Proving the absence of certain totally geodesic surfaces

Warren Shull

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Abstract

We have proven the absence of totally geodesic surfaces bounded or punctured by either the figure-eight knot or the 6_2^2 link. We've also found a case (Borromean Rings checkerboard) in which a Dehn filling changes a surface that is not totally geodesic into one that is.

1 Introduction

We begin with some established definitions. A *hyperbolic knot* is a knot whose complement admits a hyperbolic structure, such that there is a locally isometric covering map from 3-dimensional hyperbolic space to 3-dimensional spherical space *excluding* the path traced by the knot. The image of the knot complement in hyperbolic space is called the *universal cover*. A surface in a knot complement is *totally geodesic* if its image in the universal cover is a disjoint union of hyperbolic planes. It is known that every totally geodesic surface in a knot complement is either closed, or else bounded or punctured by the knot. Totally geodesic surfaces are *compact* if and only if they are not punctured by the knot. A *two-bridge* knot is a non-trivial knot that has a planar projection with two maxima (and therefore two minima).

In [1], Adams and Schoenfeld showed that two-bridge knots do not bound any compact orientable totally geodesic surfaces, and that compact non-orientable totally geodesic surfaces bounded by a knot must be incompressible. They further showed that any two-bridge knot that bounds a non-orientable surface must be expressible as $K_{p/q}$ (the notation is taken from [2]), where $\frac{p}{q} = r + [b_1, b_2, ..., b_n]$, where $b_1, b_2, ..., b_n$ are all odd integers and (due to incompressibility) never equal to 1 or -1. The notation $[b_1, b_2, ..., b_n]$ refers to the continued fraction:

$$\cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots - \frac{1}{b_n}}}}}$$

This is enough to show that the figure-eight, one of the three knots or links we consider in this paper, does not bound any compact totally geodesic surfaces. It was shown in [3] that there are no closed totally geodesic surfaces in the figure-eight complement. The only remaining type of surface to consider is one that is punctured by the figure-eight knot, and that is what we consider in this paper. Compact totally geodesic surfaces bounded by the 6_2^2 hyperbolic link, however, are not excluded by Adams and Schoenfeld since $K_{10/3}$ is the 6_2^2 link and $\frac{10}{3} = 3 + \frac{1}{3}$, so we collectively consider all surfaces bounded or punctured by the 6_2^2 link.

2 The Figure-Eight Complement

The figure-eight complement is described by Ratcliffe in [4], page 449, to be two tetrahedra, with edges identified in a particular way. Our edge labels, a and b, are an exact match to Ratcliffe's edge labels. There also exists a correspondence between face colors in our pictures and face labels in Ratcliffe's pictures, as follows:

Ratcliffe's faces A and A' correspond to our red surface.
Ratcliffe's faces B and B' correspond to our blue surface.
Ratcliffe's faces C and C' correspond to our green surface.
Ratcliffe's faces D and D' correspond to our purple surface.

In our diagram of the universal cover, the tetrahedra whose base triangles point to our left correspond to the A,B,C,D tetrahedron in Ratcliffe's diagram (we will call this the "unprimed tetrahedron"), while those pointing to the right correspond to the A', B', C', D' (we will call this the "primed tetrahedron").

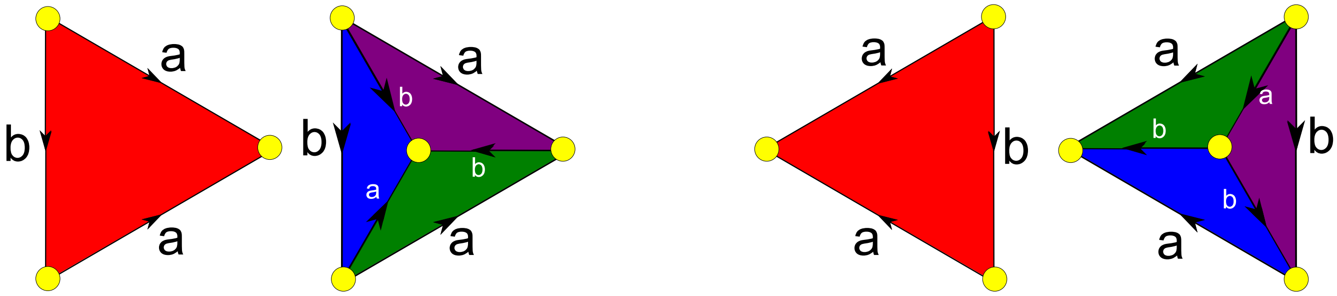

Figure 1: The two ideal tetrahedra that make up the figure-eight complement, viewed from above

On each tetrahedron, we will display the base in one picture and the other three faces in an adjacent picture. So Figure 1 shows two tetrahedra, even though it looks at first glance like four triangles.

When these two tetrahedra are repeatedly glued together, the universal cover is obtained in hyperbolic space. We represent the universal cover by sending one ideal point (per copy of a tetrahedron) infinitely far in the z-direction. We hereafter refer to this ideal vertex as ∞.

For the purposes of this paper, we will clarify what we mean by directional words like "north" and "east" with a compass rose shown in Figure 2. On the compass rose, the solid burgundy lines refer to the directions of triangle edges in our projection diagram, while the dotted orange lines refer to directions that slice those edges at right angles.

