THREE-VARIABLE BRACKET POLYNOMIAL FOR THREE BRAID KNOTS AND LINKS

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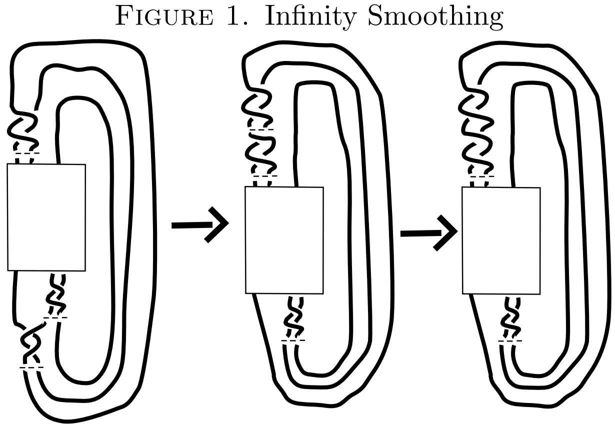
1. Introduction

In this paper alternating, three-braid knots and links are examined. A knot or link is *alternating* if its crossings switch between over and under as you move along the strands of the braid. A *braid* is a set of n-strands which are interwoven and then to make it a knot or link, the braid strands are looped back to the top of the braid, making it so that there are no loose ends [1]. A three-braid knot or link has three strands which are interwoven and then connect back together at the top. The *twists* in a braid are the individual sections formed in the braid when two strands are interwoven. A-twist can range from having just one crossing to many. An *A-twist* is a twist with all A-crossings and a *B-twist* is with all B-crossings. A braid can also have it's crossings be described as *letters* and then the entire interwoven braid described as a *braid word*. A-twist is a series of letters in the braid word. Braids have the special property where individual letters can be rotated about the strands without changing which knot or link it is.

Example 1.1.

Figure 1. Infinity Smoothing



This result can be extended to say that whole twists can move about the strands of the braid without changing the braid. Because a braid has this property, it is possible to conclude that a braid must always have an even number of twists with the exception of a one twist braid. This is because whenever there is an odd number of twists the bottom most twist can be rotated to the top and combined with the top on to reduce the twist number by exactly one, making the twist number even (DO I NEED TO PROVE THIS SECTION?).

The three-variable bracket polynomial is an *invariant* of alternating knots or links and allows us to say whether the two knots or links are isotopic. The three-variable bracket polynomial involves variables A, B, and d and assigning the values based on the skein relation of the knot or link. The exponents of A and B denote the number of A and B smoothings in any given diagram and the exponent of d represents one less the number of components in the final diagram, $|s| - 1 = m$, where m is the degree of d. Two knots are *isotropic* if they are identical under Reidimeister moves, pulling, and deforming the strands without cutting and re-glueing any of the strands. If the two polynomials are not isotropic, we can say with certainty that the two knots or links are distinct. However, if the polynomials are the same we must investigate further using other invariants of knots are links to determine whether the knots are the same or different.

The *skein* relation is formed by assigning states to each crossing. A *state* is the choice of A and B-smoothings to each crossing. A crossing is smoothed into a $zero-tangle$ when an A-smoothing is applied to a B-crossing and a B-smoothing is applied to an A-crossing. An A-crossing is said to be smoothed into an $infinity - tangle$ if an A-smoothing is applied to an A-crossing and a B-smoothing to a B-crossing.

Figure 2. Infinity Smoothing

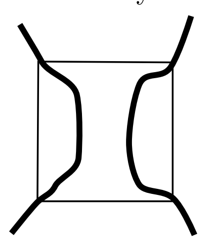
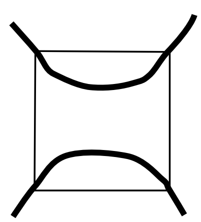


Figure 3. Zero Smoothing



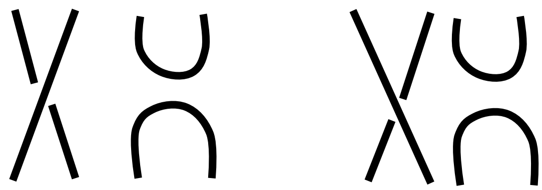
2. Looking at the bracket polynomial and determining the knot

It is possible to look at a knot and calculate it's 3-variable bracket polynomial through a series of steps. I began looking at whether the 3-variable bracket polynomial would give you any clues about the characteristics of the knot. I focused on the 3-braid alternating knots and links.

