On Relating Linking Number and Stick Number

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Abstract

This paper focuses on finding relations between linking number and stick number. We begin by examining three equivalent definitions of linking number. The equivalence of these definitions give us another way to interpret linking number for which we use to examine the linking number of two component stick links and stick numbers of two component links. We also discuss a couple of other ways to interpret the Gauss linking number integral using tools from vector calculus and possible directions to go for future study.

1 Preliminaries

Here are some terms and notation that will be used throughout this paper. Some basic topology, analysis, and manifold theory may be required later, but I will do my best to provide an intuitive description.

A mathematical *knot* K is a subset of $\mathbb{R}^3$ that is a simple closed curve. In formal terms, a knot K is a subset of $\mathbb{R}^3$ that is homeomorphic to the unit circle S^1. A *link* L with n components is n knots that may or may not pass through one another; if L is an n component link with $K_1, K_2, \ldots, K_n$ as the n components, we use the notation $L = (K_1, K_2, \ldots, K_n)$ to denote the link. By these definitions, we have that knots are links with one component, so later properties that are defined for general links will also be applicable for knots (note: if something is defined for a knot, it may not hold for a general link). When speaking about particular links, we use the traditional Alexander-Briggs notation which organizes links by their crossing number and the number of components

Since all links with n components are homeomorphic to a n copies of S^1, what distinguishes links from each other is how they sit in space. Two links may look different at first glance, but if one can play around with one link to produce the second link *without any cutting and re-gluing* (and vice-versa), then we will want to say that these two links are the "same"; they sit in space the same but just have a different physical appearance. We can talk about how two links are the "same" by the following definition: Given two links L_1 and L_2, we say L_1 and L_2 are *equivalent* links if L_1 can be continuously deformed into L_2 without any breaking or re-gluing of any components during the deformation; more formally, we say L_1 and L_2 are *equivalent*, denoted by $L_1 \sim L_2$, if there exists a bi-continuous map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi(L_1) = L_2$ and $\varphi(\mathbb{R}^3 \setminus L_1) = \mathbb{R}^3 \setminus L_2$; the map φ is sometimes called an *ambient isotopy*. With some work, one can show that the relation $\sim$ is in fact

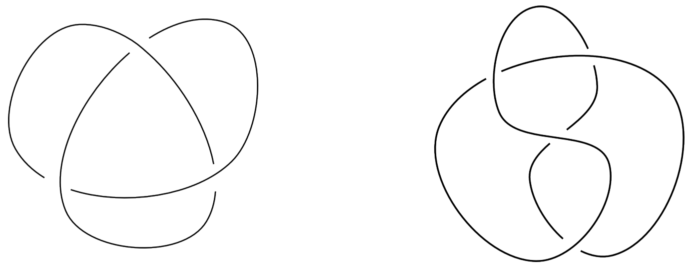


Figure 1: These two knots belong in the same link type.

an equivalence relation on links (the composition of ambient isotopies and the inverse of an ambient isotopy are again ambient isotopies). Hence, $\sim$ partitions the set of all links into equivalence classes; we say an equivalence class of links under $\sim$ is a *link type*. For example, the two knots in Figure 1 belong to the same link type.

Sometimes it is easier to work with link projections (i.e.: it is easier for us to draw a link on a board, compared to working with a link in space). Given a link L, we can analyze L on a plane by projecting L orthogonally to some vector $v \in \mathbb{R}^3$; this gives us a *planar diagram* of L which we denote by $D_v(L)$ (when the context is clear, we will use just D) with over and under crossings. A theorem of Reidemeister says that says that two links L_1 and L_2 are equivalent if and only if a finite sequence of the three Reidemeister moves can take any diagram D_1 of L_1 to any diagram of D_2 of L_2 (see Figure 2). Can you find a sequence of Reidemeister moves that takes the left diagram to the right diagram in Figure 1?

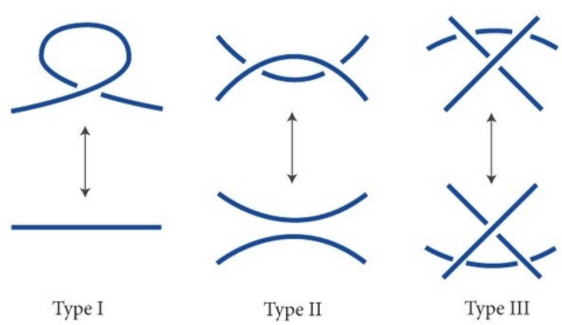


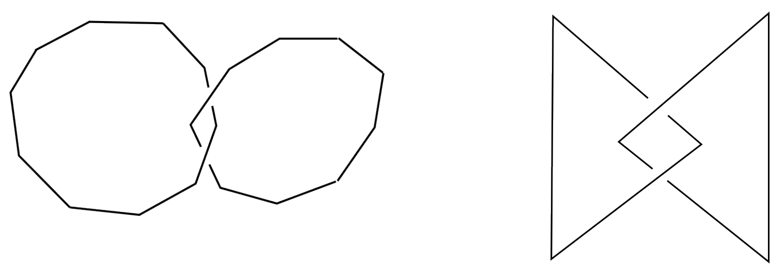
Figure 2: The three Reidemeister moves (Source: Quora)

Classifying link types using the "equivalent" definition above is not always an easy process. By definition, if we could continuously deform (without any breaking) one link into another, then we can conclude that the two links are equivalent; the difficulty in this process is finding an ambient isotopy (i.e., maneuvering one link into the other). On the other hand, if we did not succeed in continuously deforming one link into another, we *cannot* conclude anything about the two links; we would have to prove there is no such ambient isotopy (which is hard). To help us, we consider properties of links called link invariants. A *link invariant* is some property that can be represented as a function on link types; the property or function value of an invariant remains unchanged under ambient isotopy of the link. The invariant can be a number (e.g., linking number), an indicator (e.g., tricolorability), or some object (e.g., Jones polynomial). A *knot invariant* is a link invariant that only holds for knots, but not necessarily for general links. Equivalent links have the same link invariants, but the converse is *not* true: two links having the same invariants does not imply the links are equivalent. Thus, one way to prove a defined property of links is in fact a link invariant is to show the property remains unchanged under the three Reidemeister moves. If we had a property of links such that two links are equivalent if and only if the property held for both links, we say such a property is a *complete invariant*.

2 Stick Representation and Stick Number

A *polygonal link* or *stick link* is a piecewise linear link, that is, a link constructed using line segments or "sticks"; we say a *polygonal representation* or *stick representation* of a link L is a stick link that is equivalent to L. We say a *piecewise linear polygonal link* or "*stick link*" of m sticks is a stick representation of some link constructed from m sticks.

A natural question with stick links is, how many sticks does one require to construct a stick representation of a link? Clearly, if one has enough sticks, any stick representation of any link could be constructed (just take a polygonal approximation of each component). But one may not always have an infinite supply of sticks at their disposal so we want to be able to find a stick representation using the fewest number of sticks. As an example, the Hopf link could be constructed with an m-gon and an n-gon as in Figure 3a, having each component pass through the other component once. But we see that two triangles also constructs a stick representation of Hopf link but also with the *least* number of sticks as in Figure 3b; such a stick representation using the fewest number of sticks is defined to be a *minimal stick representation*.



(a) (b)

Figure 3: One could construct a stick representation of the Hopf link with an m-gon and an n-gon (a), but a minimal stick representation only requires two triangles (b).

One could now rephrase our initial question to: what is the least number of sticks needed to construct a stick representation of a link? (Equivalently, how many sticks are needed to construct a minimal stick representation?) We say the *stick number* $s(L)$ is the fewest number of sticks needed to construct a stick representation of L[1]. As we just observed, $s(\text{Hopf Link}) = 6$.

[1] One could add the requirement that the stick representation uses equal length sticks or a stick representation that lives within a cubic lattice, or perhaps both; for our purposes, we will not consider these extra requirements.

It is clear that we will need at least 3 sticks to create the unknot and furthermore, 6 sticks to create the unlink. With a little work, we see that we must require at least 6 sticks to create a nontrivial knot, i.e. the trefoil (see Figure 4). Can you see why four or five sticks is not enough?

