin(t) & 0 & \cos(t) \end{pmatrix}$$

If we instead take $R_{1213} = R_{2324} = R_{1424} = 1, R_{1334} = -1$ (which might be more symmetrically written $R_{1213} = R_{2324} = R_{3431} = R_{4142} = 1$), the algorithm in 1 produces the following noncompact one-parameter subgroup:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \overset{e^A}{\mapsto} \begin{pmatrix} \cosh(t) & 0 & \sinh(t) & 0 \\ 0 & \cosh(t) & 0 & \sinh(t) \\ \sinh(t) & 0 & \cosh(t) & 0 \\ 0 & \sinh(t) & 0 & \cosh(t) \end{pmatrix}$$

In fact, this second matrix is a circulant matrix, and as such is diagonalizable. Changing our basis, $-e_2+e_4$, $-e_1+e_3$, e_2+e_4, and e_1+e_3, we obtain a curvature tensor in which the connected component of the identity is a 1-dimensional family of diagonal matrices. The connected component of the identity could not contain points not on this curve, as the matrix fixes the curvature tensor for all $t \in R$; any path in the manifold must stay within this curve.

The usefulness of this curvature tensor arises out of the consequences of the following lemma:

Lemma 2. *Suppose $R = R_1 \oplus R_2$, is an algebraic curvature tensor on V where R_1 is an algebraic curvature tensor on V_1 and R_2 on V_2. Let $Id(G_R)$ denote the connected component of the identity matrix in G_R. Then,*

$$Id(G_R) \cong Id(G_{R_1}) \times Id(G_{R_2})$$

Proof. ($\subseteq$) We begin with the forward inclusion. By [4], if $A \in G_R$, $R = R_1 \oplus R_2$, then

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \text{ or } \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$

where P and Q are isomorphisms. Denote all $A \in G_R$ of the first kind by M_1, and all of the second kind by M_2. In the space of matrices with the usual topology, these sets are separated (that is, the closure of one is disjoint from the other), since for all elements of M_1, the off diagonal blocks are always zero, whereas for M_2, the diagonal blocks are always zero. So, the closure of M_1 will also have all zeroes on the off diagonal, and the closure of M_2 on the diagonal. Therefore, if we had a path $f : R \to G_R$ with $f(0) \in M_2$ and $f(1) = I$, then by continuity $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are separated sets. But, these sets would decompose the unit interval into separated sets, which contradicts connectedness. So, if $A \in M_2$, $A \notin \mathrm{Id}(G_R)$.

So, if we have a path of matrices $A(t)$ with $A(0) = A_0$ and $A(1) = I$, then

$$A(t) = \begin{pmatrix} P(t) & 0 \\ 0 & Q(t) \end{pmatrix}$$

That is to say, if $A \in \mathrm{Id}(G_R)$, then we can take $P(t) : R \to G_{R_1}$ and $Q(t) : R \to G_{R_2}$ as paths from $A_0|_{V_1}$ and $A_0|_{V_2}$ to the identity matrix in each. So A_0 must have been of the form

$$A_0 = \begin{pmatrix} P_0 & 0 \\ 0 & Q_0 \end{pmatrix}$$

where $P_0 \in \mathrm{Id}(G_{R_1})$ and $Q_0 \in \mathrm{Id}(G_{R_2})$.

($\supseteq$) The reverse inclusion is trivial, as if we have paths $P(t)$ and $Q(t)$ as above ending at the identity, then $A(t)$ will end at the identity matrix as well. $\square$

This almost immediately gives us the result we desire:

Theorem 2. *There exist structure groups of any dimension.*

Proof. The dimension of the structure group of $R = R_1 \oplus R_2$ is the sum of the dimensions of G_{R_1} and G_{R_2}, since the connected component of the identity remains isomorphic to a simple direct product. Since these structure groups are all manifolds by Cartan's theorem, their dimension in the connected component of the identity is the same as the dimension in every other connected component.

So, for a structure group of dimension k, we use the curvature tensor on a $4k$ dimensional vector space given by $R = \bigoplus_{n=1}^{k} R_4$, where R_4 is either of the 1-dimensional structure groups produced above on V of dimension 4. $\square$

4 Conclusions

The previous work demonstrates that the family of structure groups of algebraic curvature tensors is a rich family, with noncompact and compact groups of any desired dimension. In addition, we outlined a constructive way of producing useful information from these structure groups; computations are efficient due to the fact that only equality comparisons between indices are used in computing the matrices needed to find the Lie algebra. After that, the problem is reduced to simple linear algebra.

5 Open Questions

1. What are the possible dimensions a structure group of a curvature tensor on a given vector space can have? In dimension 3 for instance, only dimensions 3, 4, 6, and 9 are possible. What causes this restriction? What is the maximum possible dimension (other than the zero curvature tensor)?

2. How many connected components of a structure group are there? It appears that often the number could be related to subsets of the symmetric group. Is there a maximum number of connected components that can be created? What sort of new curvature components are created by the direct sum of two curvature tensors, such as those in M_2 above, and how many?

3. What sort of Lie algebras can be generated by these structure groups? Where do they fall in the classification of Lie algebras, and what does this imply about the group?

4. One could consider, instead of just groups that fix a curvature tensor, subgroups of $GL(V)$ which send the curvature tensor to a multiple of itself. Additionally, one could also change the vector space to a complex vector space, allowing for complex entries in the matrices, which could drastically change the structure.

5. When are two curvature tensors in the $GL(V)$-orbit of one another? That is to say, when is there an isomorphism $A : V \to V$ so that $A^*R_1 = R-2$?

6. Is the converse of Lemma 2 true? That is to say, if $\mathrm{Id}(G_R) \cong \mathrm{Id}(G_{R_1}) \times \mathrm{Id}(G_{R_2})$, is it necessarily true that $R = R_1 \oplus R_2$?

7. Could information about structure groups at each tangent space give information about the nature of the Riemann curvature tensor (field) and manifold? How does the structure group change with the curvature tensor? For example, on a path through the manifold, what do changes in the dimension of G_R say about the geometry of the manifold?

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References

[1] P. Gilkey, *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor.* World Scientific, River Edge, New Jersey, (2001), 45.

[2] R. Klinger, *A Basis that Reduces to Zero as many Curvature Components as Possible*, Abh. Math. Sem. Univ. Hamburg **61** (1991), 243-248.

[3] J. Palmer, *Structure Groups of Pseudo-Riemannian Algebraic Curvature Tensors*, CSUSB REU Program (2010), 4.

[4] C. Dunn, C. Franks, J. Palmer, *On the structure group of a decomposable model space*, eprint (2011) arXiv:1108.2224.