Signature Change and Linear Dependence of Curvature Tensors

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Abstract

We consider the linear dependence of 3 canonical algebraic curvature tensors and extend previous results of others by working in the higher signature setting.

1 Introduction

Let V be a finite dimensional real vector space with an inner product φ. Let's begin with some important definitions.

Definition 1. *An* inner product *satifies the following properties where u, v, and w are vectors and α is a scalar:*

1. $\varphi(u+v,w) = \varphi(u,w) + \varphi(v,w)$
2. $\varphi(\alpha \cdot v, w) = \alpha \cdot \varphi(v,w)$
3. $\varphi(v,w) = \varphi(w,v)$.

Definition 2. *Let V^* be the set of all linear functions from V to $\mathbb{R}$. An* algebraic curvature tensor *is a function $R \in \otimes^4(V^*)$, satisfying the algebraic identities of the curvature tensor of a Riemannian manifold:*

$$R(x,y,z,w) = -R(y,x,z,w) = R(z,w,x,y)$$

$$R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.$$

The last property is known as the *Bianchi Identity.*

Definition 3. *Let φ be a bilinear form on V. We say φ is* symmetric *if $\varphi(v,w) = \varphi(w,v)$ for all $v,w \in V$, and we say φ is* positive definite *if for all $v \in V$, $\varphi(v,v) \geq 0$, and equal to zero only when $v = 0$.*

Although this paper uses cases where positive definiteness is not assumed, it is still necessary to know what forms are considered positive definite as this still comes up in this paper.

Definition 4. *An inner product, φ, is* nondegenerate *if and only if, for all $x \neq 0 \in V$, there exists $y \neq 0 \in V$ such that $\varphi(x,y) \neq 0$.*

Let φ be a symmetric bilinear form on V. For all $x,y,z,w \in V$, let R_φ be defined by

$$R_\varphi(x,y,z,w) = \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w). \tag{1.a}$$

Definition 5. *$A : V \longrightarrow V$, then the* adjoint *of A, denoted A^*, with respect to φ, is characterized by the equation $\varphi(Ax, y) = \varphi(x, A^*y)$. A is said to be* self-adjoint *if $A = A^*$.*

If φ is nondegenerate and $\Psi : V \longrightarrow V$ is self-adjoint, then

$$\psi(y, x) = \varphi(\Psi y, x) = \varphi(y, \Psi^* x) = \varphi(\Psi^* x, y) = \varphi(\Psi x, y) = \psi(x, y).$$

We can also define the following

$$R_\psi(x, y, z, w) = \varphi(\Psi x, w)\varphi(\Psi y, z) - \varphi(\Psi x, z)\varphi(\Psi y, w) = R_\varphi(\Psi x, \Psi y, z, w) \tag{1.b}$$

$$R_\tau(x, y, z, w) = \varphi(Tx, w)\varphi(Ty, z) - \varphi(Tx, z)\varphi(Ty, w) = R_\varphi(Tx, Ty, z, w) \tag{1.c}$$

where Ψ and T are self-adjoint with respect to φ and Ψ, T: $V \to V$. Note that R_ψ and R_τ are algebraic curvature tensors as long as ψ and τ are symmetric [2].

Definition 6. *We define the Spec(Ψ) as the set of eigenvalues of Ψ, repeated according to multiplicity, and $|$Spec$(\Psi)|$ as the number of distinct elements of Spec(Ψ).*

The following theorem from Diaz [1] inspired this paper.

Theorem 1. *Suppose dim $V \geq 4$, φ is positive definite, Rank $\tau =$ dim V, and Rank $\psi \geq 3$. The set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent if and only if one of the following is true:*

(1) $|Spec(\psi)| = |Spec(\tau)| = 1$

(2) $Spec(\tau) = \{\eta_1, \eta_2, \eta_2, \ldots\}$, and $Spec(\psi) = \{\lambda_1, \lambda_2, \lambda_2, \ldots\}$, with $\eta_1 \neq \eta_2$, $\lambda_2^2 = \varepsilon(\delta\eta_2^2 - 1)$, and $\lambda_1 = \frac{\varepsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1)$.

Diaz worked with the assumption that φ is positive definite, *Rank* τ is equal to the dimension of the vector space so that τ^{-1} exists, and *Rank* ψ is greater than or equal to 3. We study the linear dependence of $\{R_\varphi, R_\psi, R_\tau\}$ in the event that φ is not positive definite, but merely nondegenerate. Smothers [4] explored change of signature endomorphism to show that the theorem presented by Diaz applies to the nondegenerate situation, although there is considerable difficulty in transferring the nondegenerate case to the positive definite one. According to Smothers, there exists a change of signature endomorphism that could take you from φ (not necessarily positive definite) to φ^+ (which can be positive definite).