Lemma 2.1. *For a knot with a least two crossings per twist, smoothing across the twists, turning each twist into a zero-smoothing, gives the maximum degree of d* [2].

Lemma 2.2. *To create a zero-smoothing along any vertical twist, an A-smoothing is applied to a B-crossing and a B-smoothing applied to an A-crossing* [2].

Figure 4. A-crossing (left) and B-crossing (right) to zero tangles



Lemma 2.3. *Three braid knots or links with at least two twists can be smoothed into the unknot when every twist is replaced by a zero-tangle.*

Proof. A three braid knot or link will always have exactly one twist or an even number because a property of braids is that the twists can move in a linear rotation about the strands, meaning if there appears to have an odd number of twists, there is a way to minimize the twists by at least one by moving the bottom most twist to the top and combining it with the top most twist. Since there must be at least two twists for this lemma to hold, the number of twists must an even.

First consider the base case. Three-braid knots or links with two twists, having all zero-smoothings produces the unknot.

Next consider the inductive case. Let n be the number of twists in the braid where all zero-smoothings give the unknot. Suppose two new twists are added to the bottom of the braid, and both of the twists are smoothed along the zero-smoothing. We can see that the bottom of the knot with $n + 2$ twists is isotropic to the bottom of the knot with just n twists. Therefore all three-braid knots or links with at least two twists with all zero-smoothings produces the unknot. (MODELED AFTER K. LAFFERTY) □

Theorem 2.4. *For a three-braid knot or link with n twists, c total crossings, and at least two crossings in each twist, if m is the maximum degree of d in the bracket polynomial, then c-m=n*

FIGURE 5. Choosing states for a three braid results in different number of components contributing to the bracket polynomial

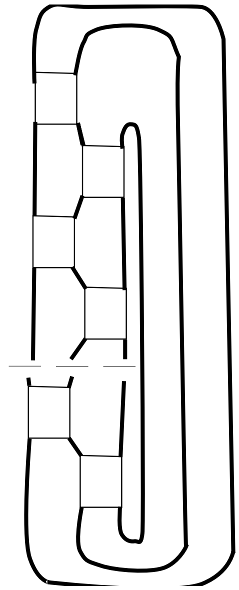
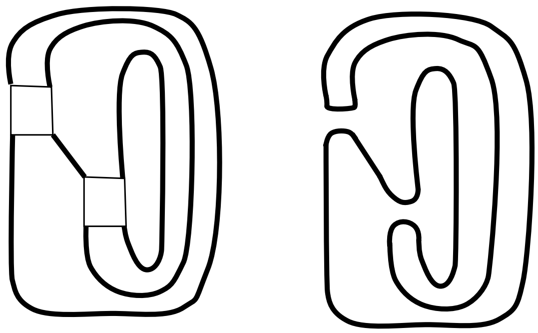
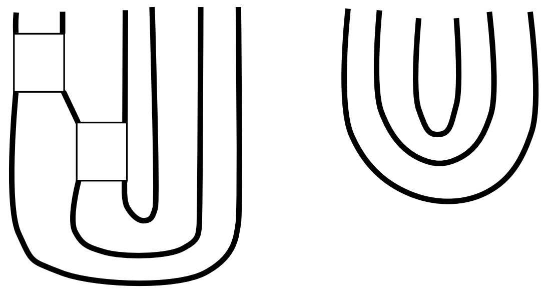