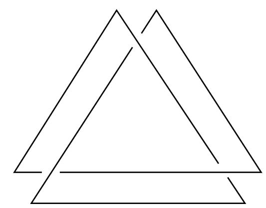


Figure 4

The stick number of some knots and links are known. For links of two components with 6 crossings or less are known (from the work of [2], [5]): $s(0_1^2) = 6$, $s(2_1^2) = 6$, $s(4_1^2) = 7$, $s(5_1^2) = 8$, and $s(6_1^2) = s(6_2^2) = s(6_3^2) = 8$.

The torus links are a class of links that can be embedded (with no self intersections) onto the 2-torus (compact orientable surface of genus 1); the torus link $T_{m,n}$ link goes through the hole of the torus m times and makes q revolutions; the $\gcd(m, n)$ is the number of components of $T_{m,n}$; the trefoil is $T_{2,3}$ and the Soloman's knot 4_1^2 is $T_{2,4}$. A well known result is $s(T_{m,m-1}) = 2m$ and can be extended to $s(T_{m,n}) = 2n$ when $2 \leq m < n < 2m$, (from [2], [4]). Finally, it has been shown that $s(T_{n,n}) = 3n$, $s(T_{n,2n}) = 4n - 1$, $s(T_{n,3n}) = 4n$ and $s(T_{n,4n}) = 5n$ (from [4], [9]). In general, the stick number of $T_{2,2n}$ is not known; an open conjecture of [9] states that $s(T_{2,2n}) = \lfloor \frac{4}{3}n + \frac{14}{3} \rfloor$.

3 Linking Number

In this section, we will discuss linking number, a link invariant for *oriented, two component* links. Given a link, we assign an orientation by choosing a direction of travel along each component. The *linking number* of an oriented, two component link roughly describes how linked the two components are; one can think of linking number as how many times one component encircles the second component in the same direction. There are several definitions of linking number (we will discuss three below) and one can show that they are all the same up to a sign and exactly the same when orientation is taken into consideration [8]. Linking number is a homotopy invariant among oriented, two component links; this means that linking number does not change under ambient isotopy of the link.

3.1 Signs of Crossings

We can compute linking number from the following algorithm (this is usually the standard definition of linking number): Given an oriented link L, we first take a projection of L onto a plane, creating an oriented link diagram which we call D_L. Let C_{D_L} denote the set of crossings between components of D_L; for each crossing $c \in C_{D_L}$, we assign the number $\varepsilon(c) = \pm 1$ where the sign is based on the standard convention: $+1$ for a right handed crossing and -1 for a left handed crossing as shown in Figure 5. The linking number of L is half the sum of these numbers:

$$lk(L) := \frac{1}{2} \sum_{c \in C_D} \varepsilon(c)$$

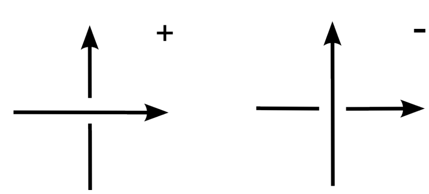


Figure 5: A right handed crossing and a left handed crossing chosen by the right handers.

One can show (by checking the three Reidemeister moves) that this definition of linking number is a link invariant among oriented link diagrams of oriented links with two components.

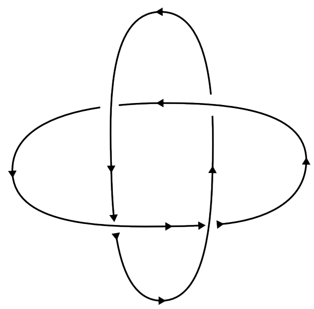


Figure 6: The 4^2_1 link with an orientation chosen by a left hander.

As an example we compute the linking number of the link 4^2_1 (Soloman's knot); Figure 6) shows an oriented diagram of 4^2_1. We see that there are four crossing between the two components; by inspection, all the crossing are negative, and so

$$lk(4^2_1) = \frac{(-1)+(-1)+(-1)+(-1)}{2} = -2$$

3.2 Gauss' Integral

The first definition of linking number we describe is via a double line integral. This is the first (known) recorded definition of linking number found in Gauss' work. It is believed that Gauss derived this formula while studying magnetism [7].

3.2.1 Definition. *Let $L = (K_1, K_2)$ be a link of two components with an assigned orientation. If $r_1 : I_1 \to \mathbb{R}^3$ and $r_2 : I_2 \to \mathbb{R}^3$ are smooth (orientation preserving) parameterizations of K_1 and K_2 respectively, then the linking number lk of L is given by:*

$$lk(L) := \frac{1}{4\pi} \int_{I_1} \int_{I_2} \frac{(r_2(t_2) - r_1(t_1)) \cdot (r_1'(t_1) \times r_2'(t_2))}{|r_2(t_2) - r_1(t_1)|^3}\, dt_2 dt_1$$

Note that the integrand is well defined since K_1 and K_2 do not intersect each other[2]. While computing linking number may be complicated or impossible using this formula, the physical interpretations of this integral and our study of stick links will give us fruitful results. One can derive this integral by interpreting the integral as the total sum of the magnetic field induced by a steady line current along K_2 while traversing along K_1 and using Biot and Savart's Law and Ampère's law from electricity and magnetism. We will come to this interpretation of the linking number integral in a later section to apply vector calculus tools.

[2]Some papers (as in [3]) define this integral using $r_1(t_1) - r_2(t_2)$ instead of $r_2(t_2) - r_1(t_1)$ in the integrand; we choose $r_2(t_2) - r_1(t_1)$ and hopefully reduce the number of minus signs in the later sections.

3.3 Degree of the Gauss Map

Let M and N be smooth manifolds of dimension m and n respectively. If $f : M \to N$ is a smooth map, the degree of f can be thought as an integer number of how many times f "wraps" M around N. As an example if $M = N = S^1$ where S^1 is unit circle, then the degree of $f : M \to N$ would be the winding number. On a first pass, the reader may choose to take the Lemmas without proof as some manifold theory is required.

- Let $f : M \to N$ be a smooth map and let $Df(p) : T_pM \to T_{f(p)}N$ denote the Jacobian matrix (or total differential) of f at $p \in M$. If $m \geq n$ and $Df(p)$ has maximal rank (i.e. $\dim(\text{im} f) = n$) for a point $p \in M$, then we say f is a *submersion* at p.

- If a smooth map $f : M \to N$ is a submersion at $p \in M$, then we say p is a *regular point* of f. We say a point $q \in N$ is a *regular value* of f if every point $p \in f^{-1}(q)$ is regular.

Here is a very useful theorem, which colloquially is known as the 'stack of records theorem' [6]. We refer the reader to [6] for the proof.

3.3.1 Theorem. *Let $f : M \to N$ be a smooth map between manifolds of the same dimension; assume further that M is compact. Then if $q \in N$ is a regular value of f, then the inverse image of q under f is finite. Moreover if $f^{-1}(q) = \{p_1, p_2, \ldots, p_r\}$ where $p_i \in M$ and r is a nonnegative integer, then for each p_i there exists a neighborhood V of q in N such that $f^{-1}(V)$ is the disjoint union of neighborhoods U_i of p_i in M, each of which are diffeomorphic to V. In symbols, we have that*

$$f^{-1}(V) = \bigcup_{i=1}^{r} U_i$$

with $f|_{U_i}$ a diffeomorphism for each i and $U_i \cap U_j = \emptyset$ if $i \neq j$.

One can think of this theorem as saying that given a smooth map $f : M \to N$ with M, compact a (regular) value on N can be covered a finite number of times from some points on M; each of these coverings are local diffeomorphisms which can preserve or reverse orientation. The degree of f, as we will see is the sum of the signs of these orientations; moreover, one can show the choice of regular value does not affect the degree of f!