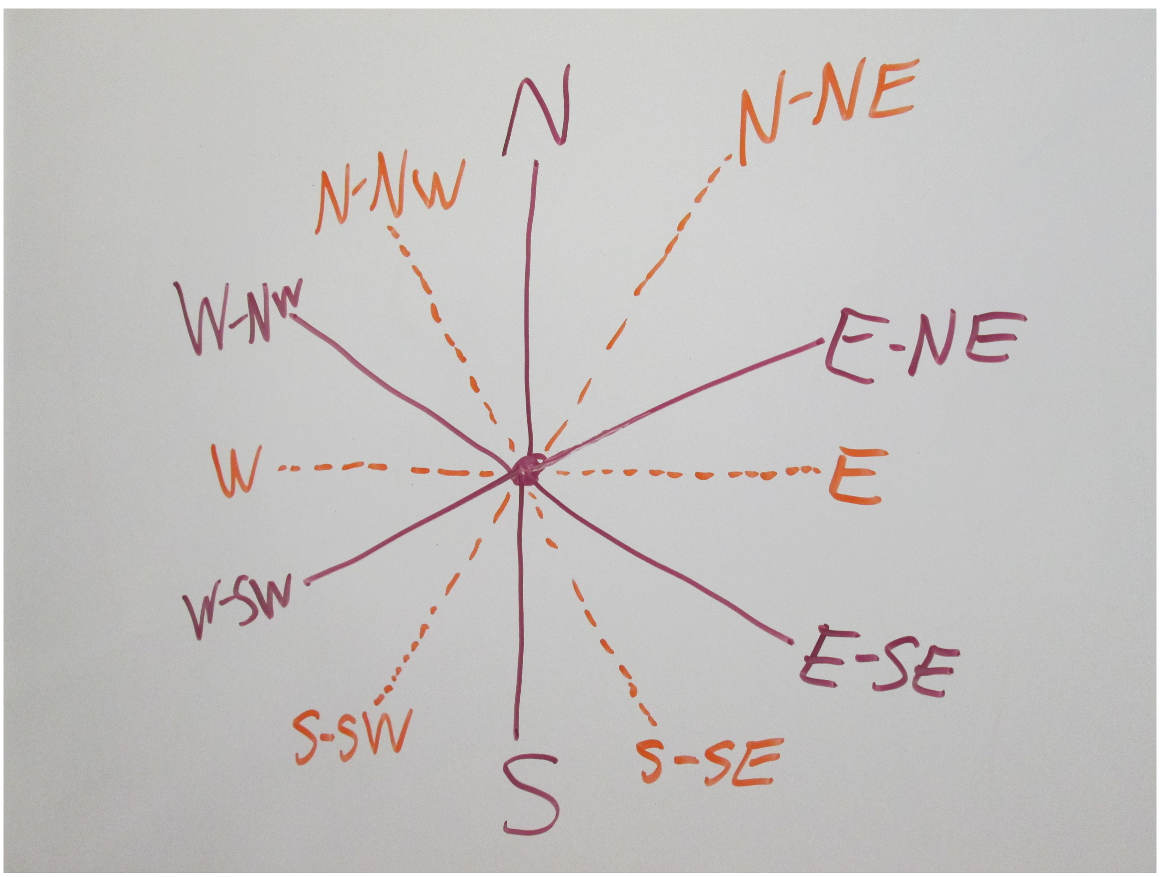

Figure 2: N = North, S = South, E = East, W = West

On our diagram, we display a coordinate system. Each vertex in the x-y plane can be uniquely determined by two coordinates, but we use three for sake of clarity and symmetry. Each ideal lattice point is denoted (J, K, L), for $J, K, L \in \mathbb{Z}$. The J-coordinate determines the placement in the east-west direction (two points have the same J-coordinate if the path from one to the other appears due north or due south in the plane of the picture). The J-coordinates are indexed across the top of the picture. The K-coordinate determines placement in the north-northwest v. south-southeast direction (two points have the same K-coordinate if their connecting path runs east-northeast to west-southwest). The K-coordinates are indexed across the lower left corner of the picture. The L-coordinate determines placement in the north-northeast v. south-southwest direction (two points have the same L-coordinate if their connecting path runs east-southeast to west-northwest). The L-coordinates are indexed across the lower right corner of the picture.

As an example, the point (0,0,0) exists, as does the point (1,0,1), but the point (1,0,0) does not exist. Between two distinct points, at least two coordinates must be different. As it turns out, for any existing point, the J-coordinate and the K-coordinate will always add up to the L-coordinate.

In referring to Euclidean distance, we will call the side length of triangles in the Euclidean projection "one unit."

We then project the universal cover onto the x-y plane for ease of visualization, resulting in a tessellation of triangles in the plane, as follows:

- The interior of a triangle in the plane represents a hyperbolic triangle with ideal vertices on the x-y plane exactly as shown, though the surface of the triangle protrudes upward (it is flat in hyperbolic space).

- What appear to be colored "edges" on the diagram are in fact triangular faces, with two vertices at the points shown in the plane, and the third vertex at ∞. We will refer to these faces from now on as "vertical faces," and to those discussed in the previous bullet point as "horizontal faces." For example, what appears to be an "edge" from $(0, 0, 0)$ to $(0, 1, 1)$ colored in purple is actually the purple *face* with vertices $(0, 0, 0)$, $(0, 1, 1)$, and ∞.

- Each vertical face stretches down only to the hyperbolic line (Euclidean semicircle) connecting the two ideal vertices in the x-y plane. The label we've given to the colored "edges" (vertical faces) in our diagram give the identity and orientation of the hyperbolic arcs which are edges of ideal tetrahedra. Capital letters represent northward orientation (either due north or angled toward the north), while lowercase letters represent southward orientation (again, either direct or angled). For example, the purple face mentioned in the previous bullet-point shares a copy of edge a with the red face on the primed tetrahedron (vertices $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, and ∞), and with the green face on the unprimed tetrahedron (vertices $(0, 0, 0)$, $(0, 1, 1)$, $(-1, 1, 0)$, and ∞) oriented toward $(0, 0, 0)$.

- At each "vertex" can be seen a yellow circle containing a capital or lowercase letter. A circle containing a capital letter represents an arrow coming out of the page (positive z-direction), while a circle containing a lowercase letter represents an arrow pointing back into the page (negative z-direction). Each of these vertical edges joints an ideal point on the x-y plane to ∞. For example, the edge connecting $(0, 0, 0)$ to ∞ is shared by two purple faces, two blue faces, one green face and one red face.

- The vertical edges each have all six of their surrounding tetrahedra shown in our representation of the universal cover. The horizontal edges (arcs) only have two of their surrounding tetrahedra shown, since showing the other four would greatly obscure the picture. As we will see, the part of the universal cover we choose to show is sufficient to prove our new result.

In Figure 3, we display our diagram of the universal cover. It is helpful to note the periodicities that arise. In particular, when the J-coordinate stays the same, but the K and L coordinates each increase by one (directly upward in the picture), the picture remains the same. On the other hand, when the K-coordinate stays the same, with J and L coordinates each increasing by 4 (east-northeast in the picture), the picture remains the same. Similarly, when the L-coordinate stays the same, with J-coordinate increasing by 4 and K-coordinate decreasing by 4 (east-southeast in the picture), the picture remains the same.

We will now define some terminology and make an observation that will be helpful, both for the figure-eight knot and for the 6_2^2 link.

Definition: If an ideal triangle R has ∞ as a vertex, it is clear that two of its three edges will be vertical Euclidean half-lines. A hyperbolic line segment or ray (a Euclidean arc centered on the ideal plane) living in R, connecting *either* the two edges incident to the ∞ vertex *or* one of those edges and the opposite ideal vertex is called a *lofted arc*. It is called *doubly lofted* in the former case, or *singly lofted* in the latter case. The *loft index* of a lofted arc is the total number of doubly lofted arcs contained in the hyperbolic line containing it, plus one if it contains at least one singly lofted arc.