FIGURE 6. Base Case: Two twists smoothed into zero-tangles creates the unknot



Proof. Using the results from the Lemma 2.1 allows us to conclude that zero smoothings in a three-braid knot or link with at least two crossings in each twist, gives us the maximum degree of d. Let c_i be the number of crossings in twist t_i. Smoothing along the zero tangle gives us $c_i - 1$ loop components. We also must consider the additional component that is created when a three braid knot with at least two crossings smoothed into zero-tangles. Therefore, smoothing along the zero tangles for all of the twists, where l is the number of loop components, gives us

FIGURE 7. Inductive Step: Two additional twists smoothed into zero-tangles



$$l = \left(\sum_{i=1}^{n} c_i\right) + 1 = \left(\sum_{i=1}^{n} c_i\right) - 1(n) + 1 \tag{1}$$

By definition we know that $l = m + 1$ where m is the degree of d. Therefore we can combine the two equations through substitution giving,

$$m = \left(\sum_{i=1}^{n} c_i\right) - n + 1 - 1 = \left(\sum_{i=1}^{n} c_i\right) - n \tag{2}$$

Since $\left(\sum_{i=1}^{n} c_i\right)$ is the crossing number, we can let $\left(\sum_{i=1}^{n} c_i\right) = c$. We can simplify the statement above we get $m = c - n$ or $c - m = n$.

□

Lemma 2.5. *$A^y B^x d^m$ where m is the highest degree of d. Where x is the total number of all A-crossings and y is the total number of all B-crossings. Let x_i be the number of A-crossings in twist t_i and y_i be the number of B crossings in twist t_i.*

$$x = (\sum_{i=1}^{n} x_i) \tag{3}$$

$$y = (\sum_{i=1}^{n} y_i) \tag{4}$$

Proof. In order to maximize the degree of d, you must smooth along the zero-tangles. In order to smooth along the zero tangles in a vertical twist, you must make A-smoothings where there are B-crossings and B-smoothings where there are A-crossings by Lemma 2.2. Since this is true, the number of A-smoothings must indicate the number of B-crossings and the number of B-smoothings must indicate the number of A-crossings.

□

Lemma 2.6. *For a three braid knot or link to be alternating, the twists must switch back and forth between A-twists and B-twists.*

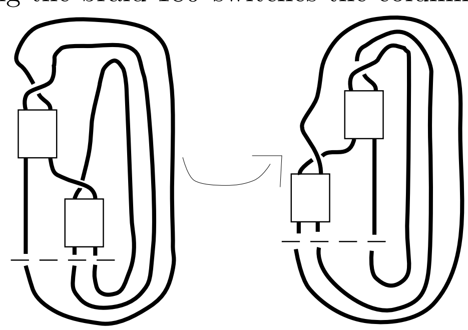
Proof. Consider the case where there are two adjacent A-twists. Where the two twist adjoin, there are either two under crossings or two over crossings. The same occurs with the case where both twists are B-twists.

Next consider the case where an A-twist is adjacent to a B-twist. When this occurs, alternating crossings are preserved. □

Lemma 2.7. *In a three braid knot or link, all of the A-twists will be involving an outer strand and the middle and located on one side of the braid and all of the B-twist will be made using the other outer string and the middle strand and located on the opposite side of the braid.*

Lemma 2.8. *We can generalize the three-braid knots or links to always look as followed. So that the braid always begins with the A-twist in the upper left corner. Refer to figure 8.*

FIGURE 8. Rotating the braid 180°switches the columns of A and B-twists



Proof. We can always have the A-crossing twist at the top of the braid because by the property of braids the twists can be rotated linearly along the strands

We can always have the A-twist on the left side of the braid because by rotating the braid 180 degrees about the z-axis, we see that it is isotropic to having the A-twist on the right side of the braid. We can always have the A-twist at the top of the braid because the property of braids we are allowed to rotate the twists about the strands of the braids, allowing us to always have an A-twist at the top of the braid. □

Lemma 2.9. *In a four twist, alternating, three-braid where all the crossings are prime, it may be possible to find the number of crossings in any given twist using the coefficient of* $(A^{(c_1+c_3]}B^{(c_2+c_4)}d^0)$.