Now say that $f : M \to N$ is a smooth map between compact, connected, oriented manifolds without boundary and of the same dimension. If $q \in N$ is a regular value of f, by the 'stacks of records' theorem we have that $f^{-1}(q)$ is finite; say, $f^{-1}(q) = \{p_1, \ldots, p_r\}$ where $p_i \in M$. We define the *index* of p_i to be

$$I(f, p_i) := \text{sgn}(\det Df(p_i))$$

We define the *degree* of f to be the sum of the indices, that is

$$\deg f := \sum_{i=1}^{r} I(f : p_i)$$

As mentioned, one can show that the degree of f does not depend on the choice of regular value $q \in N$ and also is a homotopy invariant. Now we are ready to state the second definition:

Let $L = (K_1, K_2)$ be a link of two components and let $r_1 : I_1 \to \mathbb{R}^3$ and $r_2 : I_2 \to \mathbb{R}^3$ are smooth (orientation preserving) parameterizations of K_1 and K_2 respectively. Let S^2 denote the unit sphere. For each $(p_1, p_2) \in (K_1, K_2)$ there is a point $(t_1, t_2) \in I_1 \times I_2$ (where the parametrized end points share the end points of $I_1 \times I_2$); define the map $\mathbf{n} : I_1 \times I_2 \to S^2$ by

$$\mathbf{n}(t_1, t_2) := \frac{r_1(t_1) - r_2(t_2)}{|r_1(t_1) - r_2(t_2)|}$$

3.3.2 Lemma. *With the notation from above, the map* $\mathbf{n}$ *is smooth.*

Proof. Since K_1 and K_2 do not intersect, we have that $|r_1(t_1) - r_2(t_2)| \neq 0$ for all $(t_1, t_2) \in I_1 \times I_2$; since r_1 and r_2 are smooth parameterizations and the map $v \mapsto \frac{v}{|v|}$ is smooth for nonzero $v \in \mathbb{R}^3$, we have the desired result since the composition of smooth maps is again smooth. □

3.3.3 Definition. *The linking number of L is the degree of* $\mathbf{n}$*, that is $lk(L) = \deg(\psi)$.*

The map $\mathbf{n}$ is sometimes called the *Gauss map.* One way to interpret this definition is to think of linking number as the number of signed coverings of the unit sphere S^2 by ψ. More specifically, the Gauss map ψ takes two points from two circles[3] $C_1 \times C_2$ to a vector on the unit sphere; if we vary t_1 and t_2 about their respective intervals, then ψ traces out regions on the unit sphere. We can look at regions for which a sign (assigned based on the orientation of the curves K_1 and K_2) is consistent; then adding up these regions (while keeping track of sign) gives us the signed coverings of the unit sphere. We will show later using Gauss's integral that the degree of ψ is in fact the number of these coverings (maybe up to a sign).

4 Relating Linking Number and Stick Number

We now look at some relations that have been found for stick links.

4.1 Links of Two Components with a Triangle

We begin with a simple result.

4.1.1 Proposition. *If $L = (T, K)$ is a two component link where T is a triangle and the linking number $lk(L) = n$, then $3 + 2n \leq s(L) \leq 3 + 2n + 1$.*

Proof. The upper bound $3 + 2n + 1$ comes from the sum of: the number of sticks of the triangle (3), the number of sticks that passes through the triangle in the same direction (n, since n is the linking number of L), and the number sticks to connect everything up ($n+1$). Figure 7 shows an example with linking number 6.

For the lower bound, we know that $3+n < s(L)$ because we require 3 sticks for the triangle component and we require n sticks that pass through the triangle since linking number of

[3]Technically, we define an equivalence relation $\sim_i$ that identifies the end points of the interval I_i to be the same (this is okay since if $I_i = [a, b]$, then $r_i(a) = r_i(b)$) and so the quotient space $C_i := I_i / \sim_i$ is the circle we speak about.

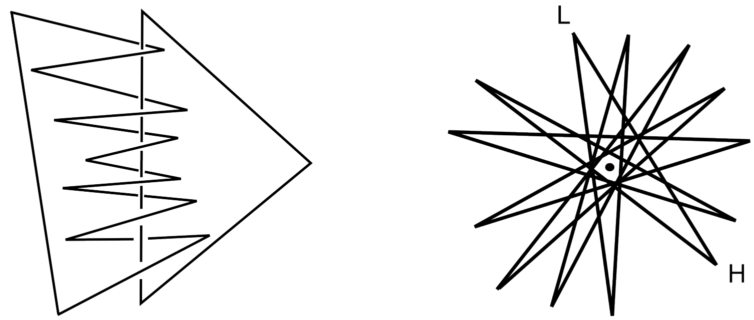


Figure 7: On the left, we have a link with linking number 6; we see that $3+2\cdot 6+1$ sticks required. On the right we see a possible top view of the same link

L is n; it's a strict inequality $3+n$ sticks does not give us a link with two components. If we were to connect up the n sticks in the most minimal way, then we would need n more sticks to have the first n sticks all oriented in the same direction of travel; thus, $3+2n$ is the minimal number of sticks required. However, it is possible that this not geometrically realizable, which is why an extra stick may be required. $\square$

For $n=1$, we see that only $6=3+2\cdot 1+1$ sticks are required: 3 for the triangle, 1 from the linking number, and 2 more to actually create a second component. Here this satisfies the upper bound. For the case $n=2$, we see that only $7=3+2\cdot 2$ sticks are required: 3 for the triangle, 2 from the linking number, and 2 more to connect everything up; here this actually satisfies the lower bound. Figure 8 shows these two cases.

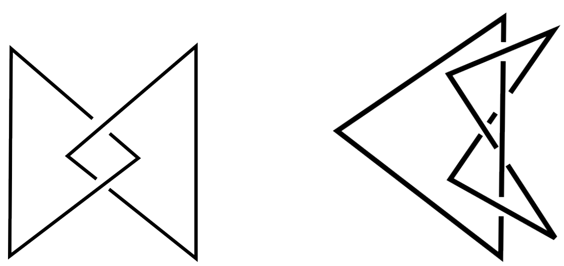


Figure 8

As an open question, one can consider the same proposition but for arbitrary n-gons; for $n\geq 4$, the n-gon is no longer planar, giving a more degrees of freedom. We can also consider wrapping the sticks around all the sides, which could give us a better lower bound.

4.2 Linking Number as Signed Coverings of S^2

As stated before, all definitions (in particular, the three listed in this paper) of linking number are equivalent up to a sign and exactly the same when orientation is considered [8]. The proof of this statement for the three definitions given above requires some work and we refer the reader to [7]. Using the equivalences of these definitions and putting together key parts of [7] will give us another interpretation of linking number. In particular, we can view linking number as the number of signed coverings of the unit sphere by the Gauss map $\mathbf{n}$, which will the main result we describe in this section. First, we have some lemmas:

4.2.1 Lemma. *Let $r_1 : I_1 \to \mathbb{R}^3$ and $r_2 : I_2 \to \mathbb{R}^3$ be simple closed curves. Let*

$$\mathbf{n}(t_1, t_2) = \frac{r_1(t_1) - r_2(t_2)}{|r_1(t_1) - r_2(t_2)|}$$

be the Gauss map. Then

$$\frac{(r_2(t_2) - r_1(t_1)) \cdot (r_1'(t_1) \times r_2'(t_2))}{|r_2(t_2) - r_1(t_1)|^3} = \mathbf{n}(t_1, t_2) \cdot (\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2))$$

where $\mathbf{n}_{t_i}(t_1, t_2)$ is the partial derivative of $\mathbf{n}(t_1, t_2)$ with respect to t_i.