Definition 7. *Let there be a change of signature endomorphism. It's* signature *is denoted by (p, q) with p time-like vectors and q space-like vectors.*

This technique involves a change of signature endomorphism with signature (4,4) in this particular case. This will be dicussed in detail in the next section. We will also point out that Smothers obtained partial results by appliying this idea.

Using the following theorems given by Diaz and Dunn [2], we reduce the equation $aR_\varphi + bR_\psi + cR_\tau = 0$ by noticing that if one or more of a, b, or c are zero, and none of φ, ψ, or τ have a rank less than 3, then the following theorem applies:

Theorem 2. *Suppose Rank $\varphi \geq 3$. The set $\{R_\varphi, R_\psi\}$ is linearly dependent if and only if $R_\psi \neq 0$, and $\varphi = \nu\psi$ for some $\nu \in \mathbb{R}$.*

Otherwise, we have a,b,c $= 0$, and we are left with the case that $R_\varphi = \varepsilon R_\psi + \delta R_\tau$ where ε and δ are a choice of signs. Below is the theorem from which Diaz and Dunn introduced this equation.

Theorem 3. *Suppose φ is positive definite,* Rank $\tau = n$, *and* Rank $\psi \geq 3$. *If $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then ψ and τ are simultaneously orthongonally diagonalizable with respect to φ.*

2 Change of Signature

Definition 8. *Let C represent our* change of signature endomorphism. *Then C is an endomorphism that exists in the vector space and has the following properties:*

1. *C is self-adjoint with respect to φ.*
2. *C^2 is the identity.*
3. *The symmetric bilinear form $\varphi^+(x,y) := \varphi(Cx,y)$ is positive definite.*
4. *On an orthonormal basis,*
$$[C] = \left[\begin{array}{c|c} -I_q & 0 \\ \hline 0 & I_p \end{array}\right]$$

Lemma 1. *Suppose Rank $\tau = dimV$, Rank $\psi \geq 3$. If $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then $R^+_{\varphi^+} = \varepsilon R^+_{\psi^+} + \delta R^+_{\tau^+}$.*

Proof. By definition of a change of signature endomorphism:

1) Using the symmetric bilinear form, $\varphi^+(x,y)$, we can construct the following:
$$R^+_{\varphi^+}(x,y,z,w) = \varphi^+(x,w)\varphi^+(y,z) - \varphi^+(x,z)\varphi^+(y,w).$$

2) Given $\varphi^+(x,y) = \varphi(Cx,y)$, we compute the following, note that $\varphi(\psi Cx,y) = \varphi(C^2\psi Cx,y) = \varphi^+(C\psi Cx,y)$:
$$\begin{aligned} R_\Psi(Cx,Cy,z,w) &= \varphi(\Psi Cx,w)\varphi(\Psi Cy,z) - \varphi(\Psi Cx,z)\varphi(\Psi Cy,w) \\ &= \varphi^+(C\Psi Cx,w)\varphi^+(C\Psi Cy,z) - \varphi^+(C\Psi Cx,z)\varphi^+(C\Psi Cy,w) \\ &= R^+_{C\Psi C}(x,y,z,w). \end{aligned}$$

Follow the same process for τ to get $R^+_{CTC}(x,y,z,w)$.

3) Using the equation found from Theorem 3, we get:
$$\begin{aligned} R_\varphi(Cx,Cy,z,w) &= \varepsilon R_\psi(Cx,Cy,z,w) + \delta R_\tau(Cx,Cy,z,w) \\ R_{\varphi^+}(x,y,z,w) &= \varepsilon R^+_{C\Psi C}(x,y,z,w) + \delta R^+_{CTC}(x,y,z,w) \\ &= \varepsilon R^+_{\psi^+}(x,y,z,w) + \delta R^+_{\tau^+}(x,y,z,w). \end{aligned}$$

□

Remark. $C\Psi C = \psi^+$ and $CTC = \tau^+$ when $C\Psi C$ is self-adjoint with respect to φ^+ if and only if C and Ψ commute, similarly for CTC. Therefore, it is not clear if $R^+_{\psi^+}$ is an algebraic curvature tensor (the same for $R^+_{\tau^+}$), so we cannot yet apply the result Diaz obtained. Only one needs to be proven to be self -adjoint in order for $R_{\varphi^+} = \varepsilon R^+_{\psi^+} + \delta R^+_{\tau^+}$ to be true.