Proof. The d^0 term is always the following for four-twist three-braid alternating knot.

$$[[(c_1+c_3)(c_2+c_4)]+c_1c_2c_3c_4]A^{(c_1+c_3]}B^{(c_2+c_4)}d^0 \tag{5}$$

We know the value of $(c_1 + c_3)(c_2 + c_4) = xy$ where where x and y are the total number of A-crossings and B-crossings from looking at d^m, where m is the highest degree of d. Using this piece of information and the theorem of prime factorization, every integer is the unique product of primes we can attempt to find the number of crossings in each individual twist.

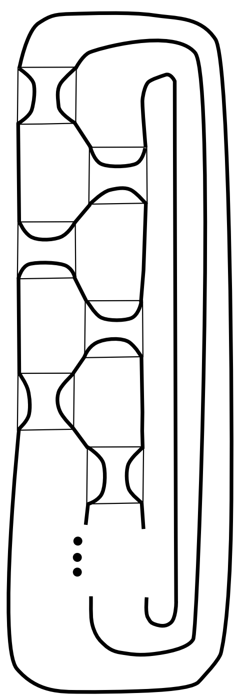
□

3. Determining the number of components given a specific ordering of states

Definition 3.1. *A state word is the ordering of the chosen states (infinity tangle or zero tangle) for each twist. Each letter in the word corresponds to a specific state for each twist. Let i represents an infinity-tangle and let z represents a zero-tangle*

Example 3.2. *The following state word can be written as* $< i, z, z, z, i, i, ... >$.

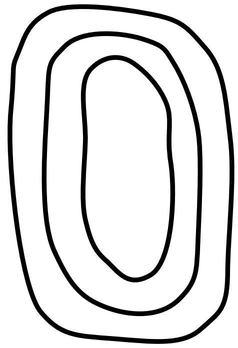
Figure 9. State Word example



Lemma 3.3. *If all twists are smoothed into infinity-tangles, the number of components in the final diagram,* $|s| = 3$.

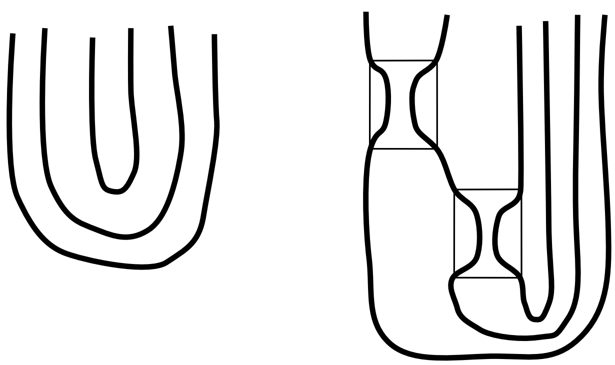
Proof. First consider case one, where there is only one infinity-tangle. When it is smoothed, it creates three unknot components.

Figure 10. Base Case: No twists create three unknot components



Next, consider case two, where there are an even number of infinity-tangles. Consider the base case where there are no twists in the three-braid. Refer to Figure 10.

Figure 11. Inductive Step: Adding two infinity twists and there still remains three unknot components



Next, consider the inductive step (Figure 11). Let n be the pairs of twists in the three-braid where all of the infinity-tangles give exactly three components. Suppose one new pair is added to the bottom of the knot and the two twists are into infinity-tangle. It will always be a pair of two individual twists because a braid must maintain an even number of twists

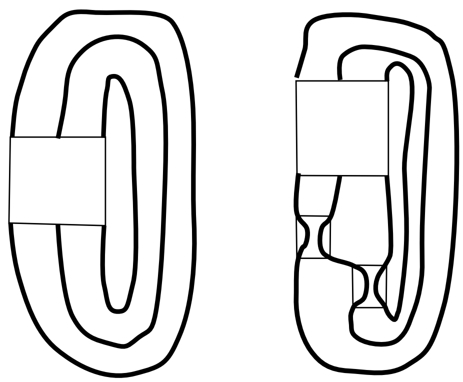
We see that the knot with $n + 2$ twists is isotropic to the knot with n twists. Therefore smoothing along all the infinity-tangles will create three components.