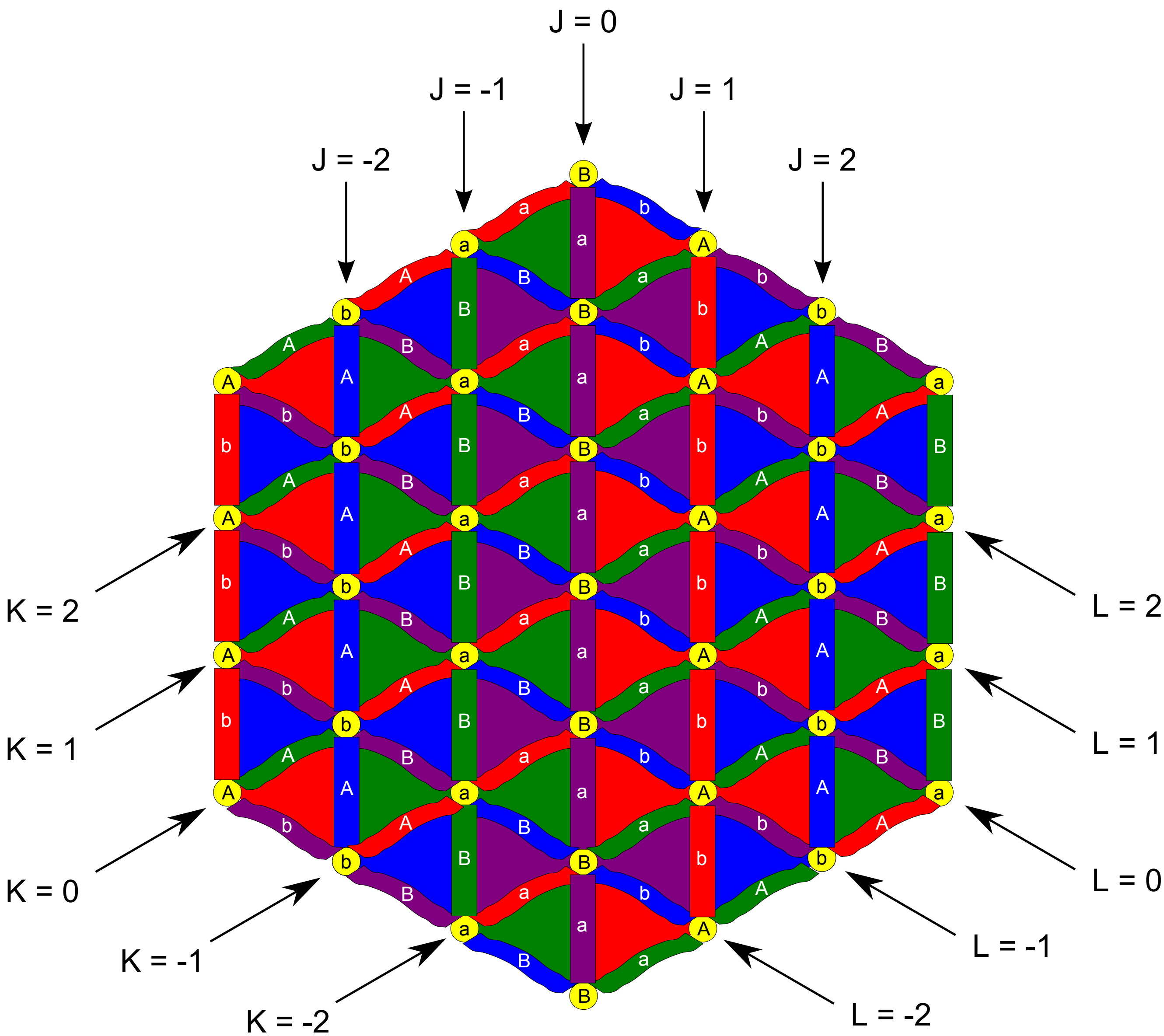

Figure 3: The universal cover of the figure-eight knot

Lemma 2.1 *If the universal cover of a hyperbolic knot or link (in the upper-half space model) appears as an equilateral triangular lattice (the shape of the figure-eight and* 6_2^2 *complements) with Euclidean period* $\leq n$ *in any direction (considering the side length of the Euclidean triangle as one unit), then no totally geodesic surface can intersect a copy of a face of an ideal tetrahedron in a lofted arc of loft index* n*.*

Proof: By way of contradiction, assume the knot complement's universal cover appears in the upper-half space model of $\mathbb{H}^3$ as an equilateral triangular lattice with Euclidean period at most n in some direction. Furthermore, assume a totally geodesic surface T intersects a copy F_0 of a face F of one of the ideal tetrahedra in a lofted arc of index n. Since T is a disjoint union of hyperbolic planes, T_0 shall denote the hyperbolic plane that intersects F_0. It can be verified that the Euclidean diameter of a hyperbolic line containing a lofted arc is strictly greater than the loft index of the lofted arc. Since $T_0 \cap F_0$ is a lofted arc of loft index n, it follows that the hyperbolic line containing $T_0 \cap F_0$ has Euclidean diameter strictly greater than n, and therefore T_0 itself has Euclidean diameter strictly greater than n. But the Euclidean period p, where $p \leq n$, implies that T contains another Euclidean hemisphere, congruent to T_0, with its center exactly p units from the center of T_0. Call this hemisphere T_1. It follows that T_0 and T_1, each having radius greater than $\frac{n}{2}$, have as the sum of their radii some number $r > n \geq p$, where p is the distance between their centers. Therefore T_0 and T_1 must intersect in some Euclidean semicircle, making T *not* totally geodesic.

The above Lemma will be used extensively in Theorem 2.2 and Theorem 3.2. The proofs of both will repeatedly use the fact that totally geodesic surfaces, which appear as disjoint unions of hyperbolic planes in the universal cover, can never intersect themselves; all the contradictions we find will involve a surface–assumed to be totally geodesic–intersecting itself.

The following theorem will verify the known result that there are no compact totally geodesic surfaces bounded by the figure-eight knot, and will add the new result that there are no totally geodesic surfaces *punctured* by the figure-eight knot.

Theorem 2.2 *There are no totally geodesic surfaces bounded or punctured by the figure-eight knot.*

Proof: The universal cover is displayed above. By Mostow's Rigidity Theorem, this geometric representation of the universal cover will yield the same set of totally geodesic surfaces as any other geometric representation.

We will use proof by contradiction. Assume there is a totally geodesic surface bounded punctured by the figure-eight knot. Call this surface T. In the universal cover, T must map to the disjoint union of vertical half-planes and hemispheres centered on the x-y plane. The image of T in the universal cover, which is this disjoint union, we will refer to as $\tilde{T}$.