It is important to note that $C\Psi C = \psi^+$ means $C\Psi C = \Psi^+$ and similarly, $CTC = \tau^+$ means $CTC = T^+$.
If $C\Psi C$ is self-adjoint, then the following will hold:
$$\begin{aligned} \varphi^+(C\Psi Cx,y) &= \varphi^+(x,C\Psi Cy) \\ \varphi(\Psi Cx,y) &= \varphi(Cx,C\Psi Cy) \\ \varphi(\Psi Cx,y) &= \varphi(x,\Psi Cy) \\ &= \varphi(C\Psi x,y) \\ &\Rightarrow \Psi C = C\Psi. \end{aligned}$$

Similarly if CTC is self-adjoint:
$$\begin{aligned} \varphi^+(CTCx,y) &= \varphi^+(x,CTCy) \\ \varphi(TCx,y) &= \varphi(Cx,CTCy) \\ \varphi(TCx,y) &= \varphi(x,TCy) \\ &= \varphi(CTx,y) \\ &\Rightarrow TC = CT. \end{aligned}$$

Which would mean that either Ψ and C or T and C commute.

Suppose the signature of $\varphi = (4,4)$ and the basis is $\{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\}$, then we have the following representations of φ and ψ:

$$\varphi = \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The calculation of ψ and C is illustrated below.

$$\psi = \left[\begin{array}{c|c} A & B \\ \hline -B^T & D \end{array}\right] \text{ and } C = \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right]$$

$$\psi C = \left[\begin{array}{c|c} -A & B \\ \hline B^T & D \end{array}\right] \text{ and } C\psi = \left[\begin{array}{c|c} -A & -B \\ \hline -B^T & D \end{array}\right]$$

We assume

$$\psi = \left[\begin{array}{cccc|cccc} \lambda_1 & 0 & 0 & 0 & a & b & c & d \\ 0 & \lambda_2 & 0 & 0 & e & f & g & h \\ 0 & 0 & \lambda_3 & 0 & i & j & k & l \\ 0 & 0 & 0 & \lambda_4 & m & n & o & p \\ \hline -a & -e & -i & -m & \lambda_5 & 0 & 0 & 0 \\ -b & -f & -j & -n & 0 & \lambda_6 & 0 & 0 \\ -c & -g & -k & -o & 0 & 0 & \lambda_7 & 0 \\ -d & -h & -l & -p & 0 & 0 & 0 & \lambda_8 \end{array}\right]$$

and similarly,

$$\tau = \left[\begin{array}{cccc|cccc} \eta_1 & 0 & 0 & 0 & \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ 0 & \eta_2 & 0 & 0 & \bar{e} & \bar{f} & \bar{g} & \bar{h} \\ 0 & 0 & \eta_3 & 0 & \bar{i} & \bar{j} & \bar{k} & \bar{l} \\ 0 & 0 & 0 & \eta_4 & \bar{m} & \bar{n} & \bar{o} & \bar{p} \\ \hline -\bar{a} & -\bar{e} & -\bar{i} & -\bar{m} & \eta_5 & 0 & 0 & 0 \\ -\bar{b} & -\bar{f} & -\bar{j} & -\bar{n} & 0 & \eta_6 & 0 & 0 \\ -\bar{c} & -\bar{g} & -\bar{k} & -\bar{o} & 0 & 0 & \eta_7 & 0 \\ -\bar{d} & -\bar{h} & -\bar{l} & -\bar{p} & 0 & 0 & 0 & \eta_8 \end{array}\right]$$

Consider the decomposition of the vector space to $V = W^+ \oplus W^-$ where $W^+ = span\{f_1, \dots, f_j\}$ and $W^- = span\{e_1, \dots, e_i\}$. Now we present the following conjecture that was expected to show that the relationship between φ, ψ, and τ holds as it did in 3.

Theorem 4. *Suppose Rank τ = dim V, Rank $\psi \geq 3$, and the signature of φ is equal to dim V. Let the set $\{R_\varphi, R_\psi, R_\tau\}$ be restricted to the vector space W^+. The set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent if and only if $\varphi|_{W^+}$ is positive definite and $\psi|_{W^+}$ and $\tau|_{W^+}$ are symmetric, then $\psi|_{W^+}$ and $\tau|_{W^+}$ are simultaneously diagonalizable with respect to φ.*

Proof. Show that all the off diagonal values are zero. We start with the following calculations,

$$\begin{aligned} R^+_{\varphi^+}(e_3, e_1, e_1, f_3) &= \varphi^+(e_3, f_3)\varphi^+(e_1, e_1) - \varphi^+(e_3, e_1)\varphi^+(e_1, f_3) \\ &= 0 \cdot 0 + 0 \cdot 0 \\ &= 0. \end{aligned}$$

By computation you can see that $R^+_{\varphi^+}(e_3, e_1, e_1, f_3) = 0 = R^+_{\varphi^+}(e_3, e_2, e_2, f_3)$ and

$$\begin{aligned} R^+_{\psi^+}(e_3, e_1, e_1, f_3) &= \psi^+(e_3, f_3)\psi^+(e_1, e_1) - \psi^+(e_3, e_1)\psi^+(e_1, f_3) \\ &= -k\lambda_1 + 0 \cdot 0 \\ &= -k\lambda_1. \end{aligned}$$