□

Lemma 3.4. *An even number of infinity-tangles can be added in pairs (one in the A column and one in the B column) without changing the number of components (so long as there was an even number of twists in the original braid. If there is only one twist in the* *original braid, an odd number of infinity tangles can be added (all in pair except one) so the overall twist number is even and without changing the number of components.*

Proof. Consider the base case, a braid knot or link with some number of zero and infinity-tangles, so $|s| = m$

FIGURE 12. Base Case (left) and Inductive Step (right)



Next, consider the inductive case. Let n be the number of additional twist pairs that have been smoothed into the infinity tangles adding, this add no additional components. These n twists are added to the bottom of the braid. Suppose two new twists are added to the bottom of the braid, we we have $n + 2$ additional twists that have been smoothed into the infinity tangle. The bottom of the braid with n twists smoothed into infinity tangles is isotopic to the braid knot with $n + 2$ twists that are smoothed into infinity tangles. Thus, adding an even number of twists does not change the number of components.

Since a braid is isomorphic under rotation of it's twist about the braid stands, we can generalize the statement that says adding an number of infinity tangles to the bottom of a braid is isomorphic to the braid which does not have these infinity tangles into, adding an number of infinity tangles in pairs anywhere into a knot will be isomorphic to the braid that does not contain these infinity tangles.

Consider the case where there is just one twist in the original braid, consider adding a infinity infinity tangle, this does not change the number of components. From here the proof continues as if it had originally started with an even number of components.

□

Corollary 3.5. *When a state word has an pair of adjacent infinity tangles ie.* $< ..., i, i, ... >$*, the number of components in the state word without those letters, ie. the pairs of infinity twists, is the same. If the state word is just comprised of a single infinity tangle is* $< i >$*, this rule also applies.*

Proof. Proof for pair adjacent infinity tangles follows that of Lemma 3.4

Proof for a single infinity tangle in a state word, then the number of components is the same as if there were none. When there is one infinity tangle $|s| = 3$. When there are infinity tangles $|s| = 3$ □

Lemma 3.6. *Adding pairs of zero tangles to already existing single zero tangles or other pairs of zero tangles does not change the number of components.*

Proof. Consider the base case, where one or two zero tangles stand alone.

FIGURE 13. Base Case: one or two zero tangles

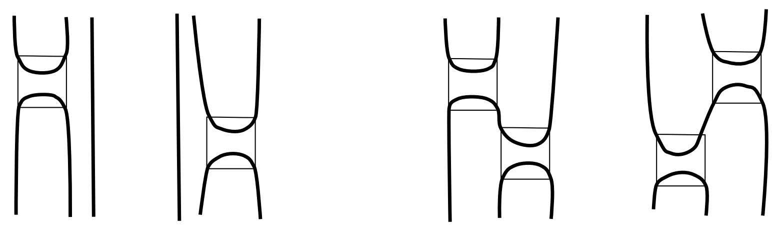
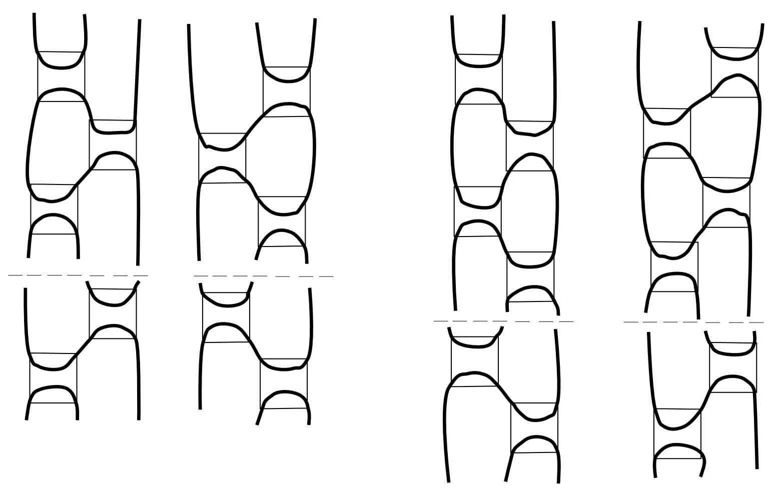