Because the surface T is bounded or punctured by the knot, its image in the tetrahedra must be bounded by at least one vertex from each tetrahedron. Meanwhile, in the universal cover, *all* the vertices of both tetrahedra are mapped to ∞ at least once. It follows that $\tilde{T}$ must contain a perfectly vertical Euclidean plane (in this paper, "vertical" always refers to the z-direction). Call this plane T_0. Now the projection of T_0 onto the x-y plane either does or does not run parallel with the (Euclidean) lines with fixed J-coordinate. Consider 4 cases:

Case 1: T_0 contains one of the faces of one of the tetrahedra.

In this case, T_0 will contain a copy of the face of a specific color, meaning that $\tilde{T}$ contains all copies of the face of that specific color. But regardless of the color, there exists a face of that color bounded by ∞, not parallel to T_0. Call this face F. The only way for $\tilde{T}$ to contain F is for another vertical half-plane in $\tilde{T}$ to contain F. Call this half-plane T_1. Thus T_1 crosses T_0, so $\tilde{T}$ crosses $\tilde{T}$, which is a contradiction.

Case 2: T_0's projection is parallel to a Euclidean line passing through all the ideal points of a fixed J-value, but does not actually pass through any ideal points with integer J-value.

In this case, thanks to periodicity, we may assume that T_0 and its projection lie to the right of the points with J-value -2, but to the left of points with J-value 2. Consider first the case where T_0's projection lies to the right of points with J-value -2, but to the left of points with J-value -1.

In this case, T_0 intersects the blue face in an arc connecting its copy of edge a (shared with the red face in the unprimed tetrahedron) to its copy of edge b pointing to the tail of a (shared with the purple face in the unprimed tetrahedron). The disjoint union $\tilde{T}$ will thus intersect every copy of the blue face in this same arc. In particular, $\tilde{T}$ will intersect the copy of the blue face with vertices $(0,0,0)$, $(-1,1,0)$, and ∞ in a lofted arc. Since the universal cover has period 1 in one direction, Lemma 2.1 tells us that $\tilde{T}$ intersects itself, verifying that T is once again *not* totally geodesic, again a contradiction.

If the plane T_0 instead sits to the right of points with J-value -1, but to the left of points with J-value 0, a similar argument applies with the green faces. The same is true for J-values between 0 and 1 (with purple faces) or between 1 and 2 (with red faces). The hemispheres leading to the contradiction, in each case, appear just to the right of T_0.

Case 3: The Euclidean projection of T_0 is not parallel to a Euclidean line passing through all the ideal points of a fixed J-value, or of fixed K-value, or of fixed L-value, but still passes through one of the ideal vertices.

First, we shall consider the case where the projection of T_0 passes through $(0,0,0)$. Thus T_0 contains a copy of edge b, so $\tilde{T}$ contains every copy of edge b. Therefore $\tilde{T}$ must contain the arc connecting $(0,-1,-1)$ to $(-1,0,-1)$, and the arc connecting $(-1,1,0)$ to $(-1,0,-1)$. It follows that T_0 cannot pass through either of these two arcs, but it must angle to the left in some direction. It therefore must pass through the arc from $(-1,1,0)$ to $(0,1,1)$, which is a copy of edge a. Then T_0 will continue off to the left, and must eventually cross the upper-half plane above the ideal vertices where $J=-1$. When it crosses this upper-half plane, it will *either* contain a copy of edge a *or* cross a copy of edge b. Since $\tilde{T}$ already contains edge b, the latter case is a contradiction with $\tilde{T}$ intersecting itself. As for the former case, since T_0 now contains a copy of edge a, $\tilde{T}$ will contain every copy of edge a, including the arc connecting $(-1,1,0)$ to $(0,1,1)$. Thus $\tilde{T}$ intersects itself somewhere along that arc, which is once again a contradiction.

Any case where the J-value of the ideal point being passed is equivalent to 0 mod 4 is equivalent under periodicity. Similar arguments apply for all other cases (that is, for all other edges leading to ∞ through which T_0 might pass).

Case 4: The Euclidean projection of T_0 is not parallel to a Euclidean line passing through all the ideal points of a fixed J-value, and does not itself pass through any ideal points visible in our universal cover diagram.

In this case, T_0 must pass through one of the copies of a leading between two consecutive ideal points sharing their J-coordinate. Without loss of generality, suppose T_0 passes through the edge oriented from $(0,1,1)$ to $(0,0,0)$, and thus passes through the purple face with those two vertices and ∞. Consider two sub-cases:

Case 4a: T_0 passes through the edge oriented from $(1,0,1)$ to $(0,0,0)$.

In this case, T_0 will intersect the red face with vertices $(0,1,1)$, $(1,0,1)$, and $(0,0,0)$ in an arc running directly between the two copies of a on the border of the red face. It follows that $\tilde{T}$ intersects every copy of the red face in an arc running between the two copies of a on its border. In particular, $\tilde{T}$ contains an arc on each of the red faces with ∞, $(1,K,L)$, and $(1,K+1,L+1)$ (for all $K,L \in \mathbb{Z}$) as their three vertices. But T_0, as it continues to the right, must cross one of these red faces, thereby crossing the arc contained in $\tilde{T}$. Thus T_0 crosses some other part of $\tilde{T}$, a contradiction.

Case 4b: T_0 passes through the edge oriented from $(0,1,1)$ to $(1,0,1)$.

In this case, as T_0 continues to the right, it must at some point cross a copy of edge b pointing between two ideal points $(1,K,L)$ and $(1,K+1,L+1)$ for some $K,L \in \mathbb{Z}$. Just before it crosses such a face, the previous vertical face it crosses must be the blue face with vertices $(0,K+1,L)$, $(1,K,L)$, and ∞. Between these two faces, T_0 will intersect a purple face in an arc connecting the two copies of b. It follows that $\tilde{T}$ intersects every copy of the purple face in an arc running between the two copies of b on its border. In particular, $\tilde{T}$ contains a "horizontal" arc (a hyperbolic non-vertical line) on the purple face through which we initially asserted that T_0 had to pass. Therefore this arc contained in $\tilde{T}$ passes through $T_0 \subset \tilde{T}$, a contradiction once again.