We can see that $R^+_{\psi^+}(e_3, e_1, e_1, f_3) = -k\lambda_1$ and by further calculation $R^+_{\psi^+}(e_3, e_2, e_2, f_3) = -k\lambda_2$. Now we can use $R^+_{\varphi^+} = \varepsilon R^+_{\psi^+} + \delta R^+_{\tau^+}$ to build the following equations:

$$0 = \varepsilon k\lambda_1 + \delta\bar{k}\eta_1 \tag{1}$$

$$0 = \varepsilon k\lambda_2 + \delta\bar{k}\eta_2. \tag{2}$$

Multiply Equation 1 by λ_2 and Equation 2 by λ_1, then subtract the equations to get

$$\begin{aligned} 0 &= \delta\bar{k}(\eta_1\lambda_2 - \eta_2\lambda_1) \\ &= \delta\bar{k}(\eta_1\lambda_2^2 - \eta_2\lambda_1\lambda_2) \\ &= \delta\bar{k}(\eta_1\varepsilon(\delta\eta_2^2 - 1) - \eta_2\varepsilon(\delta\eta_1\eta_2 - 1)) \\ &= \bar{k}(\eta_1\eta_2^2 - \eta_1 - \eta_1\eta_2^2 + \eta_2) \\ 0 &= \bar{k}(-\eta_1 + \eta_2) \\ &\Rightarrow \bar{k} = 0. \end{aligned}$$

Since λ_1 and λ_2 are not equal (and similarly $\eta_1 \neq \eta_2$), substitute $\bar{k}$ into Equation 1 to get

$$\begin{aligned} 0 &= \varepsilon k\lambda_1 \\ 0 &= k\lambda_1 \\ &\Rightarrow k = 0. \end{aligned}$$

Repeat this process for $e, f, g, h, i, j, k, l, m, n, o, p$ and $\bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}, \bar{k}, \bar{l}, \bar{m}, \bar{n}, \bar{o}, \bar{p}$.

$$\begin{aligned} R^+_{\varphi^+}(e_2, e_1, e_1, f_1) &= 0 \\ R^+_{\psi^+}(e_2, e_1, e_1, f_1) &= -e\lambda_1 \\ R^+_{\tau^+}(e_2, e_1, e_1, f_1) &= -\bar{e}\eta_1 \\ &\vdots \\ R^+_{\varphi^+}(e_4, e_1, e_1, f_4) &= 0 \\ R^+_{\psi^+}(e_4, e_1, e_1, f_4) &= -p\lambda_1 \\ R^+_{\tau^+}(e_4, e_1, e_1, f_4) &= -\bar{p}\eta_1. \end{aligned}$$

However, when we compute a, b, c, d and $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, the results are inconclusive.

$$\begin{aligned} R^+_{\varphi^+}(e_1, e_1, e_1, f_1) &= 0 \\ R^+_{\psi^+}(e_1, e_1, e_1, f_1) &= -a\lambda_1 + a\lambda_1 = 0 \\ R^+_{\tau^+}(e_1, e_1, e_1, f_1) &= -\bar{a}\eta_1 + \bar{a}\eta_1 = 0 \\ &\vdots \\ R^+_{\varphi^+}(e_1, e_1, e_1, f_4) &= 0 \\ R^+_{\psi^+}(e_1, e_1, e_1, f_4) &= -d\lambda_1 + d\lambda_1 = 0 \\ R^+_{\tau^+}(e_1, e_1, e_1, f_4) &= -\bar{d}\eta_1 + \bar{d}\eta_1 = 0. \end{aligned}$$

Unfortunately, we can go no further and the proof remains unfinished. □

The relationship between ψ, τ, and φ almost holds as it did in the theorem presented in Diaz [1]. It is surprising to be able to apply this theorem, but not be able to conclude that all $\varphi(e_i, f_j) = 0$. Perhaps if there was another relationship between the eigenvalues of ψ and τ that would allow us to complete the proof. Based on our results we were able to find that the operators almost commute and are very close to being simultaneously diagonalizable. These results widen the applications of the theorem presented in Diaz [1] and Smothers [4].

Future Questions

1. Is it possible to complete the conjecture presented? That is, must $a = b = c = d = \bar{a} = \bar{b} = \bar{c} = \bar{d} = 0$?
2. Characterize when the hypotheses of Theorem 3 are satisfied. What happens if any of them fail in this constuction?
3. Is there a similar result about simultaneous diagonalizability for R_φ built from anti-symmetric tensors?
4. How else could results in [1] be applied?
5. What if φ is degenerate? Or, what if τ does not have full rank? For instance, what if φ is positive semi-definite, and $ker\ \varphi = ker\ \tau$?

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