Proof. This is a tedious computation and is left out. □

Using Lemma 4.2.1, we can relate the integrand in the Gauss linking integral to the degree of the Gauss map. Let $L = (K_1, K_2)$ be a link with two components, $r_i : I_i \to \mathbb{R}^3$ the parameterization of K_i, and the Gauss map $\mathbf{n}$ as defined before. We let R denote a region of $\text{im}(\mathbf{n})$ where the orientation of $\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2))$ at $\mathbf{n}(t_1, t_2)$ is consistent for each $\mathbf{n}(t_1, t_2) \in R$ (either pointing out of S^2 or pointing into S^2); we will want to take R to be the maximal possible region where orientation remains consistent. If $\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)$ at $\mathbf{n}(t_1, t_2)$ points out of S^2 for each (t_1, t_2), say R is a positive region and if $\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)$ points into S^2, say R is a negative region. We can compute the surface area of these regions using the surface integrals:

$$Area(R_\pm) = \pm \iint_{R_\pm} \mathbf{n}(t_1, t_2) \cdot (\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)) \tag{1}$$

where the sign is positive for positive regions R_+ and negative for negative regions R_-.Let $\mathcal{P}$ denote the set of all positive regions of $\text{im}(\mathbf{n})$ and $\mathcal{N}$ the set of all negative regions. We denote the *signed area* of $\text{im}(\mathbf{n})$ by $\mathcal{A} := \big[\text{Area}(\mathcal{P}) - \text{Area}(\mathcal{N})\big]$; if $\mathcal{A}$ is a multiple of 4π, then $\mathcal{A}$ is equal the area of $\mathcal{A}/4\pi$ unit spheres; in this case, we say the number of *signed coverings* of S^2 is $\mathcal{A}/4\pi$

4.2.2 Lemma. *The signed area of image of the Gauss map $\mathcal{A}$ is a multiple of 4π.*

Proof. By Lemma 3.3.2, the Gauss map $\mathbf{n}$ is a continuous map from $I_1 \times I_2$ to S^2; since $I_1 \times I_2$ is a compact set, and the continuous image of a compact set is again compact, we have that $\text{im}(\mathbf{n}) = \mathbf{n}(I_1 \times I_2)$ is a compact set as well. Furthermore, since $\mathbf{n}(I_1 \times I_2) \subseteq S^2$, this implies that $\mathbf{n}(I_1 \times I_2)$ is a closed surface (has no boundary) and so the area of $\mathbf{n}(I_1 \times I_2)$ must be some multiple of 4π, which is the signed surface area of the S^2. (If $\mathbf{n}(I_1 \times I_2)$ were not a closed surface, then $\mathbf{n}(I_1 \times I_2)$ would be homeomorphic to $\mathbb{R}^2$ which contradicts $\mathbf{n}(I_1 \times I_2)$ being compact). □

4.2.3 Lemma. $\mathbf{n}(t_1, t_2) \cdot \big(\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)\big) = \pm|\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)|$

Proof. By the definition of the dot product, we have

$$\mathbf{n}(t_1,t_2)\cdot\big(\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)\big) = |\mathbf{n}(t_1,t_2)||\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|\cos\theta$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{n}_{t_1}\times\mathbf{n}_{t_2}$. Now by definition $|\mathbf{n}(t_1,t_2)|=1$ so the latter of the above equation equals

$$= |\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|\cos\theta$$

For the positive regions, i.e. when $\mathbf{n}_{t_1}(t_1,t_2)$ and $\mathbf{n}_{t_2}(t_1,t_2)$ are (anti) parallel, we have $\theta=0$ ($\theta=\pi$), which gives us $+$ $(-)$ sign and for the negative regions, i.e. when $\mathbf{n}_{t_1}(t_1,t_2)$ and $\mathbf{n}_{t_2}(t_1,t_2)$ are anti parallel, we have $\theta=\pi$, which gives us $-$ sign. Then continuing from above we have

$$= \pm|\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|$$

and we are done. $\square$

Using these three lemmas, we come to the interpretation of linking number:

4.2.4 Proposition. *With the above notation, the degree of* $\mathbf{n}$ *is given by* $\deg\mathbf{n} = \frac{1}{4\pi}\big[Area(\mathcal{P}) - Area(\mathcal{N})\big] = \mathcal{A}/4\pi$ *where* $\mathcal{A}$ *is the signed area of the image of* $\mathbf{n}$*. Moreover, linking number is given by* $\mathcal{A}/4\pi$*, which is exactly the number of signed coverings of* S^2 *by* $\mathbf{n}$*.*

Proof. For each $R_+\in\mathcal{P}$, the signed surface area of R_+ is given by

$$\iint_{D_{R_+}} |\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|\ dt_1dt_2$$

where $D_{R_+}\subseteq I_1\times I_2$ and $\mathbf{n}(D_{R_+})=R_+$, and for each $R_-\in\mathcal{N}$, the signed surface area of R_- is

$$\iint_{D_{R_-}} -|\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|\ dt_1dt_2$$

where $D_{R_-}\subseteq I_1\times I_2$ and $\mathbf{n}(D_{R_-})=R_-$. Then we see that the total signed area is given by:

$$\begin{aligned}
\mathcal{A} &= \big[Area(\mathcal{P}) - Area(\mathcal{N})\big]\\
&= \sum_{R_+\in\mathcal{P}}\iint_{D_{R_+}} |\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|\ dt_1dt_2 + \sum_{R_-\in\mathcal{N}}\iint_{D_{R_-}} -|\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|\ dt_1dt_2\\
&= \iint_{I_1\times I_2} \pm|\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)|\ dt_1dt_2
\end{aligned}$$

where we choose the appropriate sign for each region; by Lemma 4.2.3, we have

$$= \iint_{I_1\times I_2} \mathbf{n}(t_1,t_2)\cdot\big(\mathbf{n}_{t_1}(t_1,t_2)\times\mathbf{n}_{t_2}(t_1,t_2)\big)\ dt_1dt_2$$

and the latter is equaled to

$$= 4\pi\deg\mathbf{n}$$

where the last equality comes from a proof (done in [7], Proposition 5.6). Using Lemma 4.2.1, the integrand of the last integral becomes the Gauss linking number integral; we then have that $lk(L)=\mathcal{A}/4\pi$, which is the number of signed coverings. $\square$

4.3 Upper Bound on Linking Number

Using the interpretation of linking number as the number of signed coverings of S^2, we are able to find an upper bound on linking number of a stick link. This gives us an interesting result when we appeal to stick number as we will see.

When we consider stick links of two components, we assume that the stick $L = (K_1, K_2)$ has no two parallel edges from different components; to ensure that, we assume that no vertices $v_i, v_{i+1} \in K_1$ and $w_j, w_{j+1} \in K_2$ are coplanar (if that is the case, then we "nudge" one of the vertices a bit and make one segment slightly longer). Now we can look at the image of two edges from different components $e_i \in K_1$ and $e_j \in K_2$ under the Gauss map $\mathbf{n}$. We first consider difference vectors between the edges' endpoint vertices as shown in Figure 9. Normalizing these vectors creates four vertices on the unit sphere; varying the parameters, we see the difference vectors between component edges sweep out a spherical quadrilateral as shown in Figure 10 (if we had coplanar vertices, we would have arcs on S^2 as the image of two edges). When all the edges are considered, we will have an integer

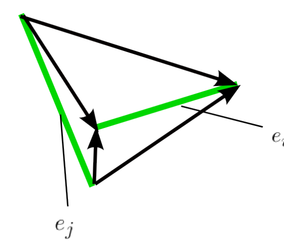


Figure 9

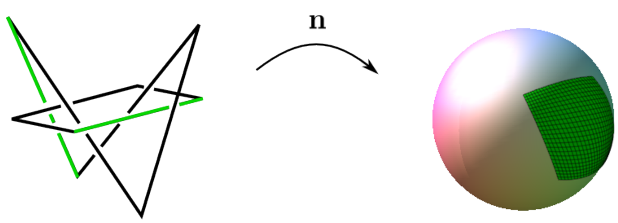


Figure 10: The Gauss map $\mathbf{n}$ takes two edges from different components of a link to a spherical quadrilateral on S^2.

number of coverings of S^2 with various spherical quadrilaterals. Below (Figure 11 we show the stick Hopf link and its coverings on the unit sphere[4].

In Figure 11, we have that the red-yellow-blue component is K_1, the black component is K_2; on K_2, let the top slant edge be e_1, the bottom slant edge e_2, and the vertical edge be e_3. Then starting from the left picture of the bottom row, we have the image of the Gauss map contribution from K_1 and e_1, from K_1 and e_2, and from K_1 and e_3 (the spheres have been rotated 180 degrees along the z-axis). The far right image shows the all the coverings; when sign is taken into consideration, we end up with one covering[5] of S^2.