Since all cases have now been considered and contradicted, there is no possible totally geodesic surface T bounded by the figure-eight knot. QED

3 6_2^2 hyperbolic link

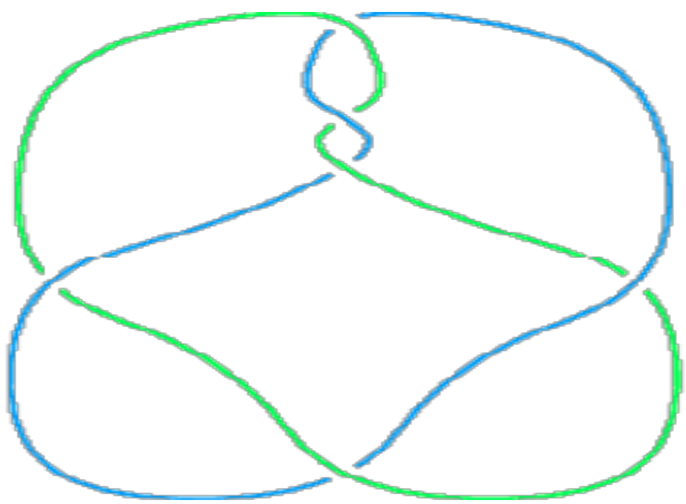

Figure 4: The 6_2^2 link, which can also be shown as $\sigma_1\sigma_2^{-1}$ with the braid axis

We now consider the 6_2^2 link, shown in figure 4. It is easily verified that the 6_2^2 link is given by the braid word $\sigma_1\sigma_2^{-1}$ as a closed braid with the braid axis. We use this fact to generate the universal cover, which we will show in the same notation we used for the figure-eight knot. We begin by building the fundamental domain, which is composed of the four tetrahedra in Figure 5.

Now, with the puncture of the braid axis sent to ∞, the part of the universal cover visible from the z-direction is shown in Figure 6. We generate our other picture by sending another vertex to ∞ and seeing which surfaces rotate around edge a and edge b, and display the result in Figure 7.

It can be easily verified that Figure 6 has period 1 in the north-south direction, and period $\sqrt{3}$ in the east-west direction, while Figure 7 has the same east-west period with a north-south period of 3. It should be noted that every vertex of every ideal tetrahedron in the fundamental domain has been sent to ∞ in exactly one of these two pictures. This means that any totally geodesic surface bounded or punctured by the 6_2^2 link must have in its universal-cover image at least one vertical Euclidean half-plane in *both* of the pictures shown above. The coordinate system works exactly as shown, just as it did in our diagram of the figure-eight complement.

As a necessary component to the proof of theorem 3.2, we will first prove the following:

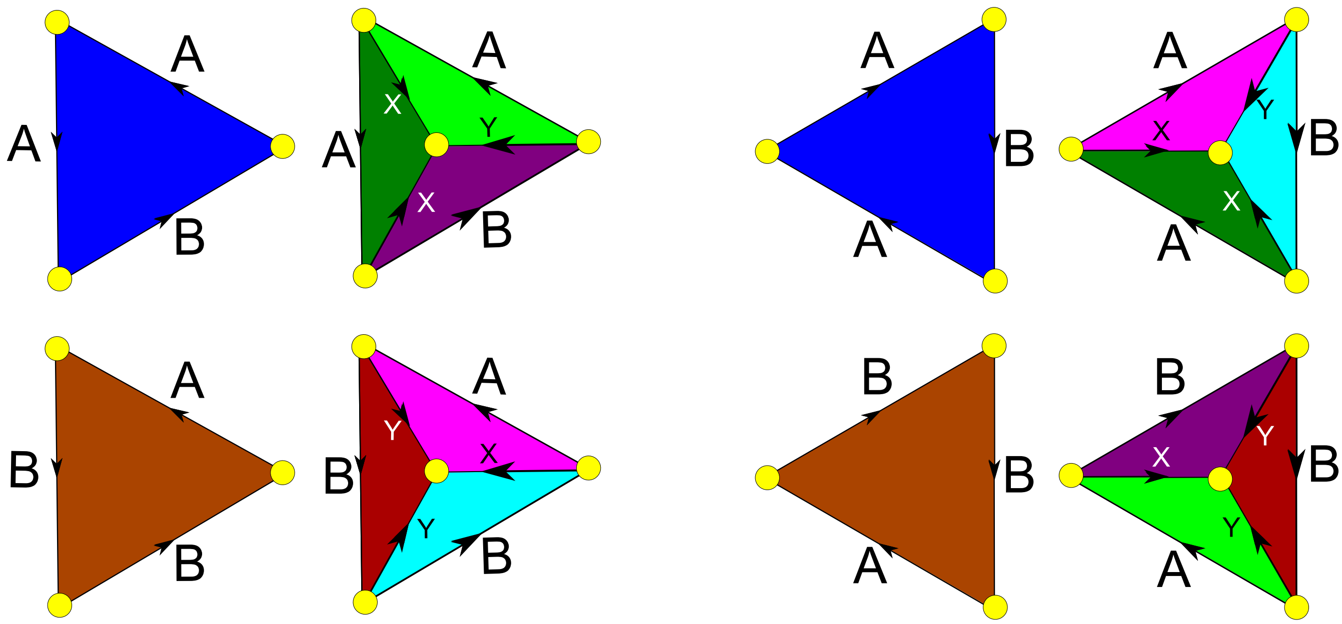

Figure 5: The four ideal tetrahedra that make up the 6_2^2 link complement

Theorem 3.1 *No totally geodesic surface bounded or punctured by the 6_2^2 link intersects a tetrahedral face in a hyperbolic line segment with endpoints on two different edges.*

Proof: Using contradiction, we begin by assuming there is such a surface. Call the surface T, and the intersected tetrahedral face F. It follows that $T \cap F$ must contain a hyperbolic line segment going with endpoints on two different edges of F. Call this line segment L_0.

The complement of the 6_2^2 link possesses symmetry between the dark blue and tan faces, and also between the dark green and burgundy faces. It further possesses symmetry between all four remaining faces (light blue, lime green, pink and purple). Therefore, we may assume without loss of generality that F is either tan, dark green, or purple. Consider 3 cases:

Case 1: F is the dark green face.

In this case, L_0 must either go from x to x, or from a to the x emerging from the tip of a, or from a to the x emerging from the tail of a. Consider three sub-cases:

Case 1a: L_0 goes from x to x.

In this case, L_0 is a lofted arc in each copy of the dark green face in Figure 6, which is a contradiction by Lemma 2.1 since its loft index must be at least 1, the north-south period of Figure 6.

Case 1b: L_0 goes from a to the x emerging from the tip of a.

In this case, L_0 appears as a lofted arc in Figure 7, along the copy of the dark green face whose vertices are $(0, -1, -1)$, $(1, -1, 0)$, and ∞. By Lemma 2.1, the loft index of L_0 must be 1, which requires the hyperbolic line containing L_0 to also contain a line segment from x to x in the copy of the dark green face whose vertices are $(-1, -1, -2)$, $(0, -1, -1)$, and ∞. Call this segment L_1. It follows that L_1 appears as a lofted arc in Figure 6, along any copy of the dark green face. Since Figure 6 has period 1 in one direction, we have once again reached a contradiction.