FIGURE 14. Inductive Step: Adding two zero tangles and the overall structure of the diagram does not change



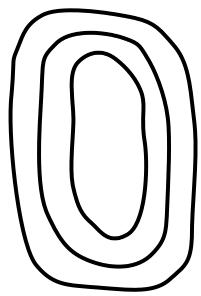
Now consider the inductive case. Let n be the number of additional zero tangle pairs. Suppose a new zero-tangle pair is added such that there are $n + 1$ zero tangle pairs. The connecting point have not changed and no components have been added from the individual connections of the zero-tangles so the number of components contributed to the three-variable bracket polynomial are the same.

□

Lemma 3.7. *Adding a single zero-tangle to column A or column B (where column A is defined by having all of the A-twists and column B by all the B-twists) each reduces the unknot components by one.*

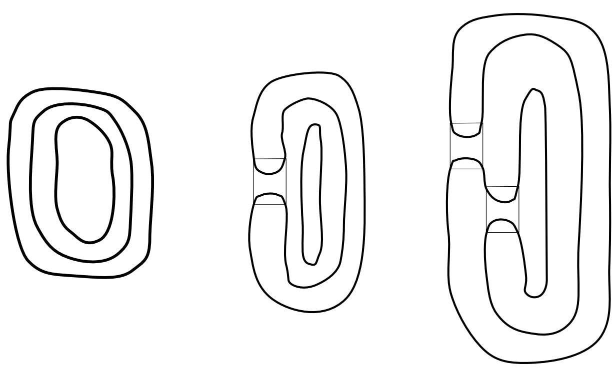
Proof. Consider the case, where there is a three-braid knot or link with no zero-tangles

FIGURE 15. No zero tangles



Next consider the case where one zero-tangle is added. When this is done, two components are being connected by the zero-smoothing. When a zero-tangle is added to both of the braid columns, two components are again reducing by one.

FIGURE 16. One or Two Zero Tangles

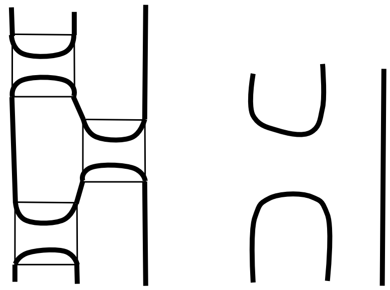


□

Lemma 3.8. *Once there is at least one zero-tangle in column A and B, the addition of anymore zero-tangles does not change the number of unknot components in the final diagram so long at they are adjacent to the existing zero-tangles (ie not broken up by any infinity-tangles).*

Proof. Follows that of 3.6. □

FIGURE 17. Base Case: Overall structure of three zero-tangles in a row

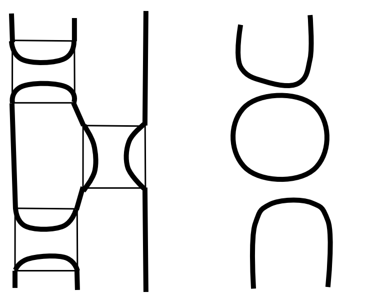


Proof.

Lemma 3.9. *When there is a sequence of (...z i z...) in a state word, an additional component is added.*

Consider the base case, where there is a string of three zero-tangles (...zzz...). Refer to Figure 17.

FIGURE 18. Inductive Case: Replacing a Zero-Tangle with an Infinity-Tangle



Now consider changing the middle most zero-tangle to an infinity-tangle. Refer to Figure 18

Having an infinity-tangle keeps the diagrams isomorphic to the diagram with all zero-tangles but with the addition of a separate component. □

Theorem 3.10. *The number of components contributed by a diagram to the bracket polynomial is*

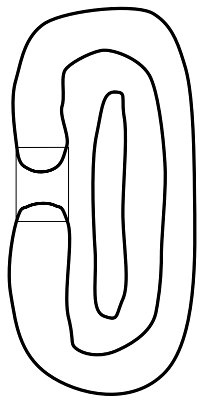
$$|s| = 3 - (q_a) - (q_b) + i_r \tag{6}$$

$$(q_i) = \begin{cases} 1 & \text{:there is a zero-tangle in column } i \\ 0 & \text{:there is NOT a zero tangle in column } i \end{cases}$$

and i_r is the number of infinity-tangles after the state word of the braid has been reduced.