These spherical quadrilaterals give us our upper bound result:

4.3.1 Proposition. *If $L = (K_1, K_2)$ is a two component stick link where K_1 is a stick knot with m sticks and K_2 is a stick knot n sticks, then $lk(L) < \frac{mn}{2}$.*

[4]These images were produced on Maple by a program written by Dr. Trapp.

[5]The sign of this covering is positive, but the Hopf link in Figure 11 is a negative Hopf link by the conventions in Figure 5. Our choice of $r_2 - r_1$ in Definition 3.2.1 may have caused this sign change.

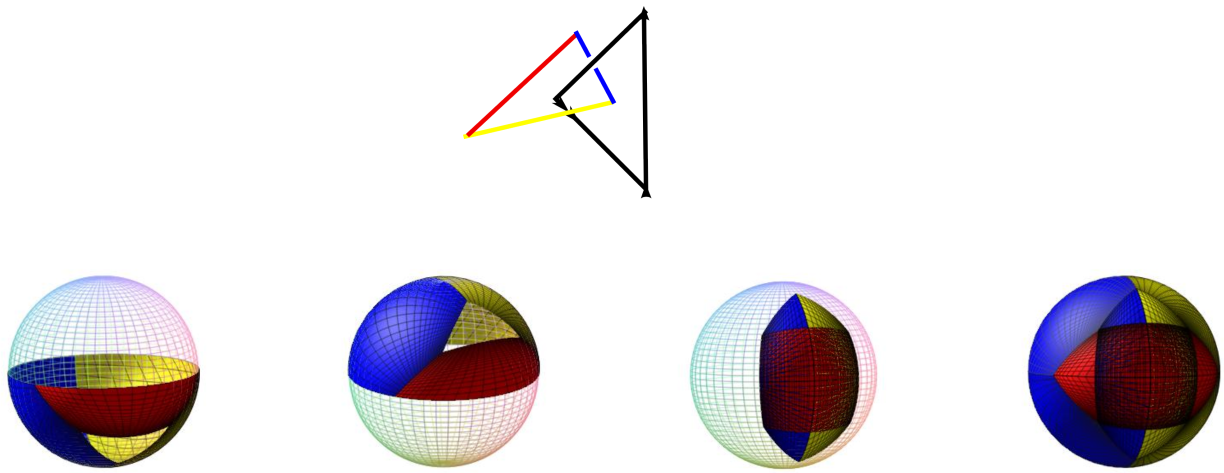


Figure 11: The coverings of S^2 of the stick Hopf link.

Proof. We have that K_1 is a polygonal knot with m sides and so we can parametrize K_1 by parametrizing each of the line segments and varying the argument of the parameter for each segment so that the overall parameterization r_1 is within an interval of $[0, m]$. Similarly, we can parametrize K_2 with r_2 so that the parameter is within $[0, n]$. Here is the construction: Let $v_0 = v_m, v_1, \ldots, v_{m-1}$ denote the vertices of K_1 with v_0 starting at any vertex, v_1 to either vertices on the side of v_0, v_2 to the vertex on the side of v_1 that is not v_0, and so on. We can parametrize each of the m line segments between adjacent vertices as follows: given v_{i-1} and v_i, let the edge between them be the i-th edge which we denote e_i; we can parametrize e_i by $r_1^i(t_1) : [i-1, i] \to \mathbb{R}^3$ where we have

$$t_1 \mapsto (v_i - v_{i-1})(t_1 - (i-1)) + v_{i-1}$$

Then taking all of the r_1^i's together, we have:

$$r_1(t_1) = \begin{cases} r_1^1(t_1) & t_1 \in [0,1] \\ r_1^2(t_1) & t_1 \in [1,2] \\ \vdots & \\ r_1^m(t_1) & t_1 \in [m-1, m] \end{cases}$$

parametrizes K_1 for $t_1 \in [0, m]$.

By our first definition of linking number, we have that

$$lk(L) = \frac{1}{4\pi} \int_0^n \int_0^m \frac{(r_2 - r_1) \cdot (r_1' \times r_2')}{|r_2 - r_1|^3} \, dt_1 dt_2$$

Substituting in our polygonal parametrizations, we get

$$= \frac{1}{4\pi} \int_0^n \sum_{i=0}^m \int_{i-1}^i \frac{(r_2 - r_1) \cdot (r_1' \times r_2')}{|r_2 - r_1|^3} \, dt_1 dt_2$$
$$= \frac{1}{4\pi} \sum_{j=1}^n \int_{j-1}^j \sum_{i=0}^m \int_{i-1}^i \frac{(r_2 - r_1) \cdot (r_1' \times r_2')}{|r_2 - r_1|^3} \, dt_1 dt_2$$

Everything is finite here so by linearity of the integral, this can be rewritten as

$$= \frac{1}{4\pi} \sum_{j=1}^n \sum_{i=1}^m \int_{j-1}^j \int_{i-1}^i \frac{(r_2 - r_1) \cdot (r_1' \times r_2')}{|r_2 - r_1|^3} \, dt_1 dt_2$$

By Lemma 4.2.1, we have that

$$= \frac{1}{4\pi} \sum_{j=1}^n \sum_{i=1}^m \int_{j-1}^j \int_{i-1}^i \mathbf{n}(t_1, t_2) \cdot \big(\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)\big) \, dt_1 dt_2$$

By Lemma 4.2.3, we have

$$= \frac{1}{4\pi} \sum_{j=1}^n \sum_{i=1}^m \int_{j-1}^j \int_{i-1}^i \pm |\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)| \, dt_1 dt_2$$

If orientation is not being considered, then we have that

$$\leq \frac{1}{4\pi} \sum_{j=1}^n \sum_{i=1}^m \int_{j-1}^j \int_{i-1}^i |\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)| \, dt_1 dt_2$$

Now for each (i, j) with $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$, the images edges e_i and e_j form some spherical quadrilateral on S^2 and by Equation 1, we have then that each double line integral $\int_{j-1}^j \int_{i-1}^i |\mathbf{n}_{t_1}(t_1, t_2) \times \mathbf{n}_{t_2}(t_1, t_2)| \, dt_1 dt_2$ is the area of some quadrilateral on S^2. We recall that given a spherical quadrilateral $\mathcal{Q}$ with angles A, B, C, D, the area of $\mathcal{Q}$ is given by excess angle formula $A + B + C + D - 2\pi$; by convention (and by our assumption of no two edges from different components being parallel) we have that $0 < A, B, C, D < \pi$; then $A + B + C + D < 4\pi$ and furthermore, $A + B + C + D - 2\pi < 2\pi$. Hence, we have that $Area(\mathcal{Q}) < 2\pi$ and so

$$< \frac{1}{4\pi} \sum_{j=1}^n \sum_{i=1}^m 2\pi = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{2} = \frac{mn}{2}$$

In addition, if m and n are both odd, then we must have $lk(L) \leq \lfloor \frac{mn}{2} \rfloor$ since $lk(L)$ is an integer value. $\square$

This result says that for a stick link of two components, the Gauss map of an edge e_i on K_1 and an edge e_j on K_2 contributes less than 2π to covering S^2 and furthermore less than $\frac{1}{2}$ to the overall linking number. An immediate corollary arises when we appeal to stick number:

4.3.2 Corollary. *Given a link L of two components, we have that $2\sqrt{2lk(L)} < s(L)$.*

Proof. First we observe that the product mn is maximal when $m = n$ (the most area a rectangle with sides length m and n can enclose is when $m = n$). Now assume we have a stick link of two components $L = (K_1, K_2)$ constructed in its minimal stick representation with m sticks for K_1 and n sticks for K_2 (so we have $m + n = s(L)$). By Proposition 4.3.1 we have that $lk(L) < \frac{mn}{2}$. By our first observation, we have that mn is maximal when $m = n = \frac{s(L)}{2}$; combining this observation and Proposition 4.3.1 gives us $lk(L) < \frac{s(L)^2}{8}$, and rewriting this inequality gives $2\sqrt{2lk(L)} < s(L)$. $\square$

Corollary 4.3.2 sets the stage for our next goal, which is improving our upper bound of $\frac{mn}{2}$ by $\frac{1}{2}$. To motivate this goal, we consider a class of links of two components that are essentially the "doubles" of the torus knots $T_{p-1,p}$ (Figure 12 shows the trefoil $T_{2,3}$ and its "double"); a consequence of the work of Jin (from [4]) states that these links satisfy[6]

$$s = 2 + 2\sqrt{1 + 4lk}$$

We note that this class of links have a lower bound on stick number that is asymptotically $4\sqrt{lk}$; our lower bound for stick number in Corollary 4.3.2 is asymptotically $2\sqrt{2lk}$, which is off by a factor of $\sqrt{2}$. Hence, if the upper bound could be improved by a factor of $\frac{1}{2}$, we would have that all stick links of two components have a sharp lower bound for stick number that is asymptotically $4\sqrt{lk}$; this would be useful for determining formulas of stick number for classes of sticks links of two components.