Case 1c: L_0 goes from a to the x emerging from the tail of a.

In this case, L_0 appears as a lofted arc in Figure 7, along the copy of the dark green face whose vertices are $(-1, -1, -2)$, $(0, -1, -1)$, and ∞. By Lemma 2.1, the loft index of L_0 must be 1, which requires the hyperbolic line containing L_0 to also contain a line segment from x to x in the copy of the dark green face whose vertices are $(0, -1, -1)$, $(1, -1, 0)$, and ∞. Call this segment L_1. It follows that L_1 appears as a lofted arc in Figure 6, along any copy of the dark green face. Since Figure 6 has period 1 in one direction, Lemma 2.1 again produces a contradiction.

It should be noted that the symmetries of the knot make Case 1 cover the green and burgundy faces.

Case 2: F is the purple face.

In this case, L_0 must either go from b to x, or from b to y, or from x to y. Consider three sub-cases:

Case 2a: L_0 goes from x to y.

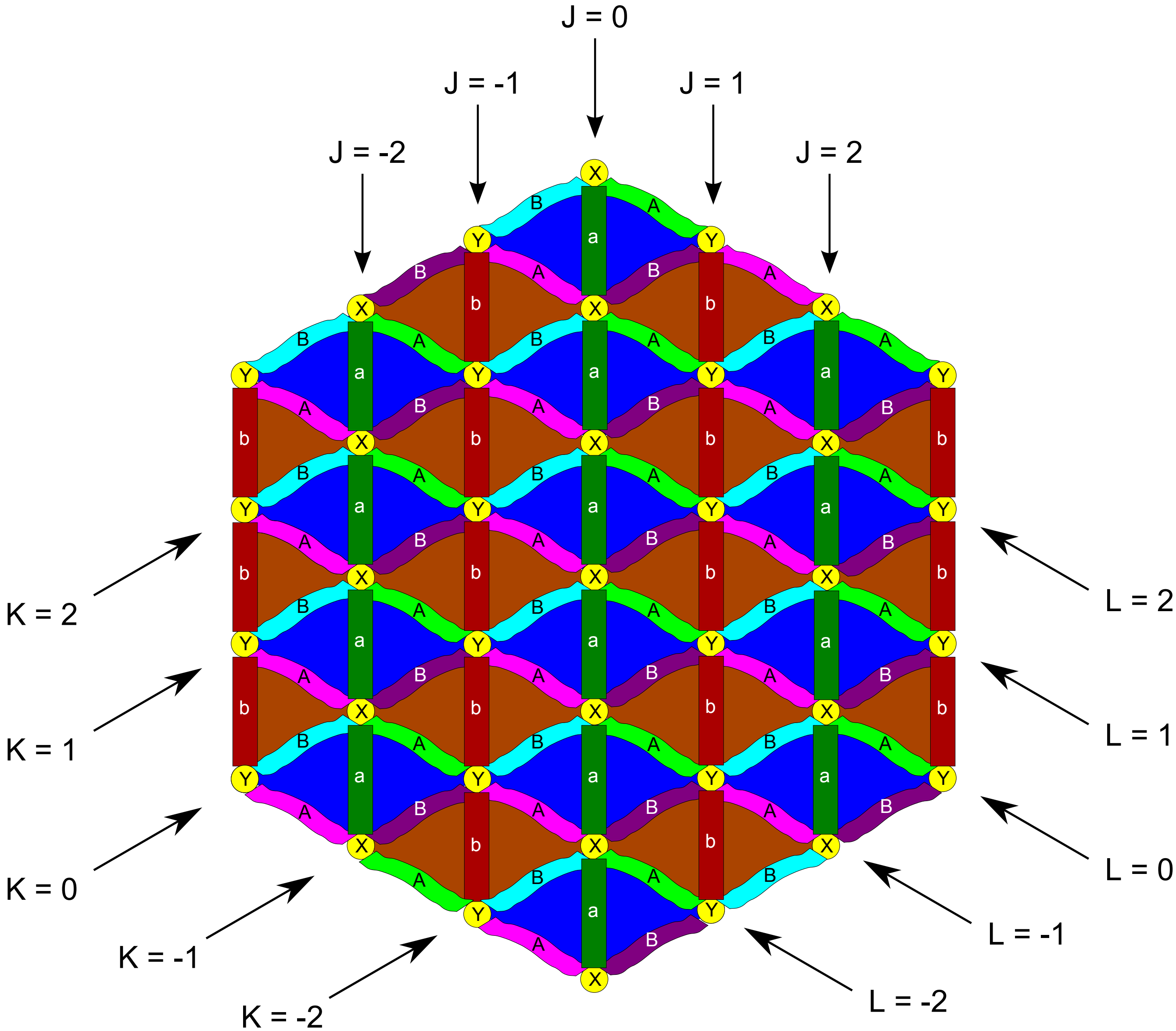

Figure 6: View of the 6^2_2 universal cover from the ideal vertex containing the braid axis

In this case, L_0 is a lofted arc in each copy of the purple face in Figure 6, which is a contradiction by Lemma 2.1 since its loft index must be at least 1, the north-south period of Figure 6.

Case 2b: L_0 goes from b to x.

In this case, L_0 appears as a lofted arc in Figure 7, along the copy of the purple face whose vertices are $(0, -1, -1)$, $(0, 0, 0)$, and ∞. By Lemma 2.1, the loft index of L_0 must be 1, which requires the hyperbolic line containing L_0 to also contain a line segment from x to y in the copy of the light blue face whose vertices are $(0, -2, -2)$, $(0, -1, -1)$, and ∞. Call this segment L_1. It follows that L_1 appears as a lofted arc in Figure 6, along any copy of the light blue face. Since Figure 6 has period 1 in one direction, Lemma 2.1 again produces a contradiction.

Case 2c: L_0 goes from b to y.

In this case, L_0 appears as a lofted arc in Figure 7, along the copy of the purple face whose vertices are $(-1, 1, 0)$, $(0, 1, 1)$, and ∞. By Lemma 2.1, the loft index of L_0 must be 1, which requires the hyperbolic line containing L_0 to also contain a line segment from x to y in the copy of the light blue face whose vertices are $(-2, 1, -1)$, $(-1, 1, 0)$, and ∞. Call this segment L_1. It follows that L_1 appears as a lofted arc in Figure 6, along any copy of the light blue face. Since Figure 6 has period 1 in one direction, Lemma 2.1 again produces a contradiction.

It should be noted that the symmetries of the knot make Case 2 cover the light blue, lime green faces.

Case 3: F is the tan face.