Proof. Consider case 1, a three-braid link with only infinity-tangles. $q_a = 0$ and $q_b = 0$ because there are no zero-tangles in column A or B. $i_r = 0$ because the number of infinity-tangles is either even and reduced to 0 or 1 which also reduces to 0 by Corollary 3.5. We know this to be true by Lemma 3.3. Note, this case includes both scenarios where there is just one infinity-tangle or an even number of infinity-tangles.

FIGURE 19. One Zero Tangles



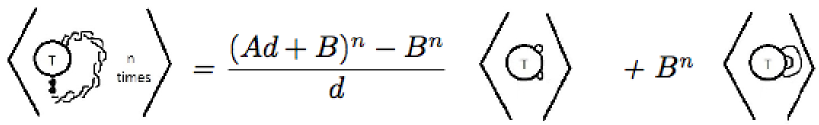
Consider case 2, when there is only one twist and it is smoothed into a zero-tangle. $|s| = 3 - 0 - 1 + 0 = 2$ by our theorem. This can be verified by figure 19.

Consider a three-braid knot or link with both zero and infinity-tangles that has already been reduced (ie no adjacent infinity tangles). The braid contains some number, n smoothing. Assume that n is even number and not 1. Consider a braid with exactly n zero-smoothings. Since there is at least one zero-smoothing in twist column A and B, $q_a = 1$ and $q_b = 1$, and since there are no infinity-tangles $i_r = 0$, therefore $|s| = 1$. By Lemma 2.3 we verify that this braid is smoothed into a single unknot component. Now going through and systematically replacing corresponding smoothings from zero-tangles to infinity-tangles based on the state word. Each time a zero-tangle is replaced by an infinity-tangle, i_r increases by 1. This corresponds and is verified by Lemma 3.9. If a whole column of zero-tangles is changed to all infinity-tangles, the corresponding column, q_a pr q_b goes from 1 to 0 and another component is added. This can be verified by Lemma 3.7. □

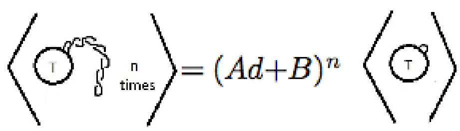
4. A Systematic Approach for Calculating the Three Variable Bracket Polynomial for Three-Braid Knots and Links

For a B-twist, the three variable bracket polynomial is given by,

Lemma 4.1. *For a B-twist, the three variable bracket polynomial is given by Equations 7 and 8*



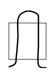
(7)



(8)

For an A-twist, the same rule applies replacing B's with A and A's with B.

It is important to note that although there are two equations, one for when the twist is connected on both ends of the tangle and the other for when only one is, this is for simplification. Equation 7 encompasses both cases. Therefore,

(9) $(Ad + B)^n <$  $>= (Ad + B)^n - B^n < ..., 0, ... > +B^n < ..., 1, ... >$

The proof for this can be done by simplification of

$$(Ad + B)^n - Bn < ..., 0, ... > +B^n < ..., 1, ... >.$$

A bracket polynomial of a diagram is the sum of every possible combination of states. Where $(Ad+B)^n - B^n < ..., 0, ... >$ is put in for all zero tangles and $B^n < ..., 1, ... >$ for all infinity tangles. Therefore by combining Lemma 4.1 and Lemma 3.10 we can systematically calculate the three variable bracket polynomial for a three braid knot or link by the following program using mathematica.

Mathematica Program

Define p_1, p_2, p_i to be the number of crossings in twist i

Define

References

[1] Adam's knot book - braids pg. 127
[2] Kelsey Lafferty
[3] Matthew Overduin