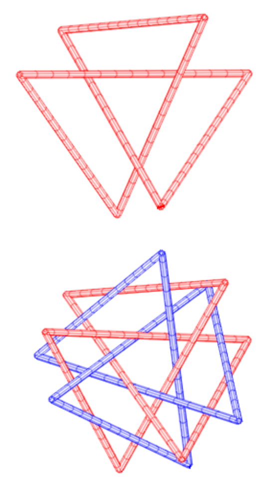


Figure 12

5 Other Interpretations of Linking Number

From here, we have been trying to improve this upper bound of $\frac{mn}{2}$ by considering orientation into the two links. This has led to several different approaches including using vector calculus tools to reinterpret the Gauss linking number integral and also using the "book foliation" of $\mathbb{R}^3$ to examine how one component sits in space with respect to an edge of the second component. We discuss each of the approaches and (hopefully) the results that came out of each result.

[6] The details of this result need to be (I think) formalized.

5.1 Foliation

The foliation approach was one of our first methods to getting a better upper bound by in incorporating orientation. If M is an n-manifold, then a *k-foliation* $\mathcal{F}$ of M (where $k < n$) is a partition of M into parallel k-submanifolds.

In our setting, we use the foliation of $\mathbb{R}^3$ defined by $\{H_\theta : \theta \in [0, 2\pi)\}$ where $H_\theta = \{(r\cos\theta, r\sin\theta, z) : r, z \in \mathbb{R}\}$ is the half plane along the z axis, θ radians from the positive x-axis.

Given an oriented stick link $L = (K_1, K_2)$ with K_1 made from m sticks and K_2 made from n sticks, for each edge $e \in K_2$, we examined how K_1 sat in space with e as the z-axis. For each θ, H_θ with e as the z-axis intersects K_1 some number of times. We define a *positive intersection* if orientation of K_1 is in the same direction of travel as the positive θ direction and *negative* if orientation of K_1 is opposite the travel of the positive θ direction; we assign each intersection i a sign $\varepsilon(i) = \pm 1$ based on the sign of the intersection and we defined $w_e(\theta) := \sum_i \varepsilon(i)$ where i are the intersections of K_1 and H_θ (note that intersections can be with vertices or edges; if a vertex intersects and the connecting edges are on one side of H_θ, we say this is a *zero vertex intersection* and the sign of the intersection is 0. See Figure 13).

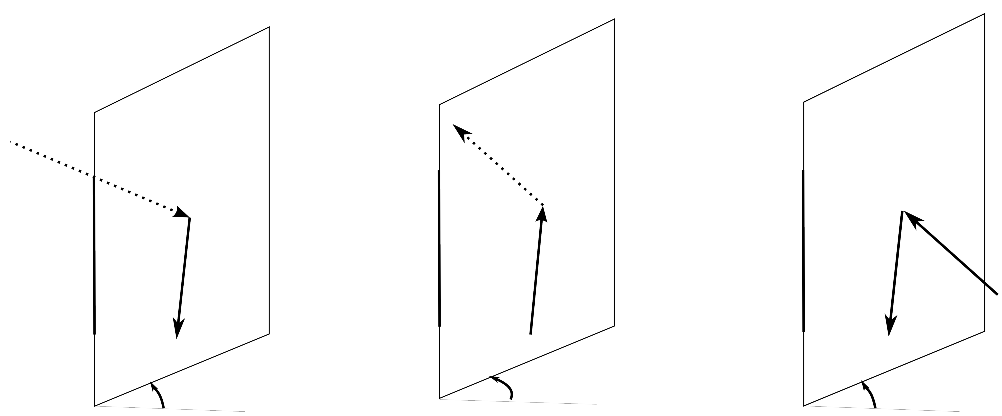


Figure 13: From left to right: a negative intersection, positive intersection, and 0 intersection

We claimed that

5.1.1 Conjecture. *Given an edge e of K_2, $w_e(\theta)$ with respect to $e \in K_2$ is consistent as θ varies.*

and defined $w_e := w_e(\theta)$ to be the *wrapping number* with respect to $e \in K_2$, although no formal proof has been written down. The proof would involve checking for zigzags and reducing them to straight line segments. In fact, if we look from above (so the z-axis pokes us in the face), we would be able to view wrapping number as the winding number of K_1 around the z-axis (equivalently, the linking number of K_1 and the z-axis connect at infinity).

We tried to relate the wrapping number and the orientation of the link. I did try to consider wrapping number with respect to an edge $e \in K_2$, with the sum of the signs of the area contributions of the Gauss map of K_1 and e, although I had no results. For a brief moment by checking some examples I did consider looking for a relation between the average of the signs of the area contributions of the image of K_1 and $e \in K_2$ under $\mathbf{n}$ and the wrapping number.

5.2 Stokes

The Stokes approach is based on interpreting Gauss' linking number integral along K_1 of a magnetic field B_{K_2} induced by a steady line current along K_2. Here is the plan: We can always find an orientable surface with boundary $\mathcal{S}_{K_1}$ that bounds K_1, namely some Seifert surface of K_1; then with distribution theory, we believe that a generalized version of Stokes' Theorem for vector fields with singularities allows us to trade our the Gauss linking number integral for an integral of the curl of the magnetic field $\nabla \times B_{K_2}$ over $\mathcal{S}_{K_1}$. A problem that could arise with using Stokes Theorem is that $B_{K_2}(r)$ is not defined on all of $\mathcal{S}_{K_1}$ which means $B_{K_2}(v)$ is not C^1 on S_{K_1}; if L is a nontrivial link, then there will be at least $lk(L)$ intersections of K_2 and $\mathcal{S}_{K_1}$ and these correspond to an indeterminate value in $B_{K_2}(r_1)$, which is why we will need some distribution theory. We start off with a lemma:

5.2.1 Lemma. *Let*

$$B(r_1) = \oint_{K_2} \frac{dr_2 \times (r_2 - r_1)}{|r_2 - r_1|^3}$$

be the magnetic field at r_1 induced by a steady line current traveling along $r_2 \in K_2$. Then we have $\nabla \times B(r_1) = \oint_{K_2} \delta^3(r_2 - r_1)$.

Proof.

$$\nabla \times B(r_1) = \nabla \times \oint_{K_2} \frac{dr_2 \times (r_2 - r_1)}{|r_2 - r_1|^3}$$

Since K_1 and K_2 do not intersect, we can see each individual component function and its partial derivative is continuous in second variable (if $r_2 = (x_2, y_2, z_2)$, then continuous in x_2, y_2, and z_2); hence we *should* be able to interchange to curl and closed loop integral

$$\begin{aligned} &= \nabla \times \oint_{K_2} \left(dr_2 \times \frac{r_2 - r_1}{|r_2 - r_1|^3} \right) \\ &= \oint_{K_2} \nabla \times \left(dr_2 \times \frac{(r_2 - r_1)}{|r_2 - r_1|^3} \right) \end{aligned}$$

and so we can pause on integration[7] and consider taking the curl of the integrand. Now using the product rule, $\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A)$, we obtain:

$$\begin{aligned} &\nabla \times \left(dr_2 \times \frac{r_2 - r_1}{|r_2 - r_1|^3} \right) \\ &\quad = \left(\frac{r_2 - r_1}{|r_2 - r_1|^3} \cdot \nabla \right) dr_2 - (dr_2 \cdot \nabla) \frac{r_2 - r1}{|r_2 - r_1|^3} + dr_2 \left(\nabla \cdot \frac{r_2 - r_1}{|r_2 - r_1|^3} \right) - \frac{r_2 - r_1}{|r_2 - r_1|^3} (\nabla \cdot dr_2) \end{aligned}$$

Now anything involving a $\nabla \cdot dr_2$ or ∇dr_2 will contribute 0 to the integrand since the curl is being taken with respect to r_1.

$$= -(dr_2 \cdot \nabla) \frac{r_2 - r_1}{|r_2 - r_1|^3} + dr_2 \left(\nabla \cdot \frac{r_2 - r_1}{|r_2 - r_1|^3} \right)$$

[7] The interchange of integral and curl was not shown, but by inspecting the first component function of $\frac{dr_2 \times (r_2 - r_1)}{|r_2 - r_1|^3}$, it did look promising. I could be wrong.

Now the second term becomes $4\pi\delta^3(r_2 - r_1)\ dr_2$ where $\delta^3(v)$ is the 3-dimension Dirac Delta Function. The claim is that the integral over the first term contributes nothing to the curl of $B(r_1)$. We see that if we rewrite the first term in to component format with $r_1 = (x_1, y_1, z_1)$ and $r_2 = (x_2, y_2, z_2)$, we obtain:

$$\begin{aligned}
-(dr_2 \cdot \nabla)\frac{r_2 - r_1}{|r_2 - r_1|^3} &= (dr_2 \cdot \nabla)\frac{r_1 - r_2}{|r_1 - r_2|^3} \\
&= (dr_2 \cdot \nabla)\left(\frac{x_1 - x_2}{|r_1 - r_2|^3}, \frac{y_1 - y_2}{|r_1 - r_2|^3}, \frac{z_1 - z_2}{|r_1 - r_2|^3}\right) \\
&= \left(dr_2 \cdot \nabla\frac{x_1 - x_2}{|r_1 - r_2|^3}, dr_2 \cdot \nabla\frac{y_1 - y_2}{|r_1 - r_2|^3}, dr_2 \cdot \nabla\frac{z_1 - z_2}{|r_1 - r_2|^3}\right) \\
&= \left(\nabla\frac{x_1 - x_2}{|r_1 - r_2|^3} \cdot dr_2, \nabla\frac{y_1 - y_2}{|r_1 - r_2|^3} \cdot dr_2, \nabla\frac{z_1 - z_2}{|r_1 - r_2|^3} \cdot dr_2\right)
\end{aligned}$$

Then the integral over this first term becomes:

$$\oint_{K_2} -(dr_2 \cdot \nabla)\frac{r_1 - r_2}{|r_1 - r_2|^3} = \left(\oint_{K_2} -\nabla\frac{x_1 - x_2}{|r_1 - r_2|^3} \cdot dr_2, \oint_{K_2} -\nabla\frac{y_1 - y_2}{|r_1 - r_2|^3} \cdot dr_2, \oint_{K_2} -\nabla\frac{z_1 - z_2}{|r_1 - r_2|^3} \cdot dr_2\right)$$

By the Gradient Theorem for closed curves, each of the components are 0 which is what we claimed. Hence, we have that $\nabla \times B(r_1) = \oint_{K_2} 4\pi\delta^3(r_2 - r_1)\ dr_2$. □

Now starting with Gauss linking number integral, we have

$$lk(L) = \frac{1}{4\pi}\int_{I_1}\int_{I_2} \frac{\big(r_2(t_2) - r_1(t_1)\big) \cdot \big(r_1'(t_1) \times r_2'(t_2)\big)}{|r_2(t_2) - r_1(t_1)|^3}\ dt_2 dt_1$$

Now use the scalar triple product identity $a \cdot (b \times c) = b \cdot (c \times a)$ to rewrite the integrand as:

$$= \frac{1}{4\pi}\int_{I_1}\int_{I_2} \frac{r_1'(t_1) \cdot \big[r_2'(t_2) \times \big(r_2(t_2) - r_1(t_1)\big)\big]}{|r_2(t_2) - r_1(t_1)|^3}\ dt_2 dt_1$$

Now $r_1'(t_1)$ only depends on t_1 and so by linearity and symmetry of the Euclidean inner product, we have

$$\begin{aligned}
&= \frac{1}{4\pi}\int_{I_1}\left[\int_{I_2} \frac{r_2'(t_2) \times \big(r_2(t_2) - r_1(t_1)\big)}{|r_2(t_2) - r_1(t_1)|^3}\ dt_2\right] \cdot r_1'(t_1)\ dt_1 \\
&= \frac{1}{4\pi}\oint_{K_1}\left[\int_{I_2} \frac{r_2'(t_2) \times \big(r_2(t_2) - r_1\big)}{|r_2(t_2) - r_1|^3}\ dt_2\right] \cdot dr_1 \\
&= \frac{1}{4\pi}\oint_{K_1}\left[\oint_{K_2} \frac{dr_2 \times \big(r_2 - r_1\big)}{|r_1 - r_2|^3}\right] \cdot dr_1
\end{aligned}$$

Now by Biot-Savart, the integral in the brackets can be interpreted as the magnetic field of a steady line current $I = 1/\mu_0$ at a point $r_1(t_1)$ where K_2 is the path of the current, that is, if $B_{K_2}(r) := \oint_{K_2} \frac{dr_2 \times (r - r_2)}{|r - r_2|^3}$

$$= \frac{1}{4\pi}\int_{K_1} B_{K_2}(r_1) \cdot dr_1$$

Now using a Seifert surface of K_1 and a Stokes Theorem for vector fields with singularities

$$= \frac{1}{4\pi} \int_{\mathcal{S}_{K_1}} \nabla \times B_{K_2}(r_1)\ dA$$

If we have Stokes' Theorem for vector fields with singularities, then we would have:

$$\stackrel{?}{=} \int_{\mathcal{S}_{K_1}} \int_{K_2} \delta^3(r_2 - r_1) \cdot dr_2\ dA$$

If $L = (K_1, K_2)$ is a stick link with m sticks for K_1 and n sticks for K_2, we could take the integral we have arrived and get

$$\begin{aligned} \int_{\mathcal{S}_{K_1}} \int_{K_2} \delta^3(r_2 - r_1) \cdot dr_2\ dA &= \int_{\mathcal{S}_{K_1}} \sum_{e \text{ edges}} \int_e \delta^3(r_2 - r_1) \cdot dr_2\ dA \\ &\leq n \int_{\mathcal{S}_{K_1}} dA \end{aligned}$$

Now if $\mathcal{S}_{K_1}$ bounds a planar surface, we could possibly get a bound possibly using the isopermetric inequality. If D is a disk that bounds the planar surface $\mathcal{S}_{K_1}$, then we would have

$$Area(\mathcal{S}_{K_1}) \leq Area(D) \leq \frac{length(D)^2}{4\pi}$$

This gives us that

$$lk(L) \leq n \int_{\mathcal{S}_{K_1}} dA = n \cdot Area(D) \leq n \cdot \frac{length(D)^2}{4\pi}$$

This could be useful for torus links $T_{2,2n}$ since it possible for one component of $T_{2,2n}$ to bound a polygon. However, I am not sure how to go from here since we need to look at length of the edges of K_1.

5.3 Divergence

The Divergence Approach is based on considering the parametrized surface $\mathcal{S}$ defined to be image of difference vectors between the two link components. More specifically, if $L = (K_1, K_2)$ is a link with two components, $r_i : I_i \to \mathbb{R}^3$ the parameterization of K_i, then we define

$$\mathcal{S}_L := \{r_1(t_1) - r_2(t_2) : t_1 \in I_1, t_2 \in I_2\}$$

to be the parametrized surface of L; in particular, we can define $\Phi : I_1 \times I_2 \to \mathbb{R}^3$ by $\Phi(t_1, t_2) := r_1(t_1) - r_2(t_2)$ to be the parametrization of $\mathcal{S}$. One can show using some differential geometry tools that

5.3.1 Lemma. *$\mathcal{S}_L$ is a smooth and closed surface.*

Viewing the integration over the two curves K_1 and K_2 as an integral over $\mathcal{S}_L$ allows us to interpret the Gauss linking number integral as a surface integral of a vector field. In particular, we know that

$$lk(L) = \frac{1}{4\pi} \int_{I_1} \int_{I_2} \frac{(r_2(t_2) - r_1(t_1)) \cdot (r_1'(t_1) \times r_2'(t_2))}{|r_1(t_1) - r_2(t_2)|^3} \, dt_2 dt_1$$

If we let $F : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the vector field defined by $F(v) := \frac{v}{|v|^3}$ for $v \in \mathbb{R}^3$, then we can rewrite the above integral into:

$$\begin{aligned} &= \frac{1}{4\pi} \int_{I_1} \int_{I_2} F(\Phi(t_1, t_2)) \cdot \big(\Phi_{t_1}(t_1, t_2) \times \Phi_{t_2}(t_1, t_2)\big) \, dt_2 dt_1 \\ &= \frac{1}{4\pi} \iint_{\mathcal{S}_L} F(v) \cdot dA \end{aligned} \tag{2}$$

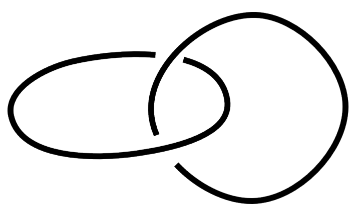
Now we want to use the Divergence Theorem the above integral to hopefully find a better bound on linking number. A problem with the Divergence Theorem requires that $\mathcal{S}_L$ is a smooth, closed, and non self-intersecting surface and unfortunately (for us), unless L is the unlink, the surface $\mathcal{S}_L$ self intersects;

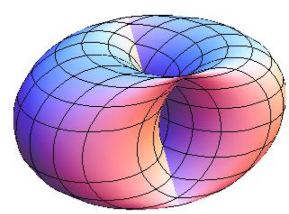
Now I think that a change of variables can bring the integral in Equation (2) to an integral over $\tilde{\mathcal{S}}_L$. It is possible to take this into $\mathbb{R}^6$ by considering the map $f : \mathcal{S}_L \to \mathbb{R}^6$ by the map

$$f(r_1 - r_2) = (r_1, r_2)$$

We can show that this map f is an immersion since K_1 and K_2 do not intersect.

Below, we see the Hopf link and $\mathcal{S}_{\text{Hopf link}}$.





When we appeal to stick links, the singular surface is a union of planes that self intersect; in fact, we know that these planar pieces are quadrilaterals if our link does not have edges from different components that are parallel. Below (Figure 14) we see the Hopf link in its minimal stick representation and its surface:

This surface can be visual by the "roof" and "base" as shown in Figure 15; note the self intersections also how the surface closes up. There is also a square at the bottom of the surface. A way to approach the Divergence Theorem approach is to consider the closed surfaces of $\mathcal{S}_L$; for the case of the stick Hopf link, as in Figure 15, one could try integrating along the top and portions of the base that form a closed surface (the portions of the base here would be the eight smaller triangles caused by the self intersections of the planes in the right picture of Figure 15) and integrating along the remaining portion of the base (the four bigger triangles cause by self intersections and the square); both these surfaces are closed surfaces, and thus we can apply the Divergence Theorem. This method could be hard to calculate for a general stick link of two components.

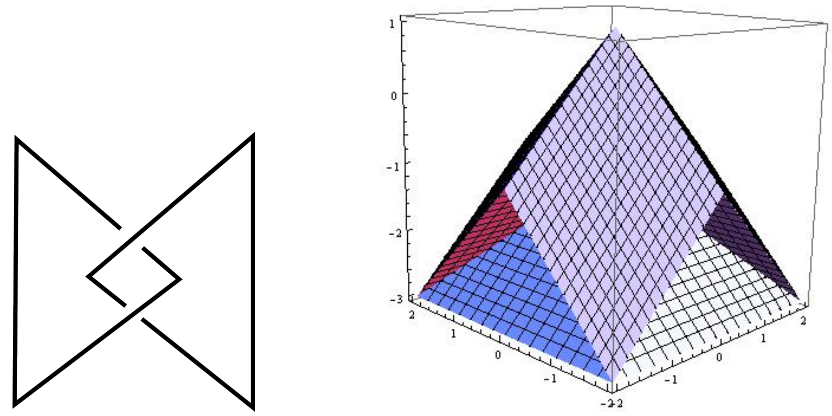


Figure 14

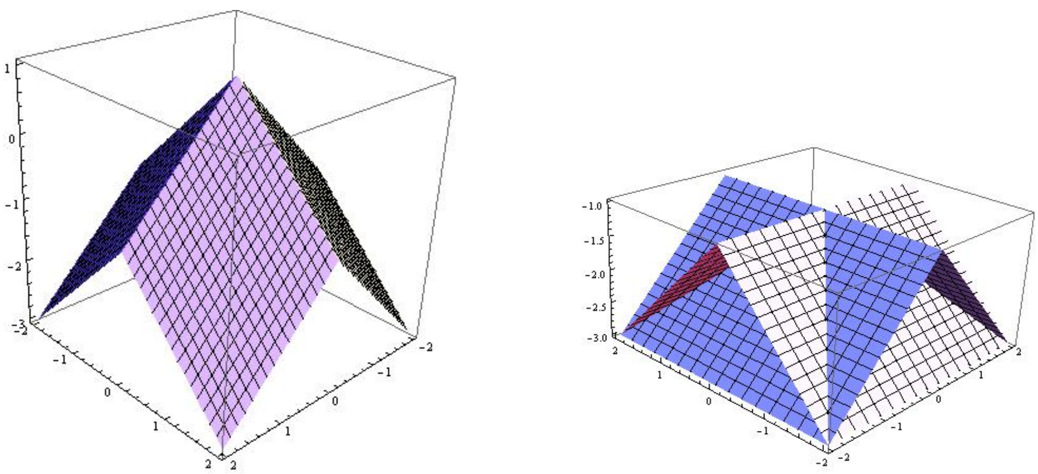


Figure 15

5.4 Concentric Surfaces

This approach looks at the surface $\mathcal{S}_L$ defined in the previous approach: Given a stick link $L := (K_1, K_2)$ parametrized by $r_i : I_i \to K_i \subset \mathbb{R}^3$, we look at the surface

$$S_L := \{r_1(t_1) - r_2(t_2) : t_1 \in I_1, t_2 \in I_2\}$$

As we said before, unless L is the unlink, the surface $\mathcal{S}_L$ will self intersect.

Another observation is that $\mathcal{S}_L$ is a closed and smooth surface that surrounds the origin. We have consider counting these concentric surfaces. A start is to ask: Given n intersecting planes, when does maximal number of closed concentric surfaces occur? We first consider showing

5.4.1 Conjecture. *Given n lines in the plane where n is odd, the maximal number of closed concentric regions (cycles) of the plane occur when the n lines intersect as an n-gon structure.*

Figure 16 shows this for $n = 5$. On the left, we see that there is only one closed region that contains the purple dot; this number remains unchanged if we moved the purple dot

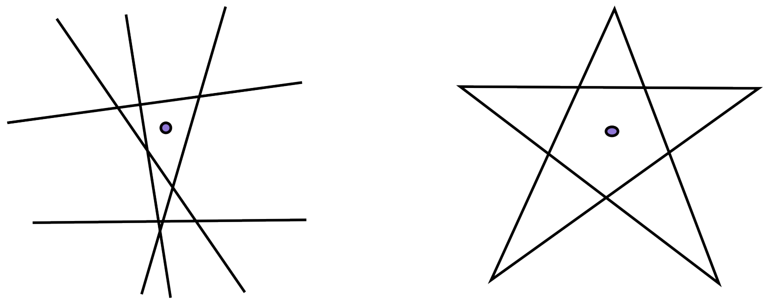


Figure 16

anywhere else in the diagram. On the right, we see that there are two regions that contains the purple dot. The next step would be to view these intersecting lines as planes; we *can* deduce that these planar pieces must be quadrilaterals (assuming that no two edges from different components are parallel) so the next question is how many more quadrilaterals must be used to close the surface? An application of cell complexes, Euler characteristic, and maybe Morse theory may also be helpful to this approach.

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