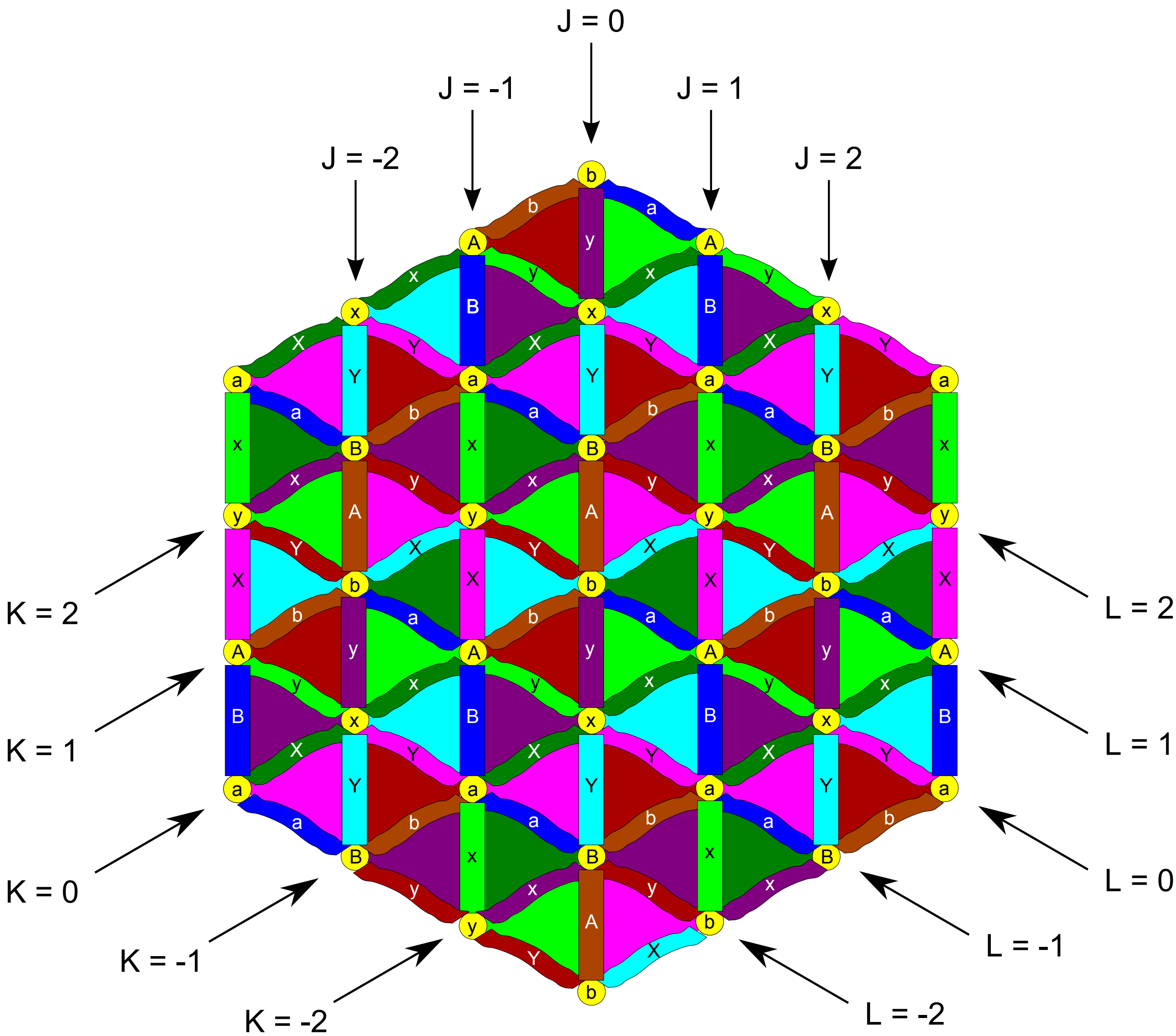

Figure 7: View of the 6_2^2 universal cover from any ideal vertex not containing the braid axis

In this case, L_0 must either go from b to b, or from a to the copy of b emerging from the tip of a, or from a to the copy of b aimed for the tail of a. Consider three sub-cases:

Case 3a: L_0 goes from b to b.

In this case, L_0 appears as a lofted arc in Figure 7, along the copy of the tan face whose vertices are $(0,1,1)$, $(0,0,0)$, and ∞. By Lemma 2.1, the loft index of L_0 must be 1, which requires the hyperbolic line containing L_0 to also contain a line segment from b to y in the copy of the purple face whose vertices are $(0,0,0)$, $(0,-1,-1)$, and ∞. But we have already established that no totally geodesic surface can intersect the purple face in any such arc, so we have again reached a contradiction.

Case 3b: L_0 goes from a to the copy of b emerging from the tip of a.

In this case, L_0 appears as a lofted arc in Figure 7, along the tan face whose vertices are $(0,1,1)$, $(0,0,0)$, and ∞. By Lemma 2.1, the loft index of L_0 must be 1, which requires the hyperbolic line containing L_0 to also contain a line segment from b to y in the copy of the light blue face whose vertices are $(0,0,0)$, $(1,0,1)$, and ∞. But we have already established that no totally geodesic surface can intersect the light blue face in any such arc, so we have again reached a contradiction.

Case 3c: L_0 goes from a to the copy of b aimed for the tail of a.

In this case, L_0 appears as a lofted arc in Figure 7, along the copy of the tan face whose vertices are $(1,1,2)$,

$(1,0,1)$, and ∞. By Lemma 2.1, the loft index of L_0 must be 1, which requires the hyperbolic line containing L_0 to also contain a line segment from b to x in the copy of the purple face whose vertices are $(-1,1,0)$, $(0,1,1)$, and ∞. But we have already established that no totally geodesic surface can intersect the purple face in any such arc, so we have again reached a contradiction.

Therefore there is no totally geodesic surface bounded or punctured by the 6_2^2 knot intersecting a tetrahedral face in a hyperbolic line segment with endpoints on two different edges. QED

Theorem 3.2 *There are no totally geodesic surfaces punctured or bounded by the* 6_2^2 *link.*

Proof: We will once again use contradiction. Assume there is a totally geodesic surface bounded or punctured by the 6_2^2 link. Call the lift of the surface to the universal cover T. Now T will be a disjoint union of hyperbolic planes, at least one of which will appear in Figure 7 as a vertical Euclidean half-plane. Call that plane T_0. By theorem [reference], T_0 will not intersect any horizontal face in a line segment from one of its edges to another, which forces it to either contain a face or bisect the dihedral angle between two faces. Consider two cases:

Case 1: T_0 contains a copy of a face.

Call the face F, and that particular copy F_0. It can be seen from Figure 7 that there are vertical copies of the dark blue face that directly intersect in an angle of $\frac{2\pi}{3}$, and the same is true of the tan face. Therefore, if T contains the dark blue face or the tan face, it will intersect itself. It is clear that F_0 is angled either due north v. south, or east-northeast v. west-southwest, or west-northwest v. east-southeast. Whichever direction F_0 is angled, T_0 is uniquely determined as the only vertical Euclidean half-plane containing it. Consider three sub-cases:

Case 1a: F_0 is angled due north-south (with constant J-coordinate).

In this case, T_0 will contain a copy of either the tan face (if the J-coordinate is even) or the dark blue face (if the J-coordinate is odd). In either case, T intersects itself, and we reach a contradiction.

Case 1b: F_0 is angled east-northeast v. west-southwest (with constant K-coordinate).

In this case, T_0 will contain a copy of the tan face, causing T to intersect itself, again a contradiction.

Case 1c: F_0 is angled west-northwest v. east-southeast (with constant L-coordinate).

In this case, T_0 will contain a copy of the dark blue face, causing T to intersect itself, again a contradiction.

Case 2: T_0 bisects the dihedral angle between two faces.

In this case, we note that some hyperbolic plane in T must appear as a vertical Euclidean half-plane in Figure 6. Call this plane T_1. Note that T_0 and T_1 are not necessarily distinct. Now T_0 must contain a vertical copy of an edge, so consider two sub-cases:

Case 2a: T_0 contains a vertical copy of either edge a or edge b.

The symmetry of the knot allows us to assume without loss of generality that T_0 contains a vertical copy of edge a. It can be seen from the picture that, regardless of which copy of edge a is contained in T_0, there will be a copy of either edge x or edge y intersecting T_0 in a point. Consider two sub-cases:

Case 2aa: T_0 is intersected by a copy of edge x.

In this case, T_1 can never be intersected by a copy of edge a, nor can it contain edge x. The only possible place for T_1 to live, therefore, is containing the burgundy face, which we know to be impossible.

Case 2ab: T_0 is intersected by a copy of edge y.

In this case, T_1 can never be intersected by a copy of edge a, nor can it contain edge y. The only possible place for T_1 to live (in Figure 6), therefore, is slicing directly from $(0,0,0)$ to $(2,-1,1)$. It follows that T intersects the tan face in a Euclidean arc from the ideal vertex at the *tip* of a to the opposite edge, and also in a Euclidean arc from the ideal vertex at the *tail* of a to its opposite edge. These two edges intersect on the tan face, forcing the surface T to intersect itself (since we already know it cannot contain the tan face).

Case 2a is now fully contradicted and thus impossible.

Case 2b: T_0 contains a vertical copy of either edge x or edge y.

In this case, it is automatically true that T contains one of the two edges, which means that T_1 also contains a copy of one of them (which must be vertical since there are no horizontal copies). When considering this case, we will consider the possible ways to angle T_1 in Figure 6. It has the same directional restrictions on it as T_0 has in Figure 7. The symmetries of the knot allow us to assume without loss of generality that T_1 contains a copy of edge x, and in fact that it contains the particular copy of edge x located at $(0,0,0)$ in Figure 6. Consider three sub-cases:

Case 2ba: T_1 angles due east-west.

In this case, the intersections of T_1 with the copies of the tan face lying between $(0,0,0)$ and $(2,-1,1)$ lead to the same contradiction found in Case 2ab.

Case 2bb: T_1 angles north-northwest v. south-southeast.

In this case, consider the intersections of T_1 with two particular copies of the tan face. One of these copies has as its vertices $(-2,3,1)$, $(-1,3,2)$, and $(-1,2,1)$, while the other's vertices are $(-1,2,1)$, $(-1,1,0)$, and $(0,1,1)$. The former copy of the tan face is intersected in the arc from the tail of a to the opposite edge of the tan face, while the latter copy is intersected in the arc from the tip of a to its opposite edge on the tan face, which once again leads to the same contradiction reached in Case 2ab.

Case 2bc: T_1 angles north-northeast v. south-southwest.

In this case, consider the intersections of T_1 with two particular copies of the dark blue face. One of these copies has as its vertices $(0,0,0)$, $(0,1,1)$, and $(1,0,1)$, while the other's vertices are $(0,0,0)$, $(0,-1,-1)$, and $(-1,0,-1)$. The former copy of the dark blue face is intersected in the arc from the tail of b to the opposite edge of the dark blue face, while the latter copy is intersected in the arc from the tip of b to its opposite edge on the dark blue face, which leads to a similar contradiction to the one used in Cases 2ab, 2ba, and 2bb.

Contradictions have now been reached in all cases. Therefore there are no totally geodesic surfaces bounded or punctured by the 6_2^2 link. QED

4 The checkerboard surface of the Borromean Rings

Another question worth considering is how totally geodesic surfaces behave under Dehn filling. To this end, we have found a totally geodesic surface present *after* Dehn filling, that was absent before Dehn filling.

The checkerboard surface of the Borromean Rings is known to be totally geodesic. The Borromean Rings can be represented by the closed braid $(\sigma_1\sigma_2^{-1})^3$. When the braid axis is added to this braid (and thus removed from the complement–the reverse of a Dehn filling), the universal cover of the resulting link can be displayed in a similar manner to the figure-eight knot or the 6_2^2 link, and the ideal tetrahedra are once again regular. Of the four faces in the Borromean Rings checkerboard surface, one of them is punctured by the braid axis. Any two of the three faces not punctured by the braid axis will meet at angles of $\frac{2\pi}{3}$, which means that the original Borromean Rings checkerboard surface is not totally geodesic, which provides a counterexample to the possible conjecture that Dehn fillings never create totally geodesic surfaces.

5 Open Questions

- Can Dehn fillings *destroy* totally geodesic surfaces while still leaving behind a hyperbolic knot or link? That is, are there totally geodesic surfaces in the complement of a hyperbolic knot or link L, that are no longer totally geodesic when some component is removed from L, even when the resulting knot/link is still hyperbolic?
- On the other hand, are there surfaces which *remain* totally geodesic through the process of Dehn filling?
- Are there more totally geodesic surfaces that have not been found?
- Which other hyperbolic knot complements are completely devoid of any totally geodesic surfaces?
  - The figure-eight knot and the 6_2^2 link both have as their continued fraction $n + \frac{1}{n}$, where $n = 2$ and $n = 3$ respectively. Can our results be generalized for all $n > 3$?
  - In the 5_2 hyperbolic knot, much symmetry is lost compared to the 6_2^2 link, even though the 5_2 knot has one less crossing. Can other methods be utilized to check the 5_2 knot for totally geodesic surfaces?
- What other restrictions on totally geodesic surfaces come from periodicity of universal covers